Auxiliary Sets of Matrices with New Step Parameter Sequences

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ABSTRACT

Constructions are given for auxiliary sets of matrices which have the previously unknown step parameter sequences (3,11,3), (2,2,8,11), (2,2,8,13), (2,2,9,11). As a consequence one obtains a projective Hjelmslev plane for each of the preceding four step parameter sequences.

1. INTRODUCTION

Every finite projective Hjelmslev plane (briefly PH-plane) possesses a canonically associated sequence of integers $(q_2, q_3, \ldots, q_n)$ called its Hjelmslev step parameter sequence. If the gross structure Π' of a PH-plane Π is a projective plane of order $r = q_1$, we call $(q_1, q_2, \ldots, q_n)$ the extended (Hjelmslev) (step parameter) sequence of Π. Until 1978 every PH-plane that had been constructed possessed a nondecreasing extended sequence. In that year Jungnickel [5] constructed (for every prime power $r \geqslant 5$) a PH-plane with extended Hjelmslev sequence (r, q_2, q_3, q_4) for which

$$r = q_2 < q_3 = r^2 > q_4 > r.$$

Later Sane [7] constructed a PH-plane with extended sequence (2,2,5,2).

To date Sane's PH-plane (and preimages of it) are the only known examples of PH-planes which satisfy $q_i > q_j = r$ for some i and j with $i < j$. In this paper we construct a second example, namely, a PH-plane with extended

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sequence (3,3,11,3). Sane's PH-plane and the one herein constructed are both of interest in relation to an observation of Törner [11, p. 12] that $q_i \geq r$ for every PH-plane and every i. (See also [10], proof of 1.1.)

Recently Sane [9] has constructed several PH-planes with extended sequences $(4,4,19,s)$ where $s<19$. The following result is thus established: for every prime power r there exists a PH-plane with extended sequence (r,r,q_3,q_4) for some pair (q_3,q_4) with $q_3>q_4$.

In addition to the PH-plane with extended sequence (3,3,11,3), we construct PH-planes with extended sequences $(2,2,2,q_4,q_5)$ where $(q_4,q_5)=$ (8,11), (8,13), (9,11), (9,13), and (9,17). The first three of the preceding five sequences are new. A (t,r) PH-plane is a PH-plane whose extended sequence $(q_1,q_2,\dots,q_n)$ satisfies $r=q_1$ and $t=q_2q_3\cdots q_n$. Up to this time $(t,2)$ PH-planes have been constructed for 72 values of $t<1000$. (See [1], [6], [7], [8].) Here we have obtained the three new values 352, 396, and 416, making a total of 75 known values.

Each of the PH-planes constructed in this paper is obtained from an "auxiliary" set of matrices (defined in the opening paragraphs of Section 2). Since such sets may be used to construct Klingenberg and Hjelmslev structures over gross structures Π' which are not necessarily projective planes, the existence of the PH-planes mentioned above is only a particular consequence of our actual results.

2. AUXILIARY SETS OF MATRICES

Let $\mathcal{A}=\{A^0,A^1,\dots,A^r\}$ be a set of square (0,1)-matrices having common order a, $A^\lambda=[a_{ij}^\lambda]$. We write $S(a)$ to denote the set $\{0,1,\dots,a-1\}$ and use the elements of $S(a)$ to index the rows and columns of the A^λ. One calls $\mathcal{A}$ a *Klingenberg auxiliary set* (briefly a *K-set*) provided that

$$A^\lambda(A^\mu)^T=(A^\lambda)^TA^\mu=J \qquad \text{when} \quad \lambda\neq\mu. \tag{2.1}$$

A K-set is said to be *cohesive* (*doubly cohesive*) if the following conditions hold with $i=1$ (with $i=2$):

$$\sum_\lambda A^\lambda(A^\lambda)^T\geq iJ, \qquad \sum_\lambda (A^\lambda)^TA^\lambda\geq iJ. \tag{2.2}$$

Here we have written A^T to denote the matrix transpose of A, J to denote a matrix with all entries equal to one, and $[a_{ij}]\geq[b_{ij}]$ to indicate that two matrices are of the same size and shape and satisfy $a_{ij}\geq b_{ij}$ for all i and j. A doubly cohesive K-set is also called an *H-set* (for Hjelmslev set).

Next let $\mathcal{A}$ and $\mathcal{B}=\{B^0, B^1,\dots,B^r\}$ be K-sets of matrices of orders a and b, respectively; $B^\lambda=[b_{ij}^\lambda]$ for λ in $S(r+1)$. Let f and g be surjective maps from $S(b)$ to $S(a)$. Then the pair (f,g) is called an *epimorphism* from $\mathcal{B}$ to $\mathcal{A}$ (and $\mathcal{A}$ is said to be an *epimorphic image* of $\mathcal{B}$) provided that the following condition holds for all λ in $S(r+1)$ and all i,j in $S(b)$:

$$(2.3)\qquad b_{ij}^\lambda=1 \quad\text{implies}\quad a_{mn}^\lambda=1,\quad\text{where } m=f(i) \text{ and } n=g(j).$$

The epimorphism (f,g) is called *proper* unless both f and g are one-to-one; one calls (f,g) an *isomorphism* if f and g are both invertible and (f^{-1},g^{-1}) is an epimorphism. Now assume that $\mathcal{A}=\{[1],\dots,[1]\}$ is the trivial K-set consisting of $r+1$ identity matrices of order one. Then there is a unique epimorphism (f,g) from $\mathcal{B}$ to $\mathcal{A}$. Let

$$(2.4)\qquad \mathcal{B}=\mathcal{A}_m \xrightarrow{(f_{m-1},g_{m-1})} \mathcal{A}_{m-1}\to\cdots\xrightarrow{(f_1,g_1)}\mathcal{A}_1=\mathcal{A}$$

be a sequence of proper epimorphisms between K-sets $\mathcal{A}_i$. Abusing language in the interest of convenience, one calls (2.4) a *K-factorization of* $\mathcal{B}$ [rather than of (f,g)]. If every $\mathcal{A}_i$ is doubly cohesive, one calls (2.4) an *H-factorization* of $\mathcal{B}$. The following lemma is proved in [2, 3.17].

LEMMA 2.5. *Let $\mathcal{B}$ be a K-set of $r+1$ matrices where $r\geq 2$. Then, if $\mathcal{B}$ has a K-factorization* (2.4), *there exists a sequence of integers* $(R_2,R_3,\dots,R_m)$ *which satisfies the following two conditions*:

(i) *for $i\geq 2$ every matrix A in $\mathcal{A}_i$ is square of order t_i^2, where $t_i=R_2R_3\cdots R_i$*;

(ii) *every row sum and every column sum in A is t_i for A in $\mathcal{A}_i$, $i\geq 2$.*

Call $\mathcal{B}$ a *(t,r) K-set*, where $t=R_2R_3\cdots R_m$. Call $\mathcal{R}=(R_2,R_3,\dots,R_m)$ *the flight parameter sequence* of the K-factorization (2.4) and *a Klingenberg flight parameter sequence of* $\mathcal{B}$; if (2.4) is an H-factorization, call $\mathcal{R}$ a *Hjelmslev flight parameter sequence* of $\mathcal{B}$.

Let $\mathcal{B}=\{B^0,B^1,\dots,B^r\}$ be a K-set of matrices with the following two K-factorizations:

$$(2.6)\qquad \mathcal{B}=\mathcal{A}_m \xrightarrow{(f_{m-1},g_{m-1})} \mathcal{A}_{m-1}\to\cdots\xrightarrow{(f_1,g_1)}\mathcal{A}_1=\mathcal{A},$$

$$(2.7)\qquad \mathcal{B}=\mathcal{B}_t \xrightarrow{(h_{t-1},k_{t-1})} \mathcal{B}_{t-1}\to\cdots\xrightarrow{(h_1,k_1)}\mathcal{B}_1=\mathcal{A}.$$

The factorizations (2.6) and (2.7) are said to be *equivalent* provided that $m = t$ and that the following condition is satisfied: there exist isomorphisms $(b_i, c_i): \mathcal{A}_i \to \mathcal{B}_i$ for all $i \leq m$ such that the following diagram commutes for all $i < m$:

$$\begin{array}{ccc} \mathcal{A}_{i+1} & \xrightarrow{(f_i, g_i)} & \mathcal{A}_i \\ {\scriptstyle (b_{i+1}, c_{i+1})} \downarrow & & \downarrow {\scriptstyle (b_i, c_i)} \\ \mathcal{B}_{i+1} & \xrightarrow{(h_i, k_i)} & \mathcal{B}_i \end{array} \tag{2.8}$$

The following two results are proved in [2, 4.7 and 4.8].

THEOREM 2.9. *All maximal K-factorizations of a given cohesive K-set of three or more matrices are equivalent.*

COROLLARY 2.10. *All maximal H-factorizations of a given H-set of three or more matrices are equivalent.*

In view of these two results one makes the following definitions. Any maximal K-factorization of a cohesive K-set $\mathcal{B}$ of three or more matrices is called a *K-solution* of $\mathcal{B}$, and the flight parameter sequence of a K-solution of $\mathcal{B}$ is called the *Klingenberg step parameter sequence* of $\mathcal{B}$. A maximal H-factorization of an H-set $\mathcal{B}$ of three or more matrices is called an *H-solution* of $\mathcal{B}$, and the flight parameter sequence of such an H-solution is called the *Hjelmslev step parameter sequence* of $\mathcal{B}$.

Our goal is to construct H-sets $\mathcal{B}$ with Hjelmslev step parameter sequences $(3,11,3)$ and $(2,2,q_4,q_5)$ where $(q_4,q_5) = (8,11),(8,13),(9,11)$. Such H-sets $\mathcal{B}$ can be used to construct the PH-planes promised in the introduction.

3. AUXILIARY SETS OF MATRICES AND FLAG MAPS

The following construction of H-sets appears in [4, pp. 74, 75]. Let Π be an affine plane of order r, and G be a group of order r (additively written, but not necessarily abelian). Use elements of $S(r^2)$ to label the points of Π; label the lines of Π by the symbols L_g^λ, g in G and λ in $S(r+1)$, in such a way that

$$L_g^\lambda \parallel L_h^\mu \quad \text{if and only if} \quad \lambda = \mu. \tag{3.1}$$

For each λ in $S(r+1)$, let $E^\lambda = [e_{ij}^\lambda]$ be the square $(0,1)$-matrix of order r^2

defined as follows: $e_{ij}^{\lambda}=1$ if and only if i and j belong to a common line L_g^{λ}. Then

(3.2) $$\mathcal{E}=\{E^0, E^1,\dots,E^r\} \quad \text{is an } H\text{-set}.$$

An H-set that is constructed from an affine plane in the preceding manner is called a *partition set*, because the subsets L_g^{λ} with fixed λ partition the set $S(r^2)$. We observe that

(3.3) $$\sum_{\lambda=0}^{r} E^{\lambda}=J+rI\geqslant J,$$

where (as usual) I denotes an identity matrix.

For λ in $S(r+1)$ and g in G, let $E_g^{\lambda}=[x_{ij}]$ be the square (0,1)-matrix of order r^2 defined as follows: for i,j in $S(r^2)$

Definition 3.4. $x_{ij}=1$ if and only if i is in L_z^{λ} and j is in L_{z+g}^{λ} for some z in G.

Clearly the following statements hold:

(3.5) $$E_0^{\lambda}=E^{\lambda},$$

(3.6) $$\sum_{g\in G} E_g^{\lambda}=J.$$

Routine calculations lead to the following conclusions:

(3.7) $$E_g^{\lambda}(E_h^{\mu})^T=(E_h^{\mu})^T E_g^{\lambda}=J \qquad \text{if} \quad \lambda\neq\mu,$$

(3.8) $$E_g^{\lambda}(E_h^{\lambda})^T=rE_{g-h}^{\lambda}, \qquad (E_g^{\lambda})^T E_h^{\lambda}=rE_{-g+h}^{\lambda}.$$

Let $A=[a_{ij}]$ be a (0,1)-matrix of size m by n. We write $F(A)$ to denote the set of *flags* of A, i.e., the set of pairs (i,j) such that $a_{ij}=1$. Call a function f: $F(A)\to G$ a *flag map*, where G is any set. The flag map is said to be *differentiated* if G is an additively written group and if the following condition and its dual both hold:

(3.9) Let $0\leqslant i<j<m$, and suppose that $\sum_{x=0}^{n-1}a_{ix}a_{jx}\neq 0$. Then, for each g in G, there is (at least) one x_g in $S(n)$ such that $(i,x_g),(j,x_g)$ are in $F(A)$ and $g=f(i,x_g)-f(j,x_g)$.

If f satisfies (3.9) for a single pair of rows i and j, we say that f is *differentiated relative to i and j*.

THEOREM 3.10. *Let r be the order of an affine plane, $\mathcal{A} = \{A^0, A^1, \ldots, A^r\}$ be an (s, r) H-set, and G be a group of order r. For each λ in $S(r+1)$ assume the existence of a differentiated flag map $f_\lambda: F(A^\lambda) \to G$. Then there exists an (rs, r) H-set $\mathcal{B} = \{B^0, B^1, \ldots, B^r\}$ which has $\mathcal{A}$ as epimorphic image.*

Proof. Recall the notation $A^\lambda = [a^\lambda_{ij}]$ for each λ in $S(r+1)$. For each λ we shall define a matrix $B^\lambda = [F^\lambda_{ij}]$ where i and j range over $S(s^2)$ and each F^λ_{ij} is a square submatrix of order r^2. If $a^\lambda_{ij} = 0$, take F^λ_{ij} to be an all-zero submatrix; if $a^\lambda_{ij} = 1$, set $F^\lambda_{ij} = E^\lambda_g$, where $g = f_\lambda(i, j)$. (The E^λ_g are defined as indicated above relative to a given affine plane Π of order r and the given G.) Let $\lambda \neq \mu$, $B^\lambda(B^\mu)^T = M = [M_{ij}]$ with i and j in $S(s^2)$. For a given pair (i, j) the definition of H-sets implies the existence of a unique integer k such that $a^\lambda_{ik} = a^\mu_{jk} = 1$. Using (3.7) one sees that G contains elements g and h such that

$$M_{ij} = F^\lambda_{ik}\left(F^\mu_{jk}\right)^T = E^\lambda_g\left(E^\mu_h\right)^T = J.$$

Then

$$B^\lambda(B^\mu)^T = J. \tag{3.11}$$

The dual argument yields

$$(B^\mu)^T B^\lambda = J. \tag{3.12}$$

Next let $B^\lambda(B^\lambda)^T$ be denoted by $N^\lambda = [N^\lambda_{ij}]$, where i and j vary over $S(s^2)$; set

$$N = \left[N_{ij}\right] = \sum_{\lambda=0}^{r} N^\lambda,$$

i, j in $S(s^2)$. Then $N^\lambda_{ii} = \Sigma_k F^\lambda_{ik}(F^\lambda_{ik})^T$. Since $a^\lambda_{ik} = 1$ for some k, there is a g in G such that $N^\lambda_{ii} \geqslant E^\lambda_g(E^\lambda_g)^T$. Then (3.8) and (3.5) imply that $N^\lambda_{ii} \geqslant rE^\lambda$, and (3.3) yields

$$N_{ii} = \sum_{\lambda=0}^{r} N^\lambda_{ii} \geqslant r \sum_{\lambda=0}^{r} E^\lambda \geqslant rJ. \tag{3.13}$$

We now consider the case $i \neq j$. Since $\mathcal{A}$ is an H-set, there is a λ such that $\Sigma_x a^\lambda_{ix} a^\lambda_{jx} \neq 0$. Since $f_\lambda = f$ is a differentiated flag map, there exist flags $(i, x_g), (j, x_g)$ with $f(i, x_g) - f(j, x_g) = g$ for each g in G. The application of

(3.8) and (3.6) yields

$$N_{ij} \geqslant N_{ij}^{\lambda} = \sum_{x} F_{ix}^{\lambda}\left(F_{jx}^{\lambda}\right)^{T} \geqslant \sum_{g \in G} rE_{g}^{\lambda} = rJ. \tag{3.14}$$

It follows from (3.13) and (3.14) that

$$\sum_{\lambda=0}^{r} B^{\lambda}(B^{\lambda})^{T} \geqslant rJ \quad \text{and} \quad \sum_{\lambda=0}^{r} (B^{\lambda})^{T} B^{\lambda} \geqslant rJ \tag{3.15}$$

by reason of duality. The assertions (3.11), (3.12), and (3.15) give the desired conclusion that $\mathfrak{B}$ is an H-set. It is clear from the construction of $\mathfrak{B}$ that $\mathfrak{A}$ is an epimorphic image and that the matrices in $\mathfrak{B}$ are of order r^2s^2. ■

REMARKS 3.16. Let the K-set $\mathfrak{A}$ of matrices of order s^2 be an epimorphic image of a K-set $\mathfrak{B}$ of matrices of order r^2s^2. Then every K-factorization of $\mathfrak{A}$ extends to a K-factorization of $\mathfrak{B}$. Consequently if $S_m = (R_2, R_3, \ldots, R_m)$ is a flight parameter sequence of $\mathfrak{A}$, the K-set $\mathfrak{B}$ will have a flight parameter sequence $S_{m+1} = (R_2, \ldots, R_m, r)$. The sequence S_{m+1} will be the Klingenberg step parameter sequence of $\mathfrak{B}$ provided that r is a prime and that S_m is the Klingenberg step parameter sequence of $\mathfrak{A}$. The latter condition can be assured, of course, by assuming that all the R_i's are prime. Analogous conclusions hold for H-factorizations and for Hjelmslev step parameter sequences of H-sets $\mathfrak{A}$ and $\mathfrak{B}$.

4. AN H-SET WITH SEQUENCE (3,11,3)

Theorem 3.10 and Remarks 3.16 can be used to construct H-sets of three and four matrices, respectively, whose Hjelmslev step parameter sequences are (2,5,2) and (3,11,3). A construction of the former has been given by Sane [7] without the machinery that we have assembled in Section 3 above. We devote the present section of this paper to constructing the latter.

We utilize the notation of (2.4): in this section with $m = 4$ and in later sections with $m = 5$. Consequently the reader must keep in mind a large number of matrices and submatrices of various sizes. We now introduce some notation which we hope will facilitate this task. Since $\mathfrak{A}_1$ is trivial, we concern ourselves only with $\mathfrak{A}_i = \{A^{i0}, A^{i1}, \ldots, A^{ir}\}$ for $i \geqslant 2$. Assume that the flight parameter sequence associated with (2.4) is $(r = q_2, q_3, \ldots, q_m)$. Then every $A^{i\lambda}$ is a square matrix whose order is the square of the integer $q_2 q_3 \cdots q_i$. We

$$A^{2\lambda} = \left[a_{ij}^{2\lambda}\right]$$

$$A^{3\lambda} = \left[B_{ij}^{3\lambda}\right] = \left[a_{ijmn}^{3\lambda}\right]$$

$$A^{4\lambda} = \left[B_{ij}^{4\lambda}\right] = \left[C_{ijmn}^{4\lambda}\right] = \left[a_{ijmnxy}^{4\lambda}\right]$$

$$A^{5\lambda} = \left[B_{ij}^{5\lambda}\right] = \left[C_{ijmn}^{5\lambda}\right] = \left[D_{ijmnxy}^{5\lambda}\right] = \left[a_{ijmnxy\alpha\beta}^{5\lambda}\right]$$

FIG. 1

shall use the notation summarized in Figure 1. The letter a (with subscripts and superscripts attached) always represents a single matrix entry of 0 or 1; the capital letters B, C, D represent square submatrices. The subscripts i, j range over $S(q_2^2)$; m, n over $S(q_3^2)$; x, y over $S(q_4^2)$; α, β over $S(q_5^2)$. To illustrate the meaning of the subscripts, we examine line 3 of Figure 1. The matrix $A^{4\lambda}$ is of order $q_2^2 q_3^2 q_4^2$; it consists of submatrices $B_{ij}^{4\lambda}$. Every $B_{ij}^{4\lambda}$ is of order $q_3^2 q_4^2$ and consists of submatrices $C_{ijmn}^{4\lambda}$. Every matrix $C_{ijmn}^{4\lambda}$ is of order q_4^2 and has entries $a_{ijmnxy}^{4\lambda}$.

We begin by taking $\mathcal{Q}_2 = \{A^{20}, A^{21}, \dots, A^{2r}\}$ to be a partition set of matrices of order r^2 (defined in the first paragraph of Section 3). For λ in $S(r+1)$ define a relation $(\sim\lambda)$ on $S(r^2)$ by the rule

$$\text{(4.1)} \qquad i(\sim\lambda)j \quad \text{if and only if} \quad a_{ij}^{2\lambda} = 1.$$

The lines belonging to a given parallel class λ partition the points of the chosen affine plane Π of order r into r disjoint sets of size r. Therefore each relation $(\sim\lambda)$ is an equivalence relation which partitions $S(r^2)$ into r disjoint classes of size r. Call the classes $(i)_\lambda$. Let $\phi_\lambda: S(r^2) \to S(r)$ be a map which satisfies the following condition:

(4.2) The restriction of ϕ_λ to each class $(i)_\lambda$ is one-to-one.

Next let $\mathcal{X} = \{X_\theta^\lambda: \lambda \text{ in } S(r+1), \theta \text{ in } S(r)\}$ be an H-set of matrices of order s^2. For each λ in $S(r+1)$, define a matrix $X^\lambda = [X_{ab}^\lambda]$ with a, b in $S(r)$ by setting $X_{ab}^\lambda = X_\theta^\lambda$, where θ is the residue of $a+b$ modulo r. Next define $A^{3\lambda} = [B_{ij}^{3\lambda}]$ (compare Figure 1) as follows:

$$\text{(4.3)} \qquad \begin{aligned} B_{ij}^{3\lambda} &= X_{ab}^\lambda \quad \text{if} \quad a_{ij}^{2\lambda} = 1, \quad \phi_\lambda(i) = a, \quad \phi_\lambda(j) = b; \\ B_{ij}^{3\lambda} &= O \quad \text{if} \quad a_{ij}^{2\lambda} = 0. \end{aligned}$$

(O denotes an all-zero matrix.)

LEMMA 4.4. *Let* $\mathcal{A}_3 = \{A^{30}, A^{31}, \ldots, A^{3r}\}$ *be constructed in the manner indicated above. Then*

(i) $\mathcal{A}_3$ *is an* (rs, r) *H-set with* $\mathcal{A}_2$ *as epimorphic image;*
(ii) *if there is a differentiated flag map* $g_\lambda : F(X^\lambda) \to G$ *for some* λ *and some group* G, *then there is a differentiated flag map* $f_\lambda : F(A^{3\lambda}) \to G$.

Proof. The easy proof of (i) is left to the reader: the construction producing $\mathcal{A}_3$ first appeared in [4]. Next assume the existence of a differentiated flag map $g_\lambda : F(X^\lambda) \to G$. For each flag (i, j, m, n) of $A^{3\lambda}$, define f_λ as follows:

DEFINITION 4.5. $f_\lambda(i, j, m, n)$ is $g_\lambda(a, b, m, n)$ where $a = \phi_\lambda(i)$ and $b = \phi_\lambda(j)$.

Now let (i, m) and (j, n) be two distinct rows of $A^{3\lambda}$ which have a nonzero inner product, and let g be in G. Let (k, z) be a column of $A^{3\lambda}$ such that (i, k, m, z) and (j, k, n, z) are in $F(A^{3\lambda})$. Setting $a = \phi_\lambda(i)$, $b = \phi_\lambda(j)$, $c = \phi_\lambda(k)$, one sees that (a, c, m, z) and (b, c, n, z) are in $F(X^\lambda)$. Then rows (a, m) and (b, n) have a nonzero inner product in X^λ. Since g_λ is a differentiated flag map, there is a column (d, w) such that

$$g_\lambda(a, d, m, w) - g_\lambda(b, d, n, w) = g. \tag{4.6}$$

There is a unique line h in the parallel class λ which contains the point i. Let l be the unique point in h which satisfies $\phi_\lambda(l) = d$. Then Definition 4.5 and 4.6 imply that $g = f_\lambda(i, l, m, w) - f_\lambda(j, l, n, w)$. We have proved that f_λ is differentiated relative to rows, and essentially the same argument proves that f_λ is differentiated relative to columns. ■

EXAMPLE 4.7. *There exists an H-set* $\mathcal{A}_4$ *of four matrices whose Hjelmslev step parameter sequence is* (3, 11, 3).

Proof. Since 3 and 11 are orders of affine planes, we may take r to be 3, s to be 11 in defining the H-sets $\mathcal{A}_2$ and $\mathcal{X}$ above. Further we may take $\mathcal{X}$ (as well as $\mathcal{A}_2$) to be a partition set. Then Lemma 4.4(i) yields the existence of an H-set $\mathcal{A}_3$ which (by the Remarks 3.16) has a Hjelmslev step parameter sequence (3, 11).

We take G to be the additive group Z_3 of integers modulo 3. We plan to prove the existence of differentiated flag maps f_λ from $F(X^\lambda)$ to Z_3 for every λ in S(4). As a consequence of Lemma 4.4(ii) and Theorem 3.10 we shall then obtain an H-set $\mathcal{B} = \mathcal{C}_4$ of matrices of order $3^2 \times 33^2$ which has $\mathcal{C}_3$ as epimorphic image. Remarks 3.16 will then guarantee that $\mathcal{C}_4$ has the sequence (3, 11, 3).

Henceforth we assume that λ is a fixed element of $S(4)$, and consequently we are able to simplify notation by suppressing superscripts λ. We have

$$X^\lambda = X = \begin{bmatrix} X_0 & X_1 & X_2 \\ X_1 & X_2 & X_0 \\ X_2 & X_0 & X_1 \end{bmatrix} \tag{4.8}$$

where the X_θ are distinct elements of $\mathcal{X}$. For θ in S(3) let $X_\theta = [x^\theta_{mn}]$, m and n varying over S(121). Let $(*\theta)$ denote the equivalence relation defined on S(121) by setting $m(*\theta)n$ if and only if $x^\theta_{mn} = 1$. Define $h_\theta\colon F(X_\theta) \to Z_3$ by the rule

$$h_\theta(m, n) = \begin{cases} 1 & \text{if } m = n, \\ 0 & \text{if } m \neq n. \end{cases}$$

Clearly h_θ is differentiated: for any two distinct rows (or columns) with nonzero inner product, eleven h_θ-differences are defined, nine equal to 0 and one each equal to 1 and 2. Next define $f\colon F(X) \to Z_3$ by the rule

$$f(a, b, m, n) = h_\theta(m, n) - ab,$$

where θ is the residue of $a + b$ modulo 3. Since h_θ is differentiated, f is also differentiated relative to any two distinct rows (a, m) and (a, m') of X which have a nonzero inner product.

Next we consider two rows (a, m) and (a', m') of X whose inner product is not zero and for which $a \neq a'$. Define a'' by demanding that $Z_3 = \{a, a', a''\}$. By permuting the labels θ on the X_θ we may assume that rows a and a' of submatrices of X appear as in the table (4.9): the key observation is that the three entries on the main diagonal of X have distinct subscripts.

(4.9)

Rows \ Columns	a	a'	a''
a	X_0	X_1	X_2
a'	X_1	X_2	X_0

LEMMA 4.10. *Let n, n', n'' be the unique elements of* S(121) *such that $x^0_{mn} = x^1_{m'n} = 1$, $x^1_{mn'} = x^2_{m'n'} = 1$, and $x^2_{mn''} = x^0_{m'n''} = 1$. Then one of the fol-*

lowing cases occurs:

(i) $m = m' = n = n' = n''$.
(ii) $m(*0)m'$; $n = m'$, n', *and* $n'' = m$ *are distinct.*
(iii) $m(*1)m'$; $n = m$, $n' = m'$, *and* n'' *are distinct.*
(iv) $m(*2)m'$; n, $n' = m$, *and* $n'' = m'$ *are distinct.*
(v) $m(*\theta)m'$ *holds for no* θ; $m \neq m'$; $m \neq n, n', n'' \neq m'$.

Proof. Since the X_θ's belong to a partition set, $x^\theta_{mm} = 1$ for each θ. Then if $m' = m$, the uniqueness of n, n', n'' implies that case (i) holds.

Next assume that $m' \neq m$ but that $m'(*0)m$. Then $x^0_{mm'} = 1 = x^1_{m'm'}$, so $m' = n$. Similarly one obtains $m = n''$. Suppose that $n' = n''$. Then $n' = m$, so $1 = x^2_{m'n'} = x^2_{m'm}$; i.e., $m'(*2)m$. Since $m'(*0)m$ also holds, it must be the case that $m' = m$. This final contradiction yields the conclusion $n' \neq n''$. The proof that $n \neq n'$ is similar. Since $n = m' \neq m = n''$, case (ii) holds.

If $m(*\theta)m'$ for $\theta = 1$ or 2 and if $m \neq m'$, similar arguments prove that either case (iii) or case (iv) must hold. Then we may assume that $m(*\theta)m'$ does not hold for any value of θ; *a fortiori*, that $m \neq m'$. If $m = n$, then $1 = x^1_{m'n} = x^1_{m'm}$ would imply $m(*1)m'$. The facts $m \neq n'$, n'' and $m' \neq n, n', n''$ are established in similar fashion. ■

To prove that f is differentiated relative to the rows (a, m) and (a', m'), we shall show that the following three f-differences are distinct:

$$K = f(a, a, m, n) - f(a', a, m', n) = h(m, n) - h(m', n) + a(a' - a),$$

$$L = f(a, a', m, n') - f(a', a', m', n') = h(m, n') - h(m', n') + a'(a' - a),$$

$$M = f(a, a'', m, n'') - f(a', a'', m', n'')$$
$$= h(m, n'') - h(m', n'') + a''(a' - a).$$

In the above expressions we have suppressed the subscripts on the functions h since all three h's are defined by the same rule. Using Lemma 4.10, one obtains the following table of values for K, L, M:

(4.11)

Case	(i)	(ii)	(iii)	(iv)	(v)
$K = a(a' - a) +$	0	2	1	0	0
$L = a'(a' - a) +$	0	0	2	1	0
$M = a''(a' - a) +$	0	1	0	2	0

Clearly K, L, M are distinct in cases (i) and (v), since a, a', a'' are distinct. Then assume that case (ii), (iii) or (iv) holds. One obtains $L-K=(a'-a)^2+1=2\neq 0$; $M-K=(a''-a)(a'-a)+2=1\neq 0$; $M-L=(a''-a')(a'-a)+1=2\neq 0$. Then the differences K, L, M are distinct, so f is differentiated relative to rows. By duality f is differentiated relative to columns. The existence of an H-set $\mathcal{Q}_4$ with Hjelmslev step parameter sequence $(3,11,3)$ now follows from the comments made in paragraph two of the proof of Example 4.7. ■

5. MORE H-SETS

In this section we construct $(t,2)$ H-sets of matrices with step parameter sequences $(2,2,q_4,q_5)$ for which q_4 is 8 or 9. Our final extension (from $\mathcal{Q}_4$ to $\mathcal{Q}_5$) involves replacing the entries $a^{4\lambda}_{ijmnxy}=1$ (See Figure 1.) by submatrices which come from some other H-set. This method forces q_5 to be less than $(q_1+1)q_2q_3q_4$. In our setting, if $q_4=8$, $q_5\leqslant 95$; and if $q_4=9$, $q_5\leqslant 107$. Below the natural upper bounds it becomes increasingly difficult to construct H-sets with given sequences as q_m becomes progressively smaller relative to q_{m-1}; the one clear exception to this rule of thumb is that sequences with $q_m=q_{m-1}$ are comparatively easy to construct. We summarize progress to date by tabulating the values of q_5 for $(t,2)$ H-sets with $q_2=q_3=2$ as follows:

(5.1)

Paper	[1], 1975	[6], 1978	[8]	This paper
For $q_4=8$	17–71	17–95*	17–95	11, 13
For $q_4=9$	19–89	17–107*	13–107	11, 13

The methods of [1] and [8] allow one to obtain all listed values for q_5 which are orders of projective planes. The methods of [6] require one to assume that q_5 is a prime power; they allow one to construct PH-planes with the desirable property of regularity, but H-sets are not obtained.

At several steps in the construction we utilize matrices E^λ_g obtained from a partition set $\mathcal{E}=\{E^0, E^1,\dots,E^r\}$ relative to the cyclic additively written group Z_r. In order to keep track of these E^λ_g, we introduce notation based on the entries in the main diagonal of the matrix displayed in Figure 1. Specifically we write C^λ_g in place of E^λ_g if $r=8$ or 9, and D^λ_g if $r=11$ or 13. We use the following notation for sets:

(5.2)
$$\mathcal{C}=\left\{C^\lambda_g:\lambda \text{ in } S(q+1),\ g \text{ in } Z_q\right\},\qquad q=8 \text{ or } 9;$$
$$\mathcal{D}=\left\{D^\lambda_g:\lambda \text{ in } S(r+1),\ g \text{ in } Z_r\right\},\qquad r=11 \text{ or } 13.$$

Now let $C=[c_{xy}]$ and $C'=[c'_{xy}]$ denote any two fixed distinct partition matrices C_0^λ and C_0^μ from $\mathcal{C}$. Our matrices $B_{ij}^{4\lambda}$ (consult Figure 1 again) will eventually be constructed from modifications of the following set of six matrices:

$$X_{00}=\begin{bmatrix} C & C' & O & O \\ C' & C & O & O \\ O & O & C' & C \\ O & O & C & C' \end{bmatrix}, \quad X_{01}=\begin{bmatrix} O & O & C & C' \\ O & O & C' & C \\ C & C' & O & O \\ C' & C & O & O \end{bmatrix},$$

$$(5.3) \qquad X_{10}=\begin{bmatrix} C & O & C' & O \\ O & C & O & C' \\ C' & O & C & O \\ O & C' & O & C \end{bmatrix}, \quad X_{11}=\begin{bmatrix} O & C & O & C' \\ C & O & C' & O \\ O & C' & O & C \\ C' & O & C & O \end{bmatrix},$$

$$X_{20}=\begin{bmatrix} C & O & O & C' \\ O & C & C' & O \\ O & C' & C & O \\ C' & O & O & C \end{bmatrix}, \quad X_{21}=\begin{bmatrix} O & C & C' & O \\ C & O & O & C' \\ C' & O & O & C \\ O & C' & C & O \end{bmatrix}.$$

LEMMA 5.4. *For each* $X_{\rho\sigma}$ *in* (5.3) *there exists a flag map* $\eta_{\rho\sigma}=\eta: F(X_{\rho\sigma})\to Z_2$ *which satisfies the following assumptions*:

(i) *The restriction of* η *to each row of* $X_{\rho\sigma}$ *is a surjection.*

(ii) *Let* (m,x) *and* (m',x') *be distinct rows of* $X_{\rho\sigma}$ *which have a nonzero inner product. Then there are two columns* (n,y) *for which* $\eta(m,n,x,y)-\eta(m',n,x',y)\neq 0$ *provided that any of the following three conditions hold*: (1) $m=m'$; (2) $x=x'$; (3) $x\neq x'$, *and the line which joins* x *and* x' (*in the affine plane* Π *used to construct* $\mathcal{C}$) *does not belong to either of the parallel classes* λ *and* μ.

(iii) *The duals of* (i) *and* (ii) *hold.*

Proof. For fixed (ρ,σ) define $\eta=\eta_{\rho\sigma}$ on an arbitrary flag (m,n,x,y) by the rule

$$\eta(m,n,x,y)=\begin{cases} 0 & \text{if } x=y \text{ and } X_{mn}^{\rho\sigma}=C, \\ 0 & \text{if } x\neq y \text{ and } X_{mn}^{\rho\sigma}=C', \\ 1 & \text{otherwise.} \end{cases}$$

Clearly condition (i) and its dual are satisfied.

Let (m, x) and (m, x') be distinct rows of $X_{\rho\sigma}$ with a nonzero inner product. We choose n so that $X^{\rho\sigma}_{mn}$ is one of C or C', selecting the submatrix for which rows x and x' coincide. The η-differences will be nonzero for columns $(n,\ y = x)$ and $(n,\ y = x')$. Secondly let (m, x) and (m', x) be distinct rows with a nonzero inner product. Then the η-differences will be nonzero for columns $(n, y = x)$ for both the appropriate choices of n.

Next assume that $x \neq x'$, that (m, x) and (m', x') are rows of $X_{\rho\sigma}$ with nonzero inner product, and that the line joining x to x' in Π does not belong to the parallel classes paired to C and C'. Then $m \neq m'$, so the inner product of the two rows is 2. Let (n, y) and (n', y') be the columns where both rows have entry 1. If $x = y$, then there would be a 1 in the matrix $X_{\rho\sigma}$ in position $(m', n, x', y) = (m', n, x', x)$. This would contradict the fact that the line joining x to x' belongs to neither of the parallel classes λ, μ. Then $x \neq y$. Similarly $x \neq y'$ and $x' \neq y, y'$. It follows that (m, n, x, y) and (m', n, x', y) are nondiagonal positions, one in C and the other in C'. Then $\eta(m, n, x, y) - \eta(m', n, x', y) \neq 0$, and the same argument shows that $\eta(m, n', x, y') - \eta(m', n', x', y') \neq 0$. We have verified assertion (ii), and the dual condition is also satisfied. ■

LEMMA 5.5. *Let $t = 3$ or 4. Then for each $X_{\rho\sigma}$ in* (5.3) *there exists a flag map $\psi_{\rho\sigma} = \psi: F(X_{\rho\sigma}) \to Z_t$ which satisfies the following assumptions*:

(i) *The restriction of ψ to each row of $X_{\rho\sigma}$ is a surjection.*

(ii) *Let (m, x) and (m', x') be rows of $X_{\rho\sigma}$ with $x \neq x'$ which possess a nonzero inner product. Then there are two columns (n, y) for which $\psi(m, n, x, y) - \psi(m', n, x', y) \neq 0$.*

(iii) *The duals of* (i) *and* (ii) *hold.*

Proof. First let $t = 3$. For fixed (ρ, σ) define $\psi = \psi_{\rho\sigma}$ on an arbitrary flag (m, n, x, y) by the rule

$$\psi(m, n, x, y) = \begin{cases} 0 & \text{if } x = y, \\ 1 & \text{if } x \neq y \text{ and } X^{\rho\sigma}_{mn} = C, \\ 2 & \text{if } x \neq y \text{ and } X^{\rho\sigma}_{mn} = C'. \end{cases}$$

Clearly condition (i) and its dual hold. The argument that (ii) holds for rows (m, x) and (m', x') with $m = m'$ is essentially the one given in the proof of (5.4).

Then let (m, x) and (m', x') possess a nonzero inner product and satisfy $m \neq m'$, $x \neq x'$. Let (n, y) be either of the two columns in which both rows have entry 1. Let s denote $\psi(m, n, x, y)$, s' denote $\psi(m', n, x', y)$. Since

$m \neq m'$, s and s' cannot be equal unless $s = s' = 0$. Since $x \neq x'$, the equalities $s = 0$ and $s' = 0$ also cannot both hold. Then condition (ii) holds. Duality yields the truth of condition (iii).

If $t = 4$, one defines ψ by the following rule: for flags (x, y) in C one sets $\psi(x, y)$ equal to 0 if $x = y$, to 1 if $x \neq y$; in C' one sets $\psi(x, y)$ equal to 2 if $x = y$, to 3 if $x \neq y$. The proof that ψ is satisfactory is left to the reader. ■

DEFINITION 5.6. From each matrix $X_{\rho\sigma}$ listed in (5.3) obtain a matrix $Y_{\rho\sigma}$ by replacing all the submatrices C' with submatrices C.

LEMMA 5.7. *Let $r = 11$ or 13. Then for each $Y_{\rho\sigma}$ defined in Definition 5.6 there exists a flag map $\phi_{\rho\sigma} = \phi\colon F(Y_{\rho\sigma}) \to Z_r$ which satisfies the following assumption and its dual: ϕ is differentiated* [*see* (3.9)] *relative to any pair of distinct rows (m, x) and (m', x') which have a nonzero inner product and which satisfy either $m = m'$ or $x = x'$.*

Proof. Clearly it suffices to prove the existence of a flag map $\phi\colon F(Y) \to Z_r$ where

$$Y = \begin{bmatrix} C & C \\ C & C \end{bmatrix}$$

and ϕ satisfies both the following condition and its dual: ϕ is differentiated relative to any pair of distinct rows (a, x) and (a', x') which have a nonzero inner product and which satisfy either $a = a'$ or $x = x'$.

Let Π be the affine plane of order q used in the definition of the partition set $\mathcal{C}$. Let $\theta\colon S(q^2) \to S(q)$ be a map defined on the points of Π that is one-to-one on each line of the parallel class λ (the class used to define the matrix C). Now for each flag (a, b, x, y) of Y set

$$\phi(a, b, x, y) = [qa + \theta(x)][qb + \theta(y)] \qquad \text{in } Z_r:$$

since $a, b, q, \theta(x), \theta(y)$ are all nonnegative integers less than r, they can be identified in the usual manner with elements of Z_r, the quotient ring of integers modulo r.

Let (a, x) and (a', x') be rows with a nonzero inner product. Then x and x' are joined by a line from parallel class λ, so the two rows coincide. Then they give rise to the ϕ-differences $h[qb + \theta(y)]$ where

$$h = [qa + \theta(x)] - [qa' + \theta(x')]$$

and (b, y) ranges over the $2q$ pairs for which (a, b, x, y) is a flag of Y. Since the restriction of θ to each line of class λ is a bijection and since $2q > r$, the factor $qb + \theta(y)$ ranges over all elements of Z_r. Since r is a prime, it suffices to prove $h \neq 0$ in Z_r in order to conclude that all elements of Z_r occur as ϕ-differences. If $a = a'$ and $x \neq x'$, then $h = \theta(x) - \theta(x') \neq 0$, since x and x' are joined by a line of the parallel class λ that is paired to C; if $a \neq a'$ and $x = x'$, then $h = q(a - a') \neq 0$, since $0 < q$, $|a - a'| < r$. We have obtained the desired conclusion for rows of Y; the conclusion for columns follows from the symmetry of Y and the definition of ϕ. ■

DEFINITION 5.8. For $q = 8$ rename the partition matrices (those with $g = 0$) from the set $\mathcal{C}$ [see (5.2)] by the symbols $C^{\rho\sigma}$ with ρ and σ in S(3). For ρ in S(3) and $\sigma = 0, 1$, obtain a matrix $X'_{\rho\sigma}$ from $X_{\rho\sigma}$ by replacing every submatrix C with $C^{\rho 0}$, every C' with $C^{\rho 1}$; obtain $Y'_{\rho\sigma}$ from $Y_{\rho\sigma}$ by replacing every submatrix C with $C^{\rho 2}$. Now set

$$A^{40} = \begin{bmatrix} X'_{00} & Y'_{00} & O & O \\ Y'_{01} & X'_{00} & O & O \\ O & O & X'_{00} & Y'_{00} \\ O & O & Y'_{01} & X'_{00} \end{bmatrix}, \qquad A^{41} = \begin{bmatrix} X'_{10} & O & Y'_{10} & O \\ O & X'_{10} & O & Y'_{10} \\ Y'_{11} & O & X'_{10} & O \\ O & Y'_{11} & O & X'_{10} \end{bmatrix},$$

$$A^{42} = \begin{bmatrix} X'_{20} & O & O & Y'_{20} \\ O & X'_{20} & Y'_{20} & O \\ O & Y'_{21} & X'_{20} & O \\ Y'_{21} & O & O & X'_{20} \end{bmatrix}.$$

Let $\mathcal{A}_4 = \{A^{40}, A^{41}, A^{42}\}$.

LEMMA 5.9. *$\mathcal{A}_4$ is an H-set with a Hjelmslev flight parameter sequence* (2,2,8).

We leave the tedious verification to the reader.

DEFINITION 5.10. For $q = 9$ one labels nine of the ten partition matrices of $\mathcal{C}$ as in Definition 5.8 and denotes the tenth by C^{03}. Define the matrices $X'_{\rho\sigma}$ and $Y'_{\rho\sigma}$ as before except for Y'_{00} and Y'_{01}. For $\sigma = 0, 1$, one now obtains $Y'_{0\sigma}$ from $X_{0\sigma}$ by trading in each submatrix C for C^{02} and each submatrix C' for C^{03} (instead of for C^{02}). Define new matrices $A^{4\lambda}$ as in Definition 5.8, using the new $X'_{\rho\sigma}$ and $Y'_{\rho\sigma}$. Set $\mathcal{B}_4$ equal to the set of new $A^{4\lambda}$'s.

LEMMA 5.11. $\mathcal{B}_4$ *is a* $(t,2)$ *H-set with Hjelmslev flight parameter sequence* $(2,2,9)$.

Again we leave the details of proof to the reader.

REMARK 5.12. In Section 6 we prove that $(2,2,8)$ is the Hjelmslev (and Klingenberg) step parameter sequence of $\mathcal{A}_4$ and that $(2,2,9)$ is the Hjelmslev (and Klingenberg) step parameter sequence of $\mathcal{B}_4$.

DEFINITION 5.13. We rename the matrices from $\mathcal{D}$ [see (5.2)]. Instead of symbols D_g^λ we use symbols $D_g^{\rho\sigma}$ with ρ in S(3) and σ in $S(3)\cup\{\infty\}$. If $r=13$, we need the additional symbols D_g^{03} and D_g^{13}.

We now obtain matrices $A^{5\lambda}$ (Figure 1 again) from the corresponding matrices $A^{4\lambda}$ of $\mathcal{A}_4$ by defining the submatrices $D_{ijmnxy}^{5\lambda}$. Specifically we set

(5.14) $D_{ijmnxy}^{5\lambda}=O$ (the zero submatrix) if $a_{ijmnxy}^{4\lambda}=0$;
$=D_0^{\lambda\sigma}$ if $a_{ijmnxy}^{4\lambda}=1$, $i=j$, and σ is the image of the flag (m,n,x,y) of $X'_{\lambda 0}$ under the map ψ of (5.5) where Z_t is identified with $S(t)$;
$=D_g^{\lambda\infty}$ if $a_{ijmnxy}^{4\lambda}=1$, $i\neq j$, $B_{ij}^{4\lambda}=Y'_{\rho\sigma}$, and g is the image of the flag (m,n,x,y) of $Y'_{\rho\sigma}$ under the image ϕ of Lemma 5.7.

In the preceding use of Lemma 5.5 one takes t to be 3 except in the following cases: when $r=13$, one uses $t=4$ for $\lambda=0$ and $\lambda=1$.

PROPOSITION 5.15. $\mathcal{A}_5=\{A^{50}, A^{51}, A^{52}\}$ *is an H-set with Hjelmslev flight parameter sequence* $(2,2,8,r)$, *where r is either* 11 *or* 13 *as one chooses.*

Proof. For $\lambda\neq\mu$ let T denote $A^{5\lambda}(A^{5\mu})^T$, $T=[T_{ij}]$, $T_{ij}=[T_{ijmn}]$, $T_{ijmn}=[T_{ijmnxy}]$. Then

$$T_{ijmnxy}=\sum_{k,p,z} D_{ikmpxz}^{5\lambda}\left(D_{jknpyz}^{5\mu}\right)^T.$$

Lemma 5.9 implies that $A^{4\lambda}A^{4\mu}=J$, so there exist k,p,z,ρ,σ,g,h such that $T_{ijmnxy}=D_{ikmpxz}^{5\lambda}(D_{jknpyz}^{5\mu})^T=D_g^{\lambda\rho}D_h^{\mu\sigma}$. Since $\lambda\neq\mu$, the condition (3.7) implies that the last product is J. Then $T=J$, and duality assures that $\mathcal{A}_5$ satisfies (2.1).

Next let Q denote $\Sigma_\lambda A^{5\lambda}(A^{5\lambda})^T$, $Q=[Q_{ij}]$, $Q_{ij}=[Q_{ijmn}]$, $Q_{ijmn}=[Q_{ijmnxy}]$. Then

$$Q_{ijmnxy}=\sum_{\lambda,k,p,z} D^{5\lambda}_{ikmpxz}\left(D^{5\lambda}_{jknpyz}\right)^T.$$

To see that $\mathcal{Q}_5$ satisfies (2.2) it suffices to prove that

(5.16) $$Q_{ijmnxy}\geqslant 2J$$

for all i, j, m, n, x, y.

Case 1: $(i,m,x)=(j,n,y)$. Then (5.14) and Lemma 5.5(i) assure that for each σ in $S(3)\cup\{\infty\}$ there exists an element $g(\sigma)$ in Z_r such that $D^{\lambda\sigma}_{g(\sigma)}$ occurs on row (i,m,x) of $A^{5\lambda}$. Consequently $Q_{iimmxx}\geqslant\Sigma_{\lambda,\sigma}D^{\lambda\sigma}_{g(\sigma)}(D^{\lambda\sigma}_{g(\sigma)})^T$. Then Definition 5.13, (3.8), (3.5) and (3.3) yield $Q_{iimmxx}\geqslant r(rI+J)\geqslant 2J$, so (5.16) holds.

Case 2: $(i,m)=(j,n)$, $x\neq y$. Then there is a unique λ such that the inner product of rows (i,m,x) and (i,m,y) is nonzero in $A^{4\lambda}$. By consulting Definition 5.8 and (5.3) one sees that within $A^{4\lambda}$ the rows (i,m,x) and (i,m,y) have nonzero inner product in a single $C^{\rho\sigma}$ (though $C^{\rho\sigma}$ will be repeated if $\sigma=2$). Assume first that $\sigma\neq 2$. Then $Q_{iimmxy}=\Sigma_z D^{5\lambda}_{iimpxz}(D^{5\lambda}_{iimpyz})^T$: here (i,p) is the unique pair such that $C^{4\lambda}_{iimp}=C^{\rho\sigma}$, and the sum is taken over all points z of the unique line in the $\rho\sigma$-parallel class which joins x and y. Then (5.14) and Lemma 5.5(ii) imply that two of the terms in the summation are of the form $D_0^{\lambda\rho}(D_0^{\lambda\sigma})^T$ where $\rho\neq\sigma$. Then (3.7) implies $Q_{iimmxy}\geqslant 2J$. Assume next that $\sigma=2$. Then (5.14), Lemma 5.7, (3.8), and (3.6) imply that $Q_{iimmxy}\geqslant rJ$.

Case 3: $i=j$, $m\neq n$, $x=y$. One uses Lemma 5.7 as in the second half of the verification for case 2.

Case 4: $i=j$, $m\neq n$, $x\neq y$. There is a (unique) λ for which rows (i,m,x) and (i,n,y) have a nonzero inner product in $A^{4\lambda}$. Within this $A^{4\lambda}$ the rows (m,x) and (n,y) have nonzero inner product in the submatrix $X'_{\lambda 0}$. Then Lemma 5.5(ii) and (5.14) yield $Q_{iimnxy}\geqslant 2J$ as in the first half of the verification for case 2.

Case 5: $i\neq j$. There is a (unique) λ for which rows (i,m,x) and (j,n,y) have a nonzero inner product in $A^{4\lambda}$, and the inner product is 2. Then Q_{ijmnxy} is the sum of two terms, each of the form $D_g^{\lambda\rho}(D_h^{\lambda\sigma})^T$ where g, h are in Z_r and exactly one of ρ,σ is ∞. Since each of the two terms is J, we have completed the verification in case 5.

Then (5.16) holds in general, so $\mathcal{Q}_5$ is doubly differentiated, i.e., is an H-set. Clearly $\mathcal{Q}_4$ is an epimorphic image of $\mathcal{Q}_5$, so Lemma 5.9 implies that $\mathcal{Q}_5$ has a Hjelmslev flight parameter sequence $(2,2,8,r)$. ∎

PROPOSITION 5.17. *There exist H-sets* $\mathfrak{B}_5 = \{A^{50}, A^{51}, A^{52}\}$ *with Hjelmslev flight parameter sequences* $(2,2,9,r)$ *for* $r = 11$ *and* 13.

Proof. Since the construction is similar to the construction which gives rise to the H-sets of Proposition 5.15, we only outline the major differences. One uses the matrices $A^{4\lambda}$ of Definition 5.10 rather than the ones from Definition 5.8. The replacement of flags in the $A^{4\lambda}$ by submatrices $D_g^{\lambda\sigma}$ is carried out in accord with (5.14) for $\lambda = 0, 1$. For $\lambda = 2$, however, we use maps η of Lemma 5.4 rather than the maps ψ of Lemma 5.5 and ϕ of Lemma 5.7. If η' is the η which replaces ϕ, we let η' map into the set $\{2, \infty\}$ rather than into S(2). One replaces the flag (i, j, m, n, x, y) by $D_0^{\lambda\sigma}$ if $i \neq j$ and σ is the image of (m, n, x, y) under η'. ■

REMARK 5.18. Further modifications in the preceding constructions will yield an H-set $\mathfrak{B}_5$ with Hjelmslev flight parameter sequence (2,2,9,17).

6. HJELMSLEV STRUCTURES

In this section we obtain Hjelmslev structures (briefly, H-structures) by using the H-sets constructed in Sections 4 and 5 above. We also prove that the Klingenberg flight parameter sequences of Section 5 are step parameter sequences.

We refer the reader to [2], which contains all relevant definitions together with an exposition of most of the known results regarding the construction of H-structures out of H-sets. Loosely, an H-structure is a triple (ϕ, Π, Π') where Π and Π' are incidence structures and $\phi: \Pi \to \Pi'$ is an epimorphism (additional properties are required). A PH-plane is an H-structure for which Π' is a projective plane. A (t, r) PH-plane is a PH-plane such that Π' has order r and every point of Π' is the image of precisely t^2 points of Π.

THEOREM 6.1 (Drake and Lenz [1, Theorem 1.8]). *Assume the existence of an H-set of* $r+1$ *matrices of order* t^2, *where r is the order of a projective plane* Π'. *Then there exists a* (t, r) *PH-plane* (ϕ, Π, Π').

COROLLARY 6.2. *There exist* (t, r) *PH-planes with* $(t, r) = (99,3)$ *and* (352,2,),(396,2),(416,2),(468,2).

Proof. Apply Theorem 6.1 to H-sets of Example 4.7 and Propositions 5.15 and 5.17. ■

Let $\Pi = (\phi, \Pi, \Pi')$ be an arbitrary H-structure. If one factors ϕ into a product $\psi_1\psi_2\cdots\psi_{n-1}$ of "K-eumorphisms," then each map ψ_{i-1} will identify sets of points of cardinality R_i^2. The sequence $(R_2, R_3, \ldots, R_n)$ is called *a Klingenberg flight parameter sequence* of Π; it is called *the Klingenberg step parameter sequence* if the factorization is maximal. If all images of Π under maps $\psi_i\psi_{i+1}\cdots\psi_{n-1}$ are H-structures, one replaces the adjective Klingenberg by *Hjelmslev* in the preceding two descriptive phrases.

PROPOSITION 6.3. *Let $\mathcal{Q}$ be an H-set of k matrices, $k \geqslant 3$; Π' be a connected incidence structure with k points on every line and k lines through every point; Π be an "exhaustive" expansion of Π' by $\mathcal{Q}$ (meaning that all k matrices from $\mathcal{Q}$ are used as replacements for the* 1*'s in each row and in each column of some incidence matrix for Π'). Then $\Pi = (\phi, \Pi, \Pi')$ is an H-structure, where ϕ is the induced map. Every Hjelmslev (Klingenberg) flight parameter sequence for $\mathcal{Q}$ is a Hjelmslev (Klingenberg) flight parameter sequence for Π.*

Proof. The first assertion is one of the assertions of [2, 3.11]. The assertion regarding Klingenberg sequences is a special case of [2, 3.15]: one uses $\Sigma_2 = \Sigma_1 = \Pi'$ and takes μ_1 to be the identity map. To obtain the assertion about Hjelmslev sequences, one must combine the results [2, 3.11] and [2, 3.15]. ■

THEOREM 6.4. *Let $\mathcal{Q}$ be a (q, q) partition set. Let $(\mu_1, \Sigma_2, \Sigma_1)$ be an H-structure whose Klingenberg and Hjelmslev step parameter sequences are $(q_2, q_3, \ldots, q_m)$ and $(p_2, p_3, \ldots, p_t)$ respectively, and Σ_3 be an exhaustive, doubly differentiated expansion of Σ_2 by $\mathcal{Q}$. Then if $\mu_2 : \Sigma_3 \to \Sigma_2$ is the induced map, $(\mu_1\mu_2, \Sigma_3, \Sigma_1)$ is an H-structure whose Klingenberg and Hjelmslev step parameter sequences are $(q_2, \ldots, q_m, q)$ and $(p_2, \ldots, p_t, q)$, respectively.*

Proof. The preceding statement is obtained from [2, 5.6] by substituting the phrase "partition set" for "H-set generated by an $(n+1)$-uniform PH-plane." Both [2, 5.5] and [2, 5.6] remain valid under this substitution. The proofs given in [2] remain valid without alteration; they can be abbreviated somewhat, however, since the altered statements correspond to the case $n = 1$ and thus do not require the use of induction. Better still, no knowledge of "n-uniform" PH-planes is required to understand the abbreviated proofs. ■

In Theorem 6.5 below we state the altered version of [2, 5.5]. This result is not needed in the present paper, but it will be helpful to supply a reference to authors of future papers.

THEOREM 6.5. *Let $\mathcal{Q}$ be a (q,q) partition set. Let $(\mu_1, \Sigma_2, \Sigma_1)$ be a neighbor cohesive K-structure whose Klingenberg step parameter sequence is $(q_2, q_3, \ldots, q_m)$, and Σ_3 be an exhaustive, differentiated expansion of Σ_2 by $\mathcal{Q}$. Then if $\mu_2: \Sigma_3 \to \Sigma_2$ is the induced map, $(\mu_1\mu_2, \Sigma_3, \Sigma_1)$ is a neighbor cohesive K-structure whose Klingenberg sequence is $(q_2, \ldots, q_m, q)$.*

COROLLARY 6.6. *Let Σ_1 be a connected incidence structure with three points on each line and three lines through each point. Then there exist H-structures (ϕ, Σ, Σ_1) whose Klingenberg and Hjelmslev step parameter sequences coincide and are equal to the sequences $(2,2,q_4,q_5)$ for each of the following pairs (q_4, q_5):* (8,11),(8,13),(9,11),(9,13),(9,17).

Proof. Any incidence matrix for Σ_1 is a sum of three permutation matrices, so it is possible to obtain an exhaustive expansion Σ of Σ_1 by any $(t,2)$ *H*-set $\mathcal{Q}$. We use the *H*-sets $\mathcal{Q}_5$ of Section 5. Proposition 6.3 guarantees that Σ has a Hjelmslev flight parameter sequence $(2,2,q_4,q_5)$. This sequence corresponds to an *H*-factorization of (ϕ, Σ, Σ_1), say to a sequence $(\psi_1, \psi_2, \psi_3, \psi_4)$ of *H*-eumorphisms. Since every *H*-factorization can be expanded to an *H*-solution and thence to a *K*-solution by Corollary 2.10 and Theorem 2.9 above, the step parameter sequences of Σ will both be $(2,2,q_4,q_5)$ if no ψ_i can be factored into a product of proper *K*-eumorphisms. Since 2 is a prime, neither ψ_1 nor ψ_2 can be so factored. Applying ψ_4 to Σ yields an incidence structure Σ_4 which is an expansion of Σ_1 by $\mathcal{Q}_4$. Applying ψ_3 to Σ_4 yields an incidence structure Σ_3. Now Σ_4 is an exhaustive, doubly differentiated expansion of Σ_3 by $\mathcal{C}$; and $\mathcal{C}$ is a partition set. Then Theorem 6.4 implies that the Klingenberg and Hjelmslev step parameter sequences of $(\psi_1\psi_2\psi_3, \Sigma_4, \Sigma_1)$ both end with the parameter q_4; i.e., that ψ_3 cannot be factored. A similar argument guarantees that ψ_5 cannot be factored. ■

COROLLARY 6.7. *For each of the five H-sets of Propositions* 5.15 *and* 5.17 *and Remark* 5.18 *the Klingenberg and Hjelmslev step parameter sequences coincide. They are the sequences of Corollary* 6.6.

Proof. Let Σ be constructed as in the proof of Corollary 6.6. If one could obtain a *K*-factorization of $\mathcal{Q}_5$ of length greater than four, then Proposition 6.3 would yield a Klingenberg flight parameter sequence of length greater than four for Σ, in contradiction to Corollary 6.6. ■

COROLLARY 6.8. *Let Σ_1 be a connected incidence structure with four points on each line and four lines through each point. Then there is an H-structure (ϕ, Σ, Σ_1) whose Klingenberg and Hjelmslev step parameter sequences are both* (3,11,3).

Proof. Let Σ be an exhaustive expansion of Σ_1 by the H-set $\mathcal{C}_4$ of Example 4.7. Apply Proposition 6.3, and observe that 3 and 11 are primes. ■

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Note added in proof. In the introduction we credited Törner with making the observation that $q_i \geqslant r$ for every i. We thank P. Y. Bacon for informing us that she had made the same observation [12, Theorem 1.3.5] at an earlier date.

In [13, p. 73] Sane asserted that PH-planes had been constructed for 70 pairs $(t,3)$ with $t < 1000$. The pair $(3 \cdot 11^2, 3)$ was counted twice, however, since this pair is constructed in [4] as well as in [13]. Since the new pair (99,3) is obtained in the present paper, the current count is now legitimately 70.

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