ELSEVIER Journal of Pure and Applied Algebra 111 (1996) 1–20

JOURNAL OF PURE AND APPLIED ALGEBRA

$*$-Autonomous categories, revisited☆

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Communicated by F.W. Lawvere; received 24 April 1994

Abstract

This paper shows how the systematic use of the "Chu construction" can both strengthen and simplify the rather complicated constructions used in the original papers and lecture notes on $*$-autonomous categories. In effect, representation data are substituted for topological and, although this loses topological information, it retains just the right amount.

1. Introduction

The main purpose in [1] was to explore the notion of a category that was both autonomous (symmetric, monoidal, closed) and also had a duality. We were interested in the case in which the duality was mediated by a dualizing object, call it $\perp$, so that the dual of an object was the internal hom into $\perp$. A number of examples were constructed, with a fair amount of work. They were mainly categories of topologized objects (more precisely, uniform space objects, but, beng groups, this made no practical difference). The categories were full subcategories defined by some sort of completeness condition that was hard to describe and even harder to verify. At the end of the monograph was a paper by P.-H. Chu that contained the essentials of his M.Sc. thesis. This paper described another, rather formal, construction of a class of $*$-autonomous categories. The main purpose of this "Chu construction" was to demonstrate that there were plenty of $*$-autonomous categories. It was never really anticipated that the construction itself would give interesting categories.

☆In the preparation of this paper, I have been assisted by a grant from the NSERC of Canada.
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SSDI 0022-4049(95)00040-2

In the meantime, J.-Y. Girard created linear logic, which can be thought of as the logic of dealing with limited resources or, more prosaically, as the logic of additive categories, just as classical and related logics are the logics of one or another class of toposes. (It is not true that all models of linear logic are additive, but very many are.) In its original formulation, at least, linear logic was to have linear conjunction, disjunction and implication as well as linear negation. These are, in fact, the tensor cotensor (see below) and duality in a $*$-autonomous category. Girard also supposed one more operation, the ! (and it dual ?; one proposal is to call them bang and whimper) that is not part of the definition of $*$-autonomous categories. The basic idea is that if A is a resource, then $!A$ is an indefinite supply of copies of A. For example, if you are programming in a functional programming language in which functions always consume their arguments, the store is still a source of an indefinite number of duplicates of any resource that is kept there. See [5] for more details.

For these reasons, $*$-autonomous categories, expecially those with !, have become more interesting as models of linear logic. It comes as some surprise and considerable interest to learn that in fact, the Chu construction can basically replace the complications of topology and completeness that characterized [1]. More precisely, it is shown that in two of the cases considered there, at least, a certain full $*$-autonomous subcategory of the Chu category (the separated, extensional objects) is fully embedded (in two ways, actually) into the category of topologized objects, preserving its $*$-autonomous structure. Moreover, the image categories are not defined by any completeness conditions, only by their topologies being either maximal or minimal for their dual spaces. Thus, the completeness conditions of the original paper are seen to have been unnecessary.

We have not looked at the other examples from [1], but there is little reason to doubt that the results would be the same.

I must mention that this new look at the Chu construction was motivated by results of V. Pratt and his students who discovered some remarkable properties of these categories. I should also mention that D. Pavlovic has discovered that in many cases of interest, the ! operator exists for the separated extensional Chu category. Thus, these categories are models of the full linear logic.

2. The Chu construction

In this section, we will be dealing with a category $\mathscr{V}$ that is autonomous and, we assume without further mention, has pullbacks.

Suppose $\mathscr{V}$ is a symmetric monoidal closed (hereafter called autonomous) category and K is an object of $\mathscr{V}$. Then the category we denote Chu $(\mathscr{V}, K)$ has as objects $(V, V', \langle -, - \rangle)$ where $\langle -, - \rangle : V \otimes V' \to K$ is a morphism of $\mathscr{V}$. We will normally omit explicit mention of the pairing $\langle -, - \rangle$. A morphism $(f, f') : (V, V') \to (W, W')$ is a pair of arrows of $\mathscr{V}, f : V \to W$ and $f' : W' \to V'$ that satisfy the identity symbolized by the equation $\langle fv, w' \rangle = \langle v, f'w' \rangle$, which is to say that the diagram

$$
\begin{array}{ccc}
V \otimes W' & \xrightarrow{1 \otimes f'} & V \otimes V' \\
{\scriptstyle f \otimes 1}\downarrow & & \downarrow{\scriptstyle \langle -,- \rangle} \\
W \otimes W' & \xrightarrow[\langle -,- \rangle]{} & K
\end{array}
$$

commutes. Then $\mathscr{A} = \mathrm{Chu}(\mathscr{V}, K)$ is self-dual, the duality being the one that takes (V, V') to (V', V) and similarly on arrows. Moreover, $\mathscr{A}$ is also autonomous. To see that, let us denote by $V \multimap W$ the internal hom in $\mathscr{V}$ of V into W.

First, we observe we can enrich $\mathscr{A}$ over $\mathscr{V}$ by internalizing the definition of the external hom. Namely, we define $\mathscr{V}((V, V'), (W, W'))$ by noting that the pairing of an object (V, V') determines by exponential adjoint both a map $V \to (V' \multimap K)$ and a map $V' \to (V \multimap K)$. Using these, along with the isomorphisms $V \multimap (W' \multimap K) \cong V \otimes W' \multimap K \cong W' \multimap (V \multimap K)$, define $\mathscr{V}((V, V'), (W, W'))$ by requiring that the diagram

$$
\begin{array}{ccc}
\mathscr{V}((V,V'),(W,W')) & \longrightarrow & V \multimap W \\
\downarrow & & \downarrow \\
W' \multimap V' & & V \multimap (W' \multimap K) \\
\downarrow & & \downarrow \\
W' \multimap (V \multimap K) & \longrightarrow & (V \otimes W') \multimap K
\end{array}
$$

be a pullback determines not only an object but also an arrow $\mathscr{V}((V, V'), (W, W')) \to ((V \otimes W') \multimap K)$ and thus an object $(\mathscr{V}((V, V'), (W, W')), (V \otimes W'))$ of $\mathscr{A}$, which is defined to be the object $(V, V') \multimap (W, W')$. This defines the closed structure. The monoidal structure is given by $(V, V') \otimes (W, W') = ((V, V) \multimap (W', W))^*$, which is $(V \otimes W, \mathscr{V}((V, V'), (W', W))$. The unit for the tensor is given by $(\top, K)$, where $\top$ is the unit for the tensor in $\mathscr{V}$ and $\top \otimes K \to K$ is the canonical isomorphism and the dualizing object is $(K, \top)$. It is easily shown that $(V', V) \cong ((V, V') \multimap (K, \top)$ and that we have a $*$-autonomous category. Details are found in [3].

Following Pratt, we will call an object of $\mathrm{Chu}(\mathscr{V}, K)$ a *Chu space* (of $\mathscr{V}$, with respect to K). There are a couple of reasons for calling it a space. For one thing, the main thrust of this paper is to show that it is equivalent to spaces in some cases. For another, an object of, for example, Chu (**Set**, 2) is a set S, together with a set S' equipped with a map $S' \to 2^S$. Should that map be injective (the extensional case described below), then S' is simply a set of subsets of S and so the beginnings of a topology.

2.1. Separated and extensional Chu spaces. Aside from the symmetric version of the definition of Chu space, there are two one-sided versions. The first views such a space as having objects V and V' and an arrow $V' \to V \multimap K$. If this arrow is injective, then V' can be viewed as an object of (not necessarily all) arrows $V \to K$. Two arrows are equal if and only if they are equal at every "element" of V. The term that logicians use to describe this property of functions is extensionality, so we will call the Chu space extensional when $V' \to V \multimap K$ is monic. More generally, given a class $\mathscr{M}$ of morphisms (usually monomorphisms), we say the Chu object is $\mathscr{M}$-extensional if $V' \to V \multimap K$ belongs to $\mathscr{M}$.

Similarly, we can view a Chu space as an object V equipped with an arrow $V \to V' \multimap K$ and ask that this arrow be monic and think of that as meaning it separates points of V. Accordingly, we will say that the Chu space is *separated* (resp., $\mathscr{M}$-separated) if that map is monic (resp., belongs to $\mathscr{M}$).

2.2. The separated reflection and extensional coreflection. We suppose now chosen a factorization system $\mathscr{E}/\mathscr{M}$ that will not change. In order to avoid over complicated notation, we will write $\mathrm{Chu}_s(\mathscr{V}, K)$, $\mathrm{Chu}_e(\mathscr{V}, K)$ and $\mathrm{Chu}_{se}(\mathscr{V}, K)$ for the full subcategories of $\mathscr{M}$-separated $\mathscr{M}$-extensional, and both $\mathscr{M}$-separated and $\mathscr{M}$-extensional objects, respectively.

2.3. Proposition. *Suppose $\mathscr{E}/\mathscr{M}$ is a factorization system on $\mathscr{V}$ and also suppose that for all $e \in \mathscr{E}$, $e \multimap K \in \mathscr{M}$. Then the inclusion $\mathrm{Chu}_s(\mathscr{V}, K) \to \mathrm{Chu}(\mathscr{V}, K)$ has a left adjoint s and the inclusion $\mathrm{Chu}_e(\mathscr{V}, K) \to \mathrm{Chu}(\mathscr{V}, K)$ has a right adjoint e and these operations on $\mathrm{Chu}(\mathscr{V}, K)$ commute.*

This is proved in [2, Proposition 5.2] under slightly different assumptions. The arguments remain valid, however. In that paper, what we are calling separated and extensional are called left and right separated, respectively.

The result of this is that the inclusion $\mathrm{Chu}_{se}(\mathscr{V}, K) \to \mathrm{Chu}(\mathscr{V}, K)$ has a left inverse and the former category is complete if $\mathscr{V}$ is. See [2] for details.

The following result was proved in that paper under the assumption that $\mathscr{V}$ was cartesian closed. It has since been extended to the general case by Pavlovic [8].

2.4. Theorem. *Suppose that $\mathscr{V}$ has cofree coalgebras (commutative, associative, unitary) for its tensor product and has complete subobject lattices. Then for a factorization system $\mathscr{E}/\mathscr{M}$ satisfying the condition that $e \in \mathscr{E}$ implies $e \multimap K \in \mathscr{M}$, the category $\mathrm{Chu}_{se}(\mathscr{V}, K)$ has cofree coalgebras as well.*

3. Linearly topologized vector spaces

By a linearly topologized vector space, we mean a hausdorff topological vector space that has a neighborhood base at 0 consisting of open vector subspaces. Let us call the category of linearly topologized spaces **LTS**.

In this section, we show that when $\mathscr{V}$ is the category of discrete vector spaces over a discrete field and $\mathscr{E}$ and $\mathscr{M}$ are the classes of epimorphisms and monomorphisms, respectively, then the category $\mathrm{Chu}_{se}(\mathscr{V}, K)$ is equivalent to two different subcategories of **LTS**.

There is an obvious functor $F: \mathbf{LTS} \to \mathrm{Chu}_{se}(\mathscr{V}, K)$ that takes the space A to $(|A|, \mathrm{Hom}(A, K))$. The first factor is the discrete space underlying A and the second is the space of continuous linear functionals on A.

3.1. Linearly compact and sublinearly compact spaces. A vector space is called *linearly compact* if it has the finite intersection property with respect to closed affine subsets. We list various properties of linearly compact spaces that we need. These facts are all proved in [7], where linear compactness was first defined.

(LC1) A product of linearly compact vector spaces is linearly compact.

(LC2) A closed subspace of a linearly compact vector space is linearly compact.

(LC3) The quotient of a linearly compact vector space by a closed subspace is linearly compact.

(LC4) A linearly compact space is discrete if and only if it is finite dimensional.

(LC5) A vector space over a field K is linearly compact if and only if it is isomorphic, algebraically and topologically, to a power of K.

Let us call a space sublinearly compact if it is isomorphic to a subspace of a linearly compact space. In light of (LC5), this means that its topology is the weak topology for a set of continuous functionals. Let **LC** denote the category of linearly compact spaces and **SLC** denote the full subcategory of sublinearly compact spaces. Notice that in any object (V, V') of $\mathrm{Chu}_{se}(\mathscr{V}, K)$, we may view V' as a set of functionals on V. We define a functor $W: \mathrm{Chu}_{se}(\mathscr{V}, K) \to \mathbf{LC}$ that assigns to each (V, V') the space V equipped with the weak topology for the functionals in V'. Its image is clearly included in **SLC**.

3.2. Strongly topologized spaces. We now define a second functor $S: \mathrm{Chu}_{se}(\mathscr{V}, K) \to$ **LC** as follows. Let $A = S(V, V')$ denote the space whose underlying vector space is V and in which a linear subspace $U \subseteq V$ is open when $U^{\perp} \subseteq V'$. Recall that we are thinking of V' as a set of functionals on V, so this simply says that every functional on V that vanishes on U is in V'. This does define a topology since if both $U_1^{\perp}$ and $U_2^{\perp}$ are included in V', so is their sum which is $(U_1 \cap U_2)^{\perp}$. Every functional in V' is continuous in this topology. For, if $\phi \neq 0$ is in V', then $U = \ker \phi$ is a subspace of codimension 1, whose annihilator is therefore of dimension 1 and thus generated by ϕ. Thus, $U^{\perp} \subseteq V'$ and so U is open. It follows that since (V, V') is separated, there is, for each $v \in V$, a $\phi \in V'$ such that $\phi(v) \neq 0$ and since ϕ is continuous, its kernel is an open subspace that does not contain v and so the topology is hausdorff.

These topologies are not the same in general. In fact, let V be an infinite-dimensional vector space and V' the space of all continuous functionals on V. Then every linear subspace is open by that definition. Thus, $S(V, V')$ is discrete. But no infinite-dimensional discrete space can be in **SLC** since in the weak topology for the set of all functionals, the set of all linear subspaces of finite codimension is a topology in which all functionals are continuous and is clearly the weakest such.

We will often refer to these topologies as the weak and strong topologies and say that a space is weakly or strongly topologized if it has one or the other.

3.3. Theorem. *$S \dashv F \dashv W$ and the adjunction morphisms $SF \to \mathrm{id} \to WF$ are isomorphisms.*

Proof. Let (V, V') be an object of $\mathrm{Chu}_{\mathrm{se}}(\mathscr{V}, K)$. Suppose that A is an object of **LTS** and that $(f, f'):(|A|, \mathscr{V}(A, K)) \to (V, V')$. Then $f:|A| \to V$ and whenever $\phi \in V'$, $\phi \circ f$ is a continuous linear functional on A. Now let $U \subseteq W(V, V')$ be an open subspace. Then there are a finite number of functionals $\phi_1, \phi_2, \dots, \phi_n$ such that $U = \ker(\phi_1) \cap \ker(\phi_2) \cap \cdots \cap \ker(\phi_n)$. Since all of $\phi_1 \circ f, \phi_2 \circ f, \dots, \phi_n \circ f$ are continuous, it follows that

$$f^{-1}(U) = f^{-1}(\ker(\phi_1)) \cap f^{-1}(\ker(\phi_2)) \cap \cdots \cap f^{-1}(\ker(\phi_n))$$

is open in A. Thus, f is continuous. Now suppose that $f: A \to W(V, V')$ is a continuous linear map. Suppose that $\phi \in V'$. Then $\phi \circ f$ is a continuous functional on A so that $(f, - \circ f):(|A|, \mathscr{V}(A, K)) \to (V, V')$ is a map of $\mathrm{Chu}_{\mathrm{se}}(\mathscr{V}, K)$. This shows that $F \dashv W$.

Suppose that $(f, f'):(V, V') \to (|A|, \mathscr{V}(A, K))$ is a map of $\mathrm{Chu}_{\mathrm{se}}(\mathscr{V}, K)$. Let U be an open linear subspace of A. Then A/U is discrete which means that every linear functional on $|A|$ that vanishes on U is in $\mathscr{V}(A, K)$. But then $f' = - \circ f$ takes every such functional into V' which means that every functional on V that vanishes on $f^{-1}(U)$ is in V' and hence that $f^{-1}(U)$ is open in $S(V, V')$. Now suppose that $f: S(V, V') \to A$ is continuous. Let ϕ be a functional on A. Then $\phi \circ f$ is a continuous functional on V. Thus, $\ker(\phi)$ is open in $S(V, V')$. Thus, every linear functional on V that vanishes on $\ker(\phi)$, of which $\phi \circ f$ is one, belongs to V'. Thus, $(f, - \circ f):(V, V') \to (|A|, \mathscr{V}(V, K))$ is a morphism. This shows that $S \dashv F$.

$SF \to \mathrm{id}$ is an isomorphism if and only if F is full and faithful if and only if $\mathrm{id} \to WF$ is an isomorphism, so it suffices to show that one of these holds. So let (V, V') be an object of $\mathrm{Chu}_{\mathrm{se}}(\mathscr{V}, K)$. Then every continuous linear functional ϕ on $S(V, V')$ has a kernel that is open, which means that every functional on V that vanishes on $\ker(\phi)$, of which ϕ is one, belongs to V' and so $V' = \mathscr{V}(S(V, V'), K)$. □

3.4. Corollary. *If A is an object of* **LTS**, *then SFA and WFA have the same underlying vector space and the same set of continuous linear functionals as A. Moreover, among topologies on the underlying vector space of A that have the same set of continuous linear functionals, SFA is the finest and WFA is the coarsest.*

Proof. It is part of the definition that SFA and WFA have the same underlying vector space as A. Moreover, the set of linear functionals is the second component of $FA = FSFA = FWFA$. The adjunction arrows $SFA \to A \to WFA$ imply that the topology of A lies between those of SFA and WFA. If B has the same underlying vector space and the same set of continuous functionals, then $FA = FB$ so that $SFA = SFB \to B \to WFB = WFA$. □

3.5. Corollary. *The image of W is the category* **SLC**.

Proof. The definition of an object of **SLC** is that it has the weak topology for a set of functionals. In the weak topology for all the functionals, a set is open if and only if it has the form $\ker\phi_1 \cap \cdots \cap \ker\phi_n$ for linear functionals $\phi_1, \ldots, \phi_n$. But the kernel of any linear functional is continuous in the given topology, so such an intersection is also open. Conversely, suppose a linear subspace is open in the topology induced by an embedding $A \subseteq C$, where C is linearly compact. Then it is the intersection with A of an open subspace of C. But any open subspace of C has finite codimension, which implies that it is the intersection of the kernels of a finite number of linear functionals on C, a fortiori on A. But then the given subspace is open in the weak topology for the functionals. □

It is not as easy to describe the image of S. Suffice it to say that A belongs to the image if and only if there is no stronger topology on A that has the same set of functionals as A. For example, the discrete spaces clearly have that property, while a discrete space is in **SLC** if and only if it is finite dimensional.

3.6. The $*$-autonomous structure. Since $\mathrm{Chu}_{se}(\mathscr{V}, K)$ is a $*$-autonomous category with bang and whimper, the same is necessarily true of the equivalent categories **SLC** and the full image of S. In the case of **SLC** it is fairly easy to describe the $*$-autonomous structure. The dual of A is simply $\mathrm{Hom}(A, K)$, topologized with the weak topology induced on it by the evaluations at elements of A.

3.7. Proposition. *If B is weakly topologized, then a homomorphism $|A| \to |B|$ underlies a continuous homomorphism if and only if it induces a map* $\mathrm{Hom}(B, K) \to \mathrm{Hom}(A, K)$.

Proof. Suppose $f: |A| \to |B|$ is not continuous from A to B. Then there is an open subspace $U \subseteq B$ such that $f^{-1}(U)$ is not open in A. Since B is weakly topologized, there are functionals $\phi_1, \ldots, \phi_n$ on B such that $U = \ker\phi_1 \cap \cdots \cap \ker\phi_n$. But then at least one of $f \circ \phi_1, \ldots, f \circ \phi_n$ is discontinuous, else $f^{-1}(U) = \ker(f \circ \phi_1) \cap \cdots \cap \ker(f \circ \phi_n)$ would be open in A. □

This proposition implies that

$$\begin{array}{ccc} \mathrm{Hom}(A,B) & \longrightarrow & \mathrm{Hom}(B,K) \multimap \mathrm{Hom}(A,K) \\ \downarrow & & \downarrow \\ |A| \multimap |B| & \longrightarrow & \mathrm{Hom}(|\mathrm{A}| \otimes \mathrm{Hom}(B,K), K) \end{array}$$

is a pullback, so that $FA \multimap FB$ is, according to the construction of [2], the extensional reflection of $(\mathrm{Hom}(A, B), |A| \otimes \mathrm{Hom}(B, K))$. The structure map

$$\mathrm{Hom}(A, B) \otimes |A| \otimes \mathrm{Hom}(B, K) \to K$$

takes $f \otimes a \otimes \beta \mapsto \beta(f(a))$ for $f: A \to B$, $a \in |A|$ and $\beta: B \to K$. Thus, in **SLC**, $A \multimap B$ is $\mathrm{Hom}(A, B)$ with the weak topology for the set of functionals determined as above by $|A| \otimes \mathrm{Hom}(B, K)$.

3.8. In fact, in this case, the object $(\mathrm{Hom}(A, B), |A| \otimes \mathrm{Hom}(B, K))$ is extensional. To see this, we require a lemma.

3.9. Lemma. *Let A be an object of* **LTS**. *Then for all linearly independent elements $a_1, a_2, \ldots, a_n \in A$, there is a continuous functional $\alpha: A \to K$ such that $\alpha(a_1) = 1$ and $\alpha(a_2) = \alpha(a_3) = \cdots = \alpha(a_n) = 0$.*

Proof. By induction, there is a functional α' such that $\alpha'(a_1) = 1$ and $\alpha'(a_2) = \cdots = \alpha'(a_{n-1}) = 0$. If $\alpha'(a_n) = 0$, we are through, otherwise we observe that $a_1 - a_n/\alpha'(a_n), a_2, \ldots, a_{n-1}$ is another set of $n - 1$ linearly independent elements and so there is a functional α'' with $\alpha''(a_1 - a_n/\alpha'(a_n)) = 1$ and $\alpha''(a_2) = \cdots = \alpha''(a_{n-1}) = 0$. This implies that $a''(a_1) = 1 + \alpha''(a_n)/\alpha'(a_n)$. Now we want to determine u and $v \in K$ so that $\alpha = u\alpha' + v\alpha''$ has the required properties. This requires determining u and v so that

$$u\alpha'(a_1) + v\alpha''(a_1) = 1, \qquad u\alpha'(a_n) + v\alpha''(a_n) = 0,$$

which leads to

$$u + v(1 + \alpha''(a_n)/\alpha'(a_n)) = 1, \qquad u\alpha'(a_n) + v\alpha''(a_n) = 0.$$

whose determinant is $\alpha''(a_n) - \alpha'(a_n) - \alpha''(a_n) = -\alpha'(a_n) \neq 0$ and so there is a solution. □

Now suppose that $\sum_{i=1}^n a_i \otimes \phi_i$ is a non-zero element of $|A| \otimes |B^*|$. We can suppose without loss of generality that $a_1, \ldots, a_n$ are linearly independent and that $\phi_1, \ldots, \phi_n$ are non-zero. Then choose a functional α on A such that $\alpha(a_1) = 1$ and $\alpha(a_2) = \cdots = \alpha(a_n) = 0$. Let $b \in B$ be an element such that $\phi_1(b) \neq 0$. Let $f: A \to B$ be defined by $f(a) = \alpha(a)b$. Then $\sum_{i=1}^n \phi_i(f(a_i)) = \phi_1(f(a_1)) = \phi_1(b) \neq 0$.

For the category that is the image of S, the duality and internal Hom do not have such a simple description. We can give some insight by first describing the category of locally linearly compact vector spaces.

3.10. Locally linearly compact vector spaces. We define, by analogy with locally compact groups, a linearly topologized vector space to be locally linearly compact if it has a neighborhood basis at the origin consisting of linearly compact open subspaces. For linearly topologized vector spaces, this turns out to be a very easy notion.

3.11. Proposition. *A linearly topologized vector space is locally linearly compact if and nly if it has an open linearly compact subspace if and only if it is the direct sum of a linearly compact space and a discrete space.*

Proof. The direct sum of a discrete and a linearly compact space is readily seen to be locally linearly compact. A locally linearly compact space certainly has at least one open linearly compact open subspace. So, suppose that A has at least one linearly compact open subspace C. Since C is open, A/C is discrete. But that means that any linear splitting of $A \twoheadrightarrow A/C$ is continuous. Thus, $A = C \oplus A/C$ is the direct sum as claimed. □

One consequence of this is that there is obviously a duality on the category of locally linearly compact vector spaces.

Recall that a linearly topologized vector space is strongly topologized if no finer topology has the same set of linear functionals.

3.12. Proposition. *A locally linearly compact vector space is strongly topologized.*

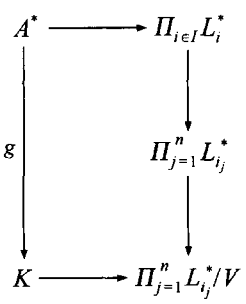
Proof. Suppose A and B are linearly topologized vector spaces with B locally linearly compact and $A \to B$ is a bijection that induces an isomorphism on the space of functionals. We may suppose without loss of generality that $|A| = |B|$. First, consider the case that A is discrete and B linearly compact. Then $B \cong K^X$ for some set X and has the further property that every linear functional on B is continuous. But if X is infinite, then any ultrafilter on X gives rise to a discontinuous functional. Thus, X is finite and B is also discrete. Next, consider the case that A is arbitrary, but B is linearly compact. Let U be an open subspace of A. Then A/U is discrete and thus every functional on A that vanishes on U is continuous on A and hence on B. But U is the intersection of the kernels of all the functionals that vanish on U and thus U is also closed in B. Then $A/U \to B/U$ is still bijective and induces a bijection on the set of continuous functionals. By the first case, B/U is discrete, whence U is open in B. For the general case, write $B = C \oplus D$ where C is linearly compact and D discrete. From continuity, it follows that D is a discrete subspace of A and C is a closed subspace of A, which, by the preceding case, must be linearly compact. It follows that $A \xrightarrow{\cong} B$. □

We now extend the duality functor to the category **LTS**. Let L^* denote the dual of a localy linearly compact vector space L. For an arbitrary linearly topologized vector space A, let A^* denote the vector space $\mathrm{Hom}(A, K)$ topologized with the weak topology for all the maps $A^* \to L^*$ induced by all $L \to A$, with L locally linearly compact. This extends the duality in the sense that if A is locally linearly compact then there is a final object in the category of locally linearly compact spaces over A, namely A itself and so the weak topology on the dual is just that of A^*.

Not every object is reflexive for this duality. Clearly, the locally linearly compact spaces are.

3.13. Theorem. *The natural map $|A| \to |A^{**}|$ is a bijection. It is not generally continuous, but the inverse $A^{**} \to A$ is. It is an isomorphism if and only if A is strongly topologized.*

Proof. We must show that every functional on A^* is evaluation at a unique element of A. Choose a set $\{f_i : L_i \to A\}$ of maps of locally linearly compact spaces to A such that A^* is embedded into $\prod L_i^*$. Suppose $g: A^* \to K$ is a continuous linear functional. Then $g^{-1}(0)$ is open in A^*, so that there is an open subspace $U \subseteq \prod L_i^*$ whose intersection with A is $g^{-1}(0)$. An open subspace in the product topology includes the product of all except a finite set of factors. Let $i_1, \ldots, i_n$ be that finite set of factors. Then there is an open linear subspace $V \subseteq L_{i_1}^* \times \cdots \times L_{i_n}^*$ such that $U = \pi^{-1}(V)$ under the projection $\pi : \prod_{i \in I} L_i^* \to \prod_{j=1}^n L_{i_j}^*$. Then we have the following commutative diagram:



Since $\prod_{j=1}^n L_{i_j}^*/V$ is discrete, the bottom arrow splits. This implies that there is a continuous linear functional $h : \prod_{j=1}^n L_{i_j}^* \to K$ such that $k \circ \pi$ extends g. Now $\prod_{j=1}^n L_{i_j}^*$ is a locally linearly compact space, so that k is evaluation at an element $l = (l_j) \in \prod_{j=1}^n L_{i_j}^*$. One now sees that g is evaluation at $\sum_{j=1}^n f_{i_j}(l_{i_j}) \in A$. The fact that this element is unique is a consequence of the sufficiency of functionals, which follows immediately from the fact that the topology is separated. This shows that the natural map is bijective.

To show that $A^{**} \to A$ is continuous, let U be an open linear subspace of A. Then $A \to A/U$ is continuous and the latter is discrete. Then $(A/U)^* \to A^*$ and finally $A^{**} \to (A/U)^{**} = A/U$ is continuous, so that U is also open in A^{**}.

It follows that $A^* = A^{***}$. In fact, they have the same elements and the dual of $A^{**} \to A$ gives the map in one direction, while the other is an instance of the same map.

If A is strongly topologized, then A^{**} must have the same topology since no stronger topology can have the same functionals. Suppose A is not strongly topologized. Let B denote A retopologized with the strong topology. I claim that $B^* = A^*$. They have the same elements, so it is sufficient to show they have the same topology. Suppose that L is locally linearly compact and $f: L \to A$ is continuous. Suppose that U is a linear subspace that is open in the topology on B. Then every functional on B that vanishes on U is continuous. Since A and B have the same functionals, it is also true of A that every functional that vanishes on U is continuous. The intersection of the kernels of all such functionals is U, so that U is closed in A and $f^{-1}(U)$ is closed in L. Then $L/f^{-1}(U)$ is locally linearly compact and every functional on it is continuous, whence $L/f^{-1}(U)$ is discrete and thus $f^{-1}(U)$ is open in L. But this means that $f: L \to B$ is continuous and thus B^* has the same topology as A^*. □

3.14. Example. This is an example to show that not every space is reflexive. Let A be an infinite-dimensional space topologized with the weak topology for all functionals. Then A is not discrete because the topology on A is that a subspace is open if and only if it has finite codimension. On the other hand, every functional is evidently continuous, so the associated strong topology is discrete.

3.15. The $*$-autonomous structure on strongly topologized spaces. What we have just seen is that the image of L can be viewed either as the strongly topologized spaces or as the spaces that are reflexive for the duality just described. It follows that full subcategory of linearly topologized vector spaces is also $*$-autonomous. The duality is the one just described. It does not seem quite possible to describe the internal hom, but we can use the result for the first part to say that $A \multimap B$ has underlying vector space $\mathrm{Hom}(A, B)$ with the topology gotten by first using the weak topology for the functionals induced by $|A| \otimes |B^*|$ and then forming the double dual . Similarly, the tensor product $A \otimes B$ has $|A| \otimes |B|$ for underlying vector space and the continuous functionals are those from $\mathrm{Hom}(A, B^*) \cong \mathrm{Hom}(B, A^*)$.

4. Abelian topological groups

The duality theory of locally compact abelian groups is well-known. In [SCAT], I showed that there are $*$-autonomous categories of topological abelian groups that are complete and cocomplete, include the locally compact groups and for which the duality reduces to the standard duality for those groups. The arguments are ad hoc and the categories are defined by somewhat obscure completeness conditions that in practice are likely impossible to verify. In this section we show how the systematic use of the category Chu(Ab, K), where Ab is the category of abelian groups and K is the circle, simplifies and clarifies everything (and, by the way, eliminates the completeness conditions).

In this section, we will let $V, V', W, \dots,$ denote abelian groups and "group" will always mean abelian group. We will denote by $A, B, A', \dots,$ topologized abelian groups. A character on A will mean a homomorphism $|A| \to K$, where $|A|$ is the discrete group underlying A. A continuous character, of course, is one that is continuous in the topology on A.

Say a topological group is subcompact if it is isomorphic to a subgroup of a compact group. We say that a group is SPLC if it is a subobject of a product of locally compact groups. **SPLC** denotes the full subcategory of all such groups. All the groups we deal with here are in this category.

An object (V, V') of Chu(Ab, K) is extensional if and only if V' is isomorphic to a group of characters on V and it is separated if those characters separate points. We will be dealing mostly with objects of $\mathrm{Chu}_{\mathrm{se}}(\mathrm{Ab}, K)$ and will tacitly suppose that V' is a group of characters on V. It is possible, and we will see examples below, for two LCR groups to have the same set of continuous characters. Thus, if we are given an object (V, V') of $\mathrm{Chu}_{\mathrm{se}}(\mathrm{Ab}, K)$, there is a poset (which conceivably could be empty) of topologies on V for which the group of continuous characters is exactly V'.

4.1. Theorem. *For any object (V, V') of $\mathrm{Chu}_{\mathrm{se}}(\mathrm{Ab}, K)$, the poset of topologies on V for which V' is the set of continuous characters is a complete lattice.*

Before proving this, we must establish some preliminary results.

4.2. Lemma. *Suppose A is a topological abelian group and $\chi : |A| \to K$ is a group homomorphism. Then χ is continuous if and only if $\chi^{-1}(-\frac{1}{4}, \frac{1}{4})$ is a neighborhood of 0 in A.*

Proof. The necessity of the condition is obvious. Suppose now that $U = \chi^{-1}(-\frac{1}{4}, \frac{1}{4})$ is a neighborhood of 0. Choose a sequence $U_2 = U, U_3, \dots, U_n$, of neighborhoods of 0 in A, such that $U_i + U_i \subseteq U_{i-1}$. I claim that $\chi^{-1}(-2^{-n}, 2^{-n}) \supseteq U_n$. This is true by assumption for $n = 2$. Assume it is true for n. Then for $a \in U_{n+1}$,

$$2\chi(a) = \chi(a + a) \in \chi(U_{n+1} + U_{n+1}) \subseteq \chi(U_n) \subseteq (-2^{-n}, 2^{-n}),$$

which is possible only if $\chi(a) \in (-2^{n+1}, 2^{-(n+1)})$ or $\chi(a) \in (\frac{1}{2} - 2^{n+1}, \frac{1}{2} + 2^{-(n+1)})$, but the latter is impossible since also

$$\chi(a) = \chi(a + 0) \in \chi(U_{n+1} + U_{n+1}) \subseteq \chi(U_n) \subseteq (-2^{-n}, 2^{-n})$$

The neighborhoods of 0 of the form $(-2^{-n}, 2^{-n})$ are neighborhood base at 0 in K and so the conclusion follows. □

4.3. Theorem. *The circle roup is injective in* **SPLC** *with respect to the class of embedded subgroups.*

Proof. Suppose $A \subseteq B$. Since B is embedded in a product of locally compact groups, we can, without loss of generality, suppose that B is a product of locally compact groups, say $B = \prod_i L_i$. Suppose $\chi: A \to K$ is a continuous character on A. Let $U = \chi^{-1}(-\frac{1}{4}, \frac{1}{4})$. Then $U = A \cap U_1$ for some neighborhood U_1 of 0 in B. Choose a neighborhood U_2 of 0 in B such that $U_2 = -U_2$ and $U_2 + U_2 \subseteq U_1$. The definition of the product topology implies that there is a cofinite subset $J \subseteq I$ such that $B_0 = \prod_{i \subseteq J} L_i \subseteq U_2$. Let $U_3 = U_2 + B_0$. It follows that $U_3 + B_0 = U_3$, that $U_3 = U_2 + B_0 \subseteq U_2 + U_2 \subseteq U_1$ and hence that $A \cap U_3 \subseteq U$. Let $p: B \to \tilde{B} = B/B_0$ be the projection, the latter equipped with the quotient topology, which topologizes it as a product of a finite number of locally compact groups, which is therefore locally compact. Topologize $\tilde{A} = A/(A \cap B_0)$ as a subspace of $\tilde{B}$ (which is not necessarily the same as the quotient topology).

First, note that $A \cap B_0$ is a subgroup of A included in U so that $\chi(A \cap B_0) \subseteq (-\frac{1}{4}, \frac{1}{4})$ and the latter includes no non-zero subgroup so that $\chi(A \cap B_0) = 0$. Thus, χ induces a possibly discontinuous character $\tilde{\chi}: \tilde{A} \to K$. Since U_3 is open and $U_3 + B_0 = U_3$, it follows that $p(U_3)$ is open in $\tilde{B}$. Therefore, if we can show that $\tilde{\chi}(\tilde{A} \cap p(U_3)) \subseteq (-\frac{1}{4}, \frac{1}{4})$, continuity follows from the preceding lemma. For $a \in A$, $p(a) \in p(U_3)$ requires that there be an element $b_0 \in B_0$ such that $a + b_0 \in U_3$. This means that $a + b_0 = u_2 + b'_0$, where $u_2 \in U_2$ and $b'_0 \in B_0$. But then $a = u_2 + (b'_0 - b_0) \in U_2 + U_2 \subseteq U_1$ and thus $a \in A \cap U_1 \subseteq U$, whence $\chi(a) \in (-\frac{1}{4}, \frac{1}{4})$.

Since $\tilde{\chi}$ is continuous, it is uniformly continuous and extends to the topological closure of $\tilde{A} \subseteq \tilde{B}$. The latter is locally compact and so is any closed subgroup and the conclusion is well known on the category of locally compact abelian groups. □

In the process of proving this, we have shown:

4.4. Proposition. *Let $A \subseteq \prod_{i \in I} B_i$. Then for any continuous character $\chi: A \to K$, there is a cofinite subset $J \subseteq I$ such that χ vanishes on $A \cap \prod_{i \in J} B_i$.*

4.5. Corollary. $\mathrm{Hom}(\prod_{i \in I} B_i, K) \cong \sum_{i \in I}(B_i, K)$.

We now turn to the proof of Theorem 4.1. It is clearly sufficient to show there is a finest and a coarsest topology for which V' is the set of continuous characters, for then the poset is obviously a complete sublattice of the lattice of all SPLC topologies on V. The latter is complete because it is a coreflective sublattice of the lattice of all topologies.

For the coarsest, let A denote the group V with the topology induced by embedding of V into the compact group $K^{V'}$. In other words, A has the weak topology for the characters in V'. No coarser topology on V can have every element of V' be continuous. We must show that no other character is continuous in this topology. So, suppose that χ is a continuous character on A. Since K is injective in **SPLC**, χ can be extended to a continuous character on $K^{V'}$, which we will continue to denote by χ.

From Corollary 4.4, there is a finite set $\chi_1, \chi_2, \dots, \chi_n \in V'$ such that χ factors through $K^{\{\chi_1, \dots, \chi_n\}} \cong K^n$. The dual of K^n is the direct sum of n copies of $\mathbb{Z}$, which means there are integers $k_1, k_2, \dots, k_n$ such that for $f: \{\chi_1, \chi_2, \dots, \chi_n\} \to K$,

$$\chi(f) = k_1 f(\chi_1) + k_2 f(\chi_2) + \cdots + k_n f(\chi_n).$$

If we apply this to the case that $f = \phi(v)$ for $v \in V$, we see that

$$\begin{aligned}\chi(\phi(v)) &= k_1 \phi(v)(\chi_1) + k_2 \phi(v)(\chi_2) + \cdots + k_n \phi(v)(\chi_n)\\ &= k_1 \chi_1(v) + k_2 \chi_2(v) + \cdots + k_n \chi_n(v)\\ &= (k_1 \chi_1 + k_2 \chi_2 + \cdots + k_n \chi_n)(v),\end{aligned}$$

which implies that $\chi/V = k_1 \chi_1 + k_2 \chi_2 + \cdots + k_n \chi_n \in V'$.

Next, we show there is a finest topology. For a group B of **SPLC**, let $B^\flat$ denote the group $|B|$ topologized with the weak topology for its continuous functionals. If A is another object of **SPLC**, say that a homomorphism $f: |A| \to |B|$ is weakly continuous if $A \to B^\flat$ is continuous. This is equivalent to supposing that for each continuous character $\chi: B \to K$, the composite $\chi \circ f$ is a continuous character on A. It is clear that every continuous homomorphism is weakly continuous.

Now let $A^\#$ denote the object A equipped with the weak topology for all the weakly continuous maps. Since every continuous map is weakly continuous, the topology on $A^\#$ is at least as fine as that of A. Before completing the argument, we show:

4.6. Proposition. *$A^\#$ has the same continuous characters as A.*

Proof. Choose a set $\{f_i: A \to B_i \mid i \in I\}$ of weakly continuous maps such that the induced $A^\# \to \prod_{i \in I} B_i$ is an embedding. From injectivity of K, it follows that any continuous character on $A^\#$ extends to a continuous character, say $\chi: \prod B_i \to K$. As above, there is a finite subset, say $\{1, \dots, k\} \subseteq I$ such that χ factors through $B_1 \times \cdots \times B_k$. **SPLC** is an additive category so that each such character has the form $\chi_1 + \cdots + \chi_k$ where, for $i = 1, \dots, k$, χ_i is a continuous character and hence so is their sum. Thus, $\chi \circ (f_i)$ is also continuous on A. □

4.7. Corollary. *For any object B of* **SPLC**, $\mathrm{Hom}(A^\#, B) \cong \mathrm{Hom}(A, B^\flat)$.

Proof. The definition of $\#$ implies that any continuous homomorphism $A \to B^\flat$ gives a continuous homomorphism $A^\# \to B$. Suppose that $f: A^\# \to B$ is continuous. Then for any continuous character $\chi: B \to K$, $\chi \circ f$ is a continuous character on $A^\#$, hence on A. Then f becomes continuous when B is retopologized by the weak topology for its continuous characters, that is $f: A \to B^\flat$ is continuous. □

Now suppose that B is the same abstract group as A and has the same set of continuous characters as A. Then the identity $A \to B^\flat$ is continuous so that $A^\# \to B$ is, which implies that the topology on B is no finer than that of A. This completes the proof of Theorem 4.1.

4.8. Corollary. *Given any topological group, there is a finest and weakest topology that has the same set of continuous characters.*

We will say that the object A of **SPLC** is weakly, resp. strongly topologized if $A = A^\flat$, resp. $A = A^\#$. Since a group is weakly topologized if and only if it is in **SC**, we already have a name for that full subcategory. We will denote the full subcategory of strongly topologized groups by **STSPLC**.

Let F: **SPLC** $\to \mathrm{Chu_{se}}(\mathrm{Ab}, K)$ denote the functor $FA = (|A|, \mathrm{Hom}(A, K))$. We denote by S, W: $\mathrm{Chu_{se}}(\mathrm{Ab}, K) \to$ **SPLC** the functors that take the object (V, V') to the group V topologized with the strong, resp. weak, topology.

4.9. Theorem. *$S \dashv F \dashv W$ and the adjunction morphismsm $SF \to \mathrm{id} \to WF$ are isomorphisms.*

Proof. Let $(f, f'){:}(|A|, \mathscr{V}(A, K)) \to (V, V')$ be given. Then $f{:}|A| \to V$ is a group homomorphism and whenever $\chi \in V'$, the composite $\chi \circ f$ is a continuous character on A. When V is given the weak topology for all its characters, f becomes continuous, so that $f{:}A \to W(V, V')$. To go the other way, we begin by noting that if χ is a continuous character on $W(V, V')$, then for any continuous $f{:}A \to W(V, V')$, every $\chi \in V'$ is a continuous character on $W(V, V')$ so that $\chi \circ f$ is continuous on A. Thus, $(f, - \circ f){:}(|A|, \mathscr{V}(A, K)) \to (V, V')$ is a morphism in $\mathrm{Chu_{se}}$. It follows that $F \dashv W$. The fact that $WF \cong \mathrm{id}$ is an immediate consequence of Theorem 4.1.

Let $S(V, V') = (W(V, V'))^\#$. Then for any object A of **SPLC**, we have the string of isomorphisms

$$\begin{aligned}\mathrm{Hom}(S(V, V'), A) &\cong \mathrm{Hom}((W(V, V'))^\#, A) \cong \mathrm{Hom}(W(V, V'), A^\flat)\\ &\cong \mathrm{Hom}(W(V, V'), WFA) \cong \mathrm{Hom}((V, V'), FA)\end{aligned}$$

since $FW \cong \mathrm{id}$ is equivalent to W being full and faithful. Thus, $S \dashv F$. It is a standard fact that if a right adjoint of a functor is full and faithful, so is any left adjoint. □

It follows that both **SC** and **STSPLC** are $*$-autonomous, as well as being equivalent to each other. They are certainly simpler to describe than the $*$-autonomous categories constructed in [SCAT]. What is surprising is that they are equivalent to the purely formal category $\mathrm{Chu_{se}}(\mathrm{Ab}, K)$, from which the topological information has been reduced the bare minimum of knowing when a character is continuous. For example, we will see later that one of these two categories includes all locally compact groups and the other one does not.

The duality and internal hom can readily be recovered from those of the Chu category. For example, the dual in **SC**, resp. **STSPLC**, of a group A is simply

Hom(A, K) with the weak, resp. strong, topology on it by A, thought of as a group of characters on Hom(A, K).

The internal hom can be described as follows. Let $\mathscr{A}$ be one of the categories **STSPLC** or **SC**. For an object A of $\mathscr{A}$, **SC**, let A^* denote the dual (with the relevant topology). Then $FA = (|A|, |A^*|)$. If B is another object of $\mathscr{A}$, then the fact that F is an equivalence on $\mathscr{A}$ implies that $\mathrm{Hom}(A, B) \cong \mathrm{Hom}(FA, FB)$. It follows that the square

$$\begin{array}{ccc} \mathrm{Hom}(A,B) & \longrightarrow & \mathrm{Hom}(|A|,|B|) \\ \downarrow & & \downarrow \\ \mathrm{Hom}(|B^*|,|A^*|) & \longrightarrow & \mathrm{Hom}(|A|\otimes|B^*|,K) \end{array}$$

is a pullback. Thinking of these as taking values in the category of discrete abelian groups, this means that the square

$$\begin{array}{ccc} \mathrm{Hom}(A,B) & \longrightarrow & |A|\multimap|B| \\ \downarrow & & \downarrow \\ |B^*|\multimap|A^*| & \longrightarrow & |A|\otimes|B^*|\multimap K \end{array}$$

is a pullback, which means that $FA \multimap FB$ is the extensional reflection of the pair $(\mathrm{Hom}(A, B), |A| \otimes |B^*|)$, with the action given by $\langle f, a \otimes \chi \rangle = \chi(f(a))$, for $f: A \to B$, $a \in A$ and $\chi \in B^*$. Thus, $A \multimap B$ is Hom(A, B) with the strong, resp. weak, topology induced on it by $|A| \otimes |B^*|$. The same considerations show that $A \otimes B$ is the hausdorff quotient of $|A| \otimes |B^*|$ with the strong, resp. weak, topology induced on it by $\mathrm{Hom}(A, B^*) \cong \mathrm{Hom}(B, A^*)$.

In this case $(\mathrm{Hom}(A, B), |A| \otimes |B^*|)$ is not extensional. You get an example by taking $A = \mathbb{R}/\mathbb{Z}$ and $B = \mathbb{Z}$. Then $\mathrm{Hom}(A, B) = 0$, while $|A| \otimes |B^*| = |\mathbb{R}/\mathbb{Z}| \otimes |\mathbb{R}/\mathbb{Z}|$. As an abstract group, $\mathbb{R}/\mathbb{Z}$ is the direct sum of $\mathbb{Q}/\mathbb{Z}$ with a torsion free divisible group of rank $2^{\aleph_0}$ which means that $|\mathbb{R}/\mathbb{Z}| \cong |\mathbb{Q}/\mathbb{Z}| \oplus |\mathbb{R}|$ and then $|\mathbb{R}/\mathbb{Z}| \otimes |\mathbb{R}/\mathbb{Z}| \cong |\mathbb{R}| \otimes |\mathbb{R}| \cong \mathbb{R}$.

4.10. Locally compact groups are strongly topologized. We have not actually given an example of a weakly continuous map that is not continuous. Here is one. The weak topology on $\mathbb{Z}$ is not discrete since no compact group can have an infinite discrete subgroup. Actually, no compact space can have an infinite uniformly discrete subspace since there would have to be a uniform cover in which each element contains at most one element of the subspace, clearly impossible. Thus, the given by the inverse of the continuous map $\mathbb{Z} \to \mathbb{Z}^\flat$ is weakly continuous, but not continuous. The only topology that makes that map continuous is the discrete topology, which is clearly strong.

In fact, all locally compact topologies are strong. The key to that is the following proposition, which is proved by Glicksberg [6] (he credits the statement, but not his proof, to unpublished work of Kaplansky's).

4.11. Proposition. *A weakly continuous map between locally compact groups is continuous.*

4.12. Corollary. *Let L be locally compact. A weakly continuous map $f: L \to A$ in* SPLC *is continuous. Thus, L is strongly topologized.*

Proof. For any $A \to L'$ with L' locally compact, the composite $L \to A \to L'$ is weakly continuous, hence continuous. Since this is true for any such $A \to L'$, the first claim follows. Since $L^{\#}$ has the coarsest topology for which every weakly continuous map is continuous, the topology on $L^{\#}$ can be no finer than that of L. Since $A^{\#} \to A$ is continuous for all A, we see that $L^{\#} = L$. □

For the rest of this section, we let A^* denote the character group topologized strongly using A as a group of characters.

4.13. Proposition. *Let $f: A \to B$ be weakly continuous. Then the induced function $B^* \to A^*$ is continuous.*

Proof. The definition of weakly continuous implies that there is a function induced from $B^* \to A^*$. This is continuous if and only if for every continuous map $L \to A$ with L locally compact, the composite $B^* \to A^* \to L^*$ is continuous. But it is clear that for every such $L \to A$, the composite $L \to A \to B$ is weakly continuous, hence from the previous proposition is continuous, from which the conclusion is clear. □

4.14. Theorem. *Let A be an object of* SPLC. *Then $A^{**} = A^{\#}$.*

Proof. If $A \to L$ is weakly continuous with L locally compact, then $L^* \to A^*$ is continuous, so that $A^{**} \to L^{**} = L$ is continuous. Thus, the homomorphism $A^{**} \to A^{\#}$ is continuous. To go the other way, we observe that by definition $A^{\#}$ is the strongest topology that has the same set of functionals as A and since A^{**} is such a topology, it follows that the other direction is continuous. □

5. Application

In this section, we are going to give an example of how systematic use of the duality can clarify and simplify some aspects of the theory of coalgebras. Here we are dealing with the category of coalgebras over a field. The field might be topologized itself, but we will ignore its topology, if any, and assume it is discrete.

5.1. The category of coalgebras. If K is a field, a K-coalgebra C is a vector space equipped with a comultiplication $\delta : C \to C \otimes C$ and a unit $\varepsilon : C \to K$ that satisfy the equations that are dual to those of an associative algebra with unit. The map δ is called the comultiplication and ε the counit. In fact, a coalgebra is an associative algebra in the dual of the category of vector spaces, that is in the category of linearly compact vector spaces.

A word is needed about the tensor product in that category. If V_1 and V_2 are linearly compact vector spaces, there is the ordinary tensor product, which is not linearly compact. On the other hand, it does have a topology, the least in which $U_1 \otimes U_2$ is open whenever U_1 is open in V_1 and U_2 is open in V_2. This topology is not linearly compact, but it is a uniform topology (as any topology on a group must be) and has a completion which is linearly compact. One way of seeing this, as well as an alternate defintion of the tensor product is to use the formula $V_1 \otimes V_2 = (V_1^* \otimes V_2^*)^*$, where the tensor on the right-hand side is the ordinary tensor product of discrete vector spaces.

A linearly compact algebra A is a linearly compact vector space, equipped with continuous maps $\mu : A \otimes A \to A$ and $\eta : K \to A$ that satisfy the usual identities for a unitary associative algebra. In point of fact, we could have used the ordinary tensor product (with the topology described in the last paragraph) since any continuous homomorphism is uniformly continuous and if its codomain is linearly compact, it is complete and so the map extends to a continuous map on the completion. Similarly, a linearly compact A-module M is a linearly compact vector space that is an A-module in such a way that the action $A \otimes M \to M$ is continuous. Again, it does not matter which tensor product is used, since M is uniformly complete.

It is now evident that the category of coalgebras is dual to the category of linearly compact algebras.

5.2. Comodules and modules. If V is a vector space over K and C is a coalgebra, a right C-comodule structure on V is given by a map $V \to C \otimes V$ that satisfies the duals of the usual identities for module. If V is a linearly compact vector space and A a linearly compact algebra, then a left A-module structure on V is given by a continuous $A \otimes V \to V$ that satisfies the usual identities. It is clear that if A is the linearly compact algebra dual to C, then the category of linearly compact left A-modules is dual to the category of left C-comodules.

It is a standard fact that every coalgebra over a field is colimit of finite-dimensional subcoalgebras. The dual theorem, that every linearly compact algebra is a filtered limit of finite-dimensional quotients is also long-known, [4], but its proof seems much more perspicuous. We give a slightly different argument.

5.3. Proposition. *Let A be a linearly compact algebra and M a linearly compact A-module. Then M is an inverse limit of a filtered family of finite-dimensional quotient A-modules.*

Proof. Let $m \neq 0$ be an element of M. The fact that any linearly compact vector space is a product of copies of the field implies there is an open linear hyperplane H with $m \notin H$. The continuity of the multiplication $A \times M \to M$ implies there are open subspaces $B \subseteq A$ and $N \subseteq M$ such that $BN \subseteq H$. Since B is open, it has finite codimension, so there is a finite set $a_1, \ldots, a_n$ of elements of A that generate A/B. For $i = 1, \ldots, n$, there is an open subspace $N_i \subseteq M$ such that $a_i N_i \subseteq H$, using continuity of translation. Then $H \cap N \cap \bigcap_{i=1}^{n} N_i$ is an open subspace of H that has the property that $AN \subseteq H$. Thus, $N' = \{n \in M \mid An \subseteq H\}$ is open in M. The fact that $1 \in A$ implies that $N' \subseteq H$, whence $m \notin N'$. Finally, it is evident that N' is a submodule. The family of all such quotients is filtered, since the intersection of two open submodules is still open. □

5.4. Corollary. *A linearly compact algebra is a filtered inverse limit of finite-dimensional quotients.*

Proof. Just take $A \otimes A^{\mathrm{op}}$ as the algebra and A as the module in the above. A submodule is a two-sided ideal, which corresponds to a quotient. □

5.5. The category of finite-dimensional representations. Let A be a linearly compact algebra. Let $\mathrm{Mod}_{\mathrm{f}}(A)$ denote the category of finite-dimensional A-modules. Let $U : \mathrm{Mod}_{\mathrm{f}}(A) \to \mathrm{Vect}$, the category of K-vector spaces by the obvious underlying functor. The ring $\mathrm{End}(U)$ of endomorphisms of U has a topology as a subring R of the linearly compact ring $\prod U(M)$, the product taken over all the objects M of $\mathrm{Mod}_{\mathrm{f}}(A)$. Since the conditions defining a natural transformation refer to only two coordinates at a time, they define a closed, hence linearly compact, subspace. It is clear that any element of A determines a natural transformation by left translation. Thus, we have a natural map $\phi : A \to R$.

5.6. Theorem. *The map $\phi : A \to R$ is an isomorphism.*

Proof. First, we show it is an injection. Given any $a \in A$, there is a two-sided ideal I of finite codimension in A that does not contain a. But then A/I is a finite-dimensional module on which the action of a is non-trivial, so that translation by a corresponds to a non-trivial element of R. To show surjectivity, we will show the image is dense and then use compactness to infer that the image is all of R. For this, it is necessary to show that given any finite number of modules the restriction of any natural transformation to the full subcategory they generate can be realized as a translation. Now the annihilator of any given module is an open left ideal and thus the common annihilator of this finite set is also an open left ideal, which is included in some open two-sided ideal I. Then these modules are all in the full subcategory of finitely generated R/I-modules and the latter is in $\mathrm{Mod}_{\mathrm{f}}(R)$. Now let $\alpha : U \to U$ be any natural transformation. Let $\alpha(R/I)(1 + I) = a + I$. For any R/I-module M and any $m \in M$, there is an $f: R/I \to M$ such that $f(1 + I) = m$. From the commutativity of

$$\begin{array}{ccc} R/I & \xrightarrow{\alpha(R/I)} & R/I \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ M & \xrightarrow[\alpha M]{} & M \end{array}$$

applied to $1 + I$, we see immediately that $\alpha M(m) = am$. Thus, α is translation by a on this finite set of modules. This shows that $A \to R$ is dense and hence surjective. □

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