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FUNCTORS ON CATEGORIES OF FINITE GROUP REPRESENTATIONS

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0. Introduction

Let k be a field, and Λ a finite-dimensional linear associative algebra over k. We assume that Λ has an identity element, and that a Λ-*module* X is left, unital and of finite dimension as a k-space. The *representation theory of* Λ is the study of such Λ-modules, or more exactly of the category $\operatorname{mod}\Lambda$ whose objects are all Λ-modules $X, X', \dots$ and whose morphisms are the Λ-*maps* (i.e. Λ-homomorphisms) between Λ-modules. We write $X \in \operatorname{mod}\Lambda$ to indicate that X is an object of $\operatorname{mod}\Lambda$; if $X, X' \in \operatorname{mod}\Lambda$ we denote the set of all morphisms $f: X \to X'$ by $(X, X')_{\operatorname{mod}\Lambda}$, or simply (X, X'). The category $\operatorname{mod}\Lambda$ is k-linear, which means that each set (X, X') is a k-space, and that composition of morphisms in $\operatorname{mod}\Lambda$ is k-bilinear. We shall meet other k-linear categories, and if $\mathfrak{C}$ is one of these, we use the notations $X \in \mathfrak{C}$ and $(X, X')_{\mathfrak{C}} = (X, X')$ in the same sense as we have indicated for the case $\mathfrak{C} = \operatorname{mod}\Lambda$. We recall that a functor $F: \mathfrak{C} \to \mathfrak{D}$ between k-linear categories $\mathfrak{C}, \mathfrak{D}$ is k-*linear* if for all $X, X' \in \mathfrak{C}$ it acts k-linearly on $(X, X')_{\mathfrak{C}}$, so that if F is, for example, contravariant, F induces a map $(X, X')_{\mathfrak{C}} \to (F(X'), F(X))_{\mathfrak{D}}$ which is k-linear. All functors which we shall meet in this work are k-linear.

In the past decade M. Auslander and I. Reiten have greatly enriched the representation theory of Λ, by embedding $\operatorname{mod}\Lambda$ in a larger category $\operatorname{Mod}(\operatorname{mod}\Lambda)$. The objects of $\operatorname{Mod}(\operatorname{mod}\Lambda)$ are the k-linear, contravariant functors $F: \operatorname{mod}\Lambda \to \operatorname{Mod} k$, where $\operatorname{Mod} k$ is the category of all k-spaces (including infinite-dimensional ones). The morphisms of $\operatorname{Mod}(\operatorname{mod}\Lambda)$ are the natural transformations $\alpha: F \to F'$ between such functors F, F'.

Notation. From now on, in the interests of brevity, we shall denote the category $\operatorname{Mod}(\operatorname{mod}\Lambda)$ by $\operatorname{Mmod}\Lambda$. Similarly the category $\operatorname{mod}(\operatorname{mod}\Lambda)$ of finitely presented objects of $\operatorname{Mmod}\Lambda$ (see Section 1) is denoted $\operatorname{mmod}\Lambda$.

The object of this paper is to study the category $\operatorname{Mmod}\Lambda$ in the case where

$\Lambda = kG$ is the group algebra over k of a finite group G. A summary of its contents follows. Section 1 is a survey, intended to be elementary, of the basic ideas of Auslander–Reiten theory. In Section 2, the processes of induction, restriction and conjugation are generalized from the mod to the Mmod categories. For example, if H is a subgroup of G, we have a functor $\mathrm{Ind}_H^G : \mathrm{Mmod}\, kH \to \mathrm{Mmod}\, kG$, which is 'adjoint' or 'dual' to the usual restriction functor $\mathrm{res}_H^G : \mathrm{mod}\, kG \to \mathrm{mod}\, kH$; if $E \in \mathrm{Mmod}\, kH$, we define $\mathrm{Ind}_H^G(E) \in \mathrm{Mmod}\, kG$ in such a way that $\mathrm{Ind}_H^G(E)(X) = E(\mathrm{res}_H^G(X))$, for all $X \in \mathrm{mod}\, kG$. The standard identities involving ind, res, cnj for modules have their counterparts in the Mmod categories, and these are proved in Section 3. Section 4 introduces relative projectivity for functors; one defines *vertex* and *source* for any indecomposable finitely presented $F \in \mathrm{Mmod}\, kG$. The theory at this point requires the analogue of the 'relative trace map', and the discussion of this occupies Section 5. Functors already show some differences from modules at this stage; for example, a projective object F of $\mathrm{Mmod}\, kG$ is not necessarily $\{1\}$-projective. The rest of the paper, apart from some technical preliminaries in Section 6, is concerned with *simple* functors. Auslander has shown that there is a correspondence $V \to SV$ which induces a bijection between the isomorphism classes of all indecomposable $V \in \mathrm{mod}\, \Lambda$, and those of all simple $SV \in \mathrm{Mmod}\, \Lambda$ (see Section 1 for definitions and references). The main results of Section 7 show that if the indecomposable modules $W \in \mathrm{mod}\, kH$ and $V \in \mathrm{mod}\, kG$ 'correspond' suitably, then $\mathrm{Ind}_H^G(SW) \cong SV$; in particular this happens if W has vertex P, $H \geq N_G(P)$ and W is a component of $\mathrm{res}_H^G(V)$, and this gives a proof of Burry–Carlson's 'strong correspondence theorem' [7, Theorem 5]. It should be said at once that this proof is much longer, and at heart is not very different from, that of Burry and Carlson; the theorems in Section 7 are offered for the new perspective provided by the 'functorial' method. In particular, one gets very easily a theorem (7.8), certainly well-known, on the induction of almost split sequences. Section 8 shows how to calculate the vertex of the simple functor SV in two special cases, viz. when G is a p-group (Example (8.1)), and when G is arbitrary but $V \in \mathrm{mod}\, kG$ is projective indecomposable (Example (8.3)). Section 9 is a short appendix, in which is sketched a proof of a theorem of Auslander (Theorem (1.4)), for which I could find no convenient reference.

1. Auslander–Reiten theory

In this section, Λ is an arbitrary finite-dimensional k-algebra, and $\mathrm{mod}\, \Lambda$, $\mathrm{Mmod}\, \Lambda$ are the categories defined in the Introduction. We shall describe some of the basic ideas of Auslander–Reiten's theory, as it applies to these categories. A much more complete survey is given by Gabriel in [10].

For each module $M \in \mathrm{mod}\, \Lambda$, we define a functor $(\cdot, M) \in \mathrm{Mmod}\, \Lambda$ (sometimes called the functor 'represented' by M) as follows: $(\cdot, M)$ takes $X \in \mathrm{mod}\, \Lambda$ to the k-space (X, M), and it takes a morphism $f : X \to X'$ in $\mathrm{mod}\, \Lambda$ to the k-map

$(f, M): (X, M') \to (X, M)$ given by the rule $(f, M)(\theta') = \theta' f$, for any $\theta' \in (X', M)$. One verifies that $(\cdot, M): \mathrm{mod}\,\Lambda \to \mathrm{Mod}\,k$, so defined, is a k-linear contravariant functor.

The next lemma is fundamental; it describes completely all the morphisms from $(\cdot, M)$ into a given, arbitrary, functor $F \in \mathrm{Mmod}\,\Lambda$. Recall that to specify a morphism $\alpha: G \to F$ in $\mathrm{Mmod}\,\Lambda$, we must give for each $X \in \mathrm{mod}\,\Lambda$ a k-map $\alpha(X): G(X) \to F(X)$, in such a way that the resulting family $\{\alpha(X)\}$ is natural in X.

(1.1) **Yoneda's lemma** ([17]; see [14, p. 61]). *If $M \in \mathrm{mod}\,\Lambda$ and $F \in \mathrm{Mmod}\,\Lambda$, then each morphism $\alpha: (\cdot, M) \to F$ is completely determined by the element $T_\alpha = \alpha(M)(1_M) \in F(M)$. (Here 1_M denotes the identity map of M onto itself.) In fact for any $X \in \mathrm{mod}\,\Lambda$, the k-map $\alpha(X): (X, M) \to F(X)$ is given by*

(1.1a) $\qquad \alpha(X)(f) = F(f)(T_\alpha)$, *for all* $f \in (X, M)$.

Conversely given any element $T \in F(M)$ there is a unique morphism $\alpha: (\cdot, M) \to F$ such that $T_\alpha = T$. The correspondence $\alpha \to T_\alpha$ defines a k-isomorphism $((\cdot, M), F)_{\mathrm{Mmod}\,\Lambda} \to F(M)$.

We sketch the proof of (1.1). Let α, T_α be as given above. For each $X \in \mathrm{mod}\,\Lambda$ and each $f \in (X, M)$, the naturality of α requires the diagram below to commute.

$$\begin{array}{ccc} (X, M) & \xrightarrow{\alpha(X)} & F(X) \\ {\scriptstyle (f, M)} \uparrow & & \uparrow {\scriptstyle F(f)} \\ (M, M) & \xrightarrow{\alpha(M)} & F(M) \end{array}$$

Since $1_M \in (M, M)$, we get $\alpha(X)(f, M)(1_M) = F(f)\alpha(M)(1_M)$, and this is (1.1a). Conversely, if $T \in F(M)$ is given, we put T in place of T_α in (1.1a), and use this to define the k-map $\alpha(X): (X, M) \to F(M)$, for all $X \in \mathrm{mod}\,\Lambda$. It is easy to check that the family $\{\alpha(X)\}$ is natural in X. The rest of the proof of (1.1) is straightforward.

Suppose $a: M \to M'$ is a morphism in $\mathrm{mod}\,\Lambda$. Define for each $X \in \mathrm{mod}\,\Lambda$ the k-map $(X, a): (X, M) \to (X, M')$ by the rule, $(X, a)(\theta) = a\theta$, for all $\theta \in (X, M)$. We check that the family $\{(X, a)\}$ is natural in X, hence defines a morphism $(\cdot, a): (\cdot, M) \to (\cdot, M')$ in $\mathrm{Mmod}\,\Lambda$. But an application of Yoneda's lemma shows that *every* morphism $\alpha: (\cdot, M) \to (\cdot, M')$ has the form $\alpha = (\cdot, a)$, for a (unique) $a: M \to M'$. We may define a covariant, k-linear functor $Y: \mathrm{mod}\,\Lambda \to \mathrm{Mmod}\,\Lambda$ by putting $Y(M) = (\cdot, M)$, $Y(a) = (\cdot, a)$. We have just proved that, for given $M, M' \in \mathrm{mod}\,\Lambda$, the map

(1.2) $\qquad Y_{M,M'}: (M, M')_{\mathrm{mod}\,\Lambda} \to ((\cdot, M), (\cdot, M'))_{\mathrm{Mmod}\,\Lambda}$

induced by Y is surjective; it is easy to see that $Y_{M,M'}$ is in fact a k-isomorphism (we shall call this the *Yoneda isomorphism*), so that Y is a full embedding (see [14, pp. 14, 15]) of $\operatorname{mod} \Lambda$ in $\operatorname{Mmod} \Lambda$; in this sense, $\operatorname{Mmod} \Lambda$ is an 'extension' of $\operatorname{mod} \Lambda$.

Another application of (1.1) gives an important

(1.3) **Proposition.** *For any* $M \in \operatorname{mod} \Lambda$, $(\cdot, M)$ *is a projective objective of* $\operatorname{Mmod} \Lambda$.

$\operatorname{Mmod} \Lambda$ is, like $\operatorname{mod} \Lambda$, a k-linear, abelian category, and the objects $F \in \operatorname{Mmod} \Lambda$ can be treated very much as if they were modules – see Auslander [1, §2]. For the reader's convenience we repeat here some definitions from [1, §2], sometimes with slight changes of terminology. If $F, G \in \operatorname{Mmod} \Lambda$, we say that F is a *subfunctor* of G (notation $F \leq G$) if (i) for each $X \in \operatorname{mod} \Lambda$, $F(X)$ is a k-subspace of $G(X)$, and (ii) for each $f: X \to X'$ in $\operatorname{mod} \Lambda$, the k-map $F(f): F(X') \to F(X)$ is just the restriction of the k-map $G(f): G(X') \to G(X)$. Notice that (ii) is possible if and only if the spaces $F(X)$ satisfy the condition (ii*) for each $f: X \to X'$, $G(f)$ maps $F(X')$ into $F(X)$.

If $F \leq G$ we may define the *quotient functor* $G/F \in \operatorname{Mmod} \Lambda$ as follows: for each $X \in \operatorname{mod} \Lambda$, $(G/F)(X) := G(X)/F(X)$, while for each morphism $f: X \to X'$ in $\operatorname{mod} \Lambda$, $(G/F)(f)$ is the k-map $G(X')/F(X') \to G(X)/F(X)$ which is induced by $G(f)$ (and notice that by (ii*), this makes sense). Each morphism $\alpha: F \to F'$ in $\operatorname{Mmod} \Lambda$ has *kernel* $\operatorname{Ker} \alpha$ and *image* $\operatorname{Im} \alpha$, which are the subfunctors of F and F' respectively given by $(\operatorname{Ker} \alpha)(X) := \operatorname{Ker} \alpha(X)$, $(\operatorname{Im} \alpha)(X) := \operatorname{Im} \alpha(X)$, for each $X \in \operatorname{mod} \Lambda$. There is an isomorphism $\bar{\alpha}: F/\operatorname{Ker} \alpha \to \operatorname{Im} \alpha$ in $\operatorname{Mmod} \Lambda$ such that for each $X \in \operatorname{mod} \Lambda$, $\bar{\alpha}(X)$ is the natural isomorphism $F(X)/\operatorname{Ker} \alpha(X) \to \operatorname{Im} \alpha(X)$. A sequence $F \xrightarrow{\alpha} F' \xrightarrow{\beta} F''$ in $\operatorname{Mmod} \Lambda$ is *exact* if $\operatorname{Ker} \beta = \operatorname{Im} \alpha$, which is the same as saying that the sequence $F(X) \xrightarrow{\alpha(X)} F'(X) \xrightarrow{\beta(X)} F''(X)$ in $\operatorname{mod} \Lambda$ is exact, for each $X \in \operatorname{mod} \Lambda$. *Short exact sequences* in $\operatorname{Mmod} \Lambda$ are defined in the expected way.

If $\{F_i : i \in I\}$ is a family of objects of $\operatorname{Mmod} \Lambda$, indexed by a set I, then the *direct sum* $F = \coprod_{i \in I} F_i$ is the object of $\operatorname{Mmod} \Lambda$ such that $F(X) := \coprod_{i \in I} F_i(X)$ for each $X \in \operatorname{mod} \Lambda$, while for each $f: X \to X'$ in $\operatorname{mod} \Lambda$, $F(f)$ is the k-map $\coprod F_i(f)$. One may define similarly the *direct product* $\prod_{i \in I} F_i$ [1, pp. 184, 185].

Finally, an object $F \in \operatorname{Mmod} \Lambda$ is *indecomposable* if $F \neq 0$ and there is no isomorphism $F \cong F_1 \amalg F_2$ with non-zero objects $F_1, F_2 \in \operatorname{Mmod} \Lambda$; equivalently, $F \neq 0$ is indecomposable if and only if the endomorphism algebra $\operatorname{End} F$ has no idempotent $\neq 1_F$ or 0.

We come now to the first non-trivial application of these ideas. Pursuing the analogy with modules, Auslander defines the *radical* $\mathbf{r}F$ of an object $F \in \operatorname{Mmod} \Lambda$ to be the intersection of all the maximal subfunctors of F [2, p. 319]. In case $F = (\cdot, M)$ for some $M \in \operatorname{mod} \Lambda$, the radical has a useful explicit description, as follows.

(1.4) **Theorem** (Auslander). *Let* $M \in \operatorname{mod} \Lambda$. *Then* $\mathbf{r}(\cdot, M)$ *is the subfunctor of* $(\cdot, M)$ *given by*

(1.4a) $\mathbf{r}(X, M) = \{f \in (X, M) \mid fg \in \operatorname{rad} \operatorname{End} M \text{ for all } g \in (M, X)\}$

for all $X \in \operatorname{mod} \Lambda$. (Here we have written $\mathbf{r}(X, M)$ for the value $\mathbf{r}(\cdot, M)(X)$ of $\mathbf{r}(\cdot, M)$ at X.)

This is implicit in [3, Corollary 1.6b, Proposition 2.1]. For another description of $\mathbf{r}(\cdot, M)$, easily seen to be equivalent to that just given, see Gabriel [10, p. 2]. A proof of (1.4) is sketched in the Appendix to this paper.

Now assume that $V \in \operatorname{mod} \Lambda$ is indecomposable (in future we shall often denote this by $V \in \operatorname{indec} \Lambda$). This implies that $(\cdot, V)$ is indecomposable, since $\operatorname{End} V \cong \operatorname{End}(\cdot, V)$ by (1.2). Assume also that $X \in \operatorname{indec} \Lambda$. Then it follows easily from Fitting's lemma and (1.4a) that $\mathbf{r}(X, V) = (X, V)$ if $X \not\cong V$. On the other hand (1.4a) shows that $\mathbf{r}(V, V) = \operatorname{rad} \operatorname{End} V$. We define next functors SV, which are (in case $\Lambda = kG$) the main concern of this paper.

Definitions. Let $V \in \operatorname{indec} \Lambda$. Define $SV \in \operatorname{Mmod} \Lambda$ to be the quotient functor $(\cdot, V)/\mathbf{r}(\cdot, V)$. Let $\Delta(V) := (\operatorname{End} V)/\operatorname{rad} \operatorname{End} V$, which is a division k-algebra, since $\operatorname{End} V$ is a local ring. Let $\delta(V) := \dim \Delta(V)$ (here and elsewhere, $\dim = \dim_k$).

Before going further, two remarks about an arbitrary $F \in \operatorname{Mmod} \Lambda$ are in order. The first is that F, like any other k-linear functor, commutes with finite direct sums. The second is that if $M \in \operatorname{mod} \Lambda$, then $F(M)$ can be regarded as a right $\operatorname{End} M$ module as follows: given $\xi \in F(M)$ and $h \in \operatorname{End} M$, one defines $\xi h := F(h)(\xi)$ [1, p. 191].

(1.5) **Theorem** (Auslander). *Let* $V \in \operatorname{indec} \Lambda$. *Then*

(i) *SV is a simple object of* $\operatorname{Mmod} \Lambda$. *Any simple object of* $\operatorname{Mmod} \Lambda$ *is isomorphic to SV for some* $V \in \operatorname{indec} \Lambda$.

(ii) *Let* $X \in \operatorname{mod} \Lambda$, *and let* $X \cong \coprod_{i \in I} X_i$, *where I is a finite index set and the* $X_i \in \operatorname{indec} \Lambda$ *(such a 'decomposition' of X is always possible). Then*

(1.5a) $\dim(SV)(X) = [V \mid X]\delta(V)$,

where $[V \mid X] = |\{i \in I \mid X_i \cong V\}|$ *is the 'multiplicity' of V in the given decomposition of X.*

(iii) $\operatorname{End}(SV) \cong \Delta(V)$, *isomorphism of k-algebras.*

Proof. (i) We must prove that any non-zero subfunctor F of SV, is equal to SV. For this it is enough to prove $F(X) = (SV)(X)$, for all $X \in \operatorname{indec} \Lambda$. If $X \not\cong V$, this holds because $(SV)(X) = (X, V)/\mathbf{r}(X, V) = 0$ (see above). If $X \cong V$ we may assume that $X = V$ and that $F(V) \neq 0$. But $F(V)$ is a right $\operatorname{End} V$ submodule of the right $\operatorname{End} V$ module $(SV) = (V, V)/\operatorname{rad} \operatorname{End} V = \Delta(V)$, and this latter is clearly a simple module. Hence $F(V) = (SV)(V)$, as required. For the second statement in (i), see [3, p. 281].

(ii) From what has been said above, it is clear that $(SV)(X) \cong [V|X]\Delta(V)$, isomorphism of k-spaces. The result follows.

(iii) Each $\theta \in \mathrm{End}(\cdot, V)$ maps $\mathbf{r}(\cdot, V)$ into itself (this means that, for all $X \in \mathrm{mod}\, \Lambda$, $\theta(X)$ maps $\mathbf{r}(X, V)$ into itself), hence θ induces an endomorphism $\bar{\theta}$ of SV. Conversely any $\phi \in \mathrm{End}\, SV$ can be lifted to some $\theta \in \mathrm{End}(\cdot, V)$, because $(\cdot, V)$ is projective. So $\theta \to \bar{\theta}$ determines a k-algebra epimorphism $\mathrm{End}(\cdot, V) \to \mathrm{End}\, SV$. The result now follows, because $\mathrm{End}(\cdot, V) \cong \mathrm{End}\, V$ by (1.2), and $\mathrm{End}\, SV$ is a division algebra by Schur's lemma.

Auslander defines an object $F \in \mathrm{Mmod}\, \Lambda$ to be *finitely generated* if there exists an exact sequence $(\cdot, V) \to F \to 0$, for some $V \in \mathrm{mod}\, \Lambda$, and to be *finitely presented* if there is an exact sequence $(\cdot, V_1) \to (\cdot, V) \to F \to 0$, for some $V_1, V \in \mathrm{mod}\, \Lambda$ [1, pp. 186, 204]. In the latter case the morphism $(\cdot, V_1) \to (\cdot, V)$ can be written $(\cdot, g)$ for some $g \in (V_1, V)$, by the Yoneda isomorphism (1.2). It is now easy to check that the sequence

$$0 \to (\cdot, V_2) \xrightarrow{(\cdot, f)} (\cdot, V_1) \xrightarrow{(\cdot, g)} (\cdot, V) \to F \to 0 \tag{1.6}$$

is exact, where $V_2 = \mathrm{Ker}\, g$ and $f: V_2 \to V_1$ is the inclusion.

Definition. The category $\mathrm{mmod}\, \Lambda$ is defined to be the full subcategory of $\mathrm{Mmod}\, \Lambda$, whose objects are the finitely presented objects of $\mathrm{Mmod}\, \Lambda$. (This category is denoted $\mathrm{mod}(\mathrm{mod}\, \Lambda)$ in [2].)

Clearly $(\cdot, M)$ is finitely presented, for all $M \in \mathrm{mod}\, \Lambda$, so that (1.6) is a projective resolution of F in $\mathrm{mmod}\, \Lambda$. This shows that $\mathrm{mmod}\, \Lambda$ has global homological dimension ≤ 2 [2, p. 327].

The fundamental theorem of Auslander–Reiten theory is that *every simple functor in* $\mathrm{Mmod}\, \Lambda$ *is finitely presented*, or equivalently, that $SV \in \mathrm{mmod}\, \Lambda$, for all $V \in \mathrm{indec}\, \Lambda$ [2, p. 319]. This is easy to prove if the indecomposable module V is projective, for in that case the sequence

$$0 \to (\cdot, \mathbf{r}V) \xrightarrow{(\cdot, g)} (\cdot, V) \to SV \to 0$$

is exact, where $\mathbf{r}V$ is the radical of V and g is the inclusion $\mathbf{r}V \to V$ ([5, Proposition 3.1a], [10, p. 4]). If V is non-projective, then the proof that SV is finitely presented is far from trivial, and is equivalent to the proof that there exists an *almost split sequence* $0 \to V_2 \xrightarrow{f} V_1 \xrightarrow{g} V \to 0$ in $\mathrm{mod}\, \Lambda$ (see [4, §4], [5, p. 443]). In fact (1.6) is a *minimal* projective resolution for $F = SV$ (for the theory of minimal projective resolutions in $\mathrm{mmod}\, \Lambda$, see [1, p. 212], [2, p. 320]) if and only if $0 \to V_2 \xrightarrow{f} V_1 \xrightarrow{g} V \to 0$ is almost split – it being understood that the morphism $(\cdot, V) \to SV$ in (1.6) is the natural epimorphism (see [10, Lemma 1.4, p. 6]).

Let $\mathrm{M'mod}\, \Lambda$ denote the category of all *covariant*, k-linear functors $E: \mathrm{mod}\, \Lambda \to \mathrm{Mod}\, k$. For any object E of $\mathrm{M'mod}\, \Lambda$, the functor DE is an object of

Mmod Λ; here $D: \text{Mod } k \to \text{Mod } k$ is the standard 'dual' functor which takes $S \in \text{Mod } k$ to $DS = \text{Hom}_k(S, k)$. In the same way, $DF \in \text{M'mod } \Lambda$, for any $F \in \text{Mmod } \Lambda$. Since D induces a genuine duality $S \cong DDS$ on the category mod k of finite-dimensional k-spaces, we have $E \cong DDE$, $F \cong DDF$ for functors E, F such that $E(X), F(X) \in \text{mod } k$, for all $X \in \text{mod } \Lambda$. One should notice that, for any $M \in \text{mod } \Lambda$, the functors $E = (M, \cdot)$, $F = (\cdot, M)$ have the property just mentioned ($(M, \cdot) \in \text{M'mod } \Lambda$ is the analogue of $(\cdot, M) \in \text{Mmod } \Lambda$); hence so do any finitely-generated E, F. An important theorem of Auslander and Reiten [2 p. 317] says that $E \in \text{M'mod } \Lambda$ is finitely presented, if and only if $DE \in \text{Mmod } \Lambda$ is finitely presented. The fact that $D(M, \cdot)$ is a finitely presented object of Mmod Λ is at the heart of Auslander and Reiten's proof that simple functors in Mmod Λ are finitely presented (see [10, p. 5]). We shall make use of the following characterization of finitely presented objects of Mmod Λ, proved in [2, pp. 318, 319].

(1.7) **Theorem** (Auslander–Reiten). *Let $F \in$ Mmod Λ. Then F is finitely presented if and only if there exist $V, X \in$ mod Λ and a morphism $\alpha : (\cdot, V) \to D(X, \cdot)$ in* Mmod Λ, *such that $F \cong$* Im α.

Yoneda's lemma (1.1) tells us that a morphism $\alpha : (\cdot, V) \to D(X, \cdot)$ is completely determined by the element $T_\alpha = \alpha(V)(1_V)$ of $D(X, V)$. So we have the rather surprising fact that any object $F \in \text{mmod } \Lambda$ is completely specified by a pair of modules $V, X \in \text{mod } \Lambda$, and a single element T_α of $D(X, V)$. This specification of objects of mmod Λ is used in Sections 6, 7 of this work. We end the present Section 1 by giving the specification of this kind for a simple functor SV ($V \in \text{indec } \Lambda$). This is used to calculate almost split sequences [4, §4].

(1.8) **Theorem** (Auslander–Reiten; see [10, p. 4]). *If $V \in$ indec λ, then a morphism $\alpha : (\cdot, V) \to D(V, \cdot)$ has the property that* Im $\alpha \cong SV$, *if and only if the element $T_\alpha = \alpha(V)(1_V)$ of $D(V, V)$ satisfies*

(1.8a) $\quad T_\alpha \neq 0, \quad T_\alpha(\text{rad End } V) = 0.$

Proof. It is clear that $\text{Im } \alpha \cong SV$ if and only if $\text{Ker } \alpha = \mathbf{r}(\cdot, V)$, i.e. if and only if for all $X \in \text{mod } \Lambda$ there holds

(*) $\quad f \in \mathbf{r}(X, V) \Leftrightarrow \alpha(X)(f) = 0, \quad \text{for all } f \in (X, V).$

By (1.1a), $\alpha(X)(f)$ is the element of $D(V, X)$ given by $\alpha(X)(f)(g) = T_\alpha(fg)$, for all $g \in (V, X)$. By (1.4a), $f \in \mathbf{r}(X, V)$ if and only if $fg \in \text{rad End } V$ for all $g \in (V, X)$. So condition (*) is equivalent to

(**) $\quad f(V, X) \leq \text{rad End } V \Leftrightarrow T_\alpha(f(V, X)) = 0, \quad \text{for all } f \in (X, V).$

It is easy to check that $f(V, X) = \{fg \mid g \in (V, X)\}$ is a right ideal of $(V, V) = \text{End } V$. Since rad End V is the unique maximal right ideal of End V, condition (**) is equivalent to (1.8a). This completes the proof of (1.8).

Remarks on notation. From now on $p = \mathrm{char}\, k$ is assumed finite.

The signs $\amalg$, $\coprod$ refer to (external) direct sums.

$V \in \mathrm{indec}\, \Lambda$ means that $V \in \mathrm{mod}\, \Lambda$ and V is indecomposable.

Suppose now that V, X are objects of $\mathfrak{C}$, where $\mathfrak{C} = \mathrm{mod}\, \Lambda$ or $\mathrm{Mmod}\, \Lambda$. Then 1_V is the identity morphism on V. $\mathbf{r}V$ is the radical of V. $V|X$ means that $X \cong V \amalg V'$, for some $V' \in \mathfrak{C}$; in this case we say V is a *summand* of X. If also V is indecomposable, we sometimes say V is a *component* of X. In that case $[V|X]$ denotes the multiplicity of V as summand of X (see (1.5). It is interesting that (1.5) gives a new proof of the Krull–Schmidt theorem for $\mathrm{mod}\, \Lambda$, since by (1.5a), $[V|X]$ is independent of the decomposition $X \cong \coprod X_i$. See Gabriel [10, pp. 3,4].) Finally we recall that for any $V \in \mathrm{indec}\, \Lambda$, we have defined

$$SV := (\cdot, V)/\mathbf{r}(\cdot, V), \qquad \Delta(V) := (\mathrm{End}\, V)/\mathrm{rad}\, \mathrm{End}\, V$$

and

$$\delta(V) := \dim \Delta(V).$$

2. Induction, restriction and conjugation

Denote by kG the group-algebra of a finite group G over our base-field k. Let H be a subgroup of G, and g an element of G. The familiar representation-theoretic operations of induction, restriction and conjugation can be expressed by the following functors

(2.1) $\qquad \mathrm{ind}_H^G : \mathrm{mod}\, kH \to \mathrm{mod}\, kG,$

(2.2) $\qquad \mathrm{res}_H^G : \mathrm{mod}\, kG \to \mathrm{mod}\, kH,$

(2.3) $\qquad \mathrm{cnj}_{H,g} : \mathrm{mod}\, kH \to \mathrm{mod}\, k({}^gH) \qquad ({}^gH = gHg^{-1}).$

Thus ind_H^G takes each $W \in \mathrm{mod}\, kH$ to the induced kG-module $kG \otimes_{kH} W$ (which we denote W^G), and it takes each kH-map $\alpha : W \to W'$ to the kG-map $\alpha^G = 1_{kG} \otimes \alpha : W^G \to W'^G$. The functor res_H^G takes each $V \in \mathrm{mod}\, kG$ to the restricted kH-module V_H (i.e., V regarded as kH-module), and it takes each kG-map $\beta : V \to V'$ to $\beta_H = \beta : V_H \to V'_H$. Finally $\mathrm{cnj}_{H,g}$ takes each $W \in \mathrm{mod}\, kH$ to the $k({}^gH)$-module ${}^gW$, where ${}^gW$ is the k-space W, given the following ${}^gH$-action (which we denote $*$ to avoid confusion) $(ghg^{-1}) * w = hw$, for $h \in H$, $w \in W$; $\mathrm{cnj}_{H,g}$ takes each kH-map $\alpha : W \to W'$ to ${}^g\alpha = \alpha$, which is clearly also a $k({}^gH)$-map from ${}^gW \to {}^gW'$. It is easy to check that these three functors are all k-linear, covariant and exact.

Suppose for the moment that Ω, Λ are any k-algebras and that

$$u : \mathrm{mod}\, \Omega \to \mathrm{mod}\, \Lambda$$

is a k-linear, covariant functor. We shall define a functor

$$\mathbf{u} : \mathrm{Mmod}\, \Lambda \to \mathrm{Mmod}\, \Omega$$

as follows. If $F \in \mathrm{Mmod}\, \Lambda$, then clearly $F \cdot u : \mathrm{mod}\, \Omega \to \mathrm{mod}\, k$ is k-linear and contravariant, so is an object of Mod(mod Ω), which we denote $\mathbf{u}(F)$. Thus $\mathbf{u}(F)\in$ Mmod Ω is given by

$$\text{(2.4)}\qquad \mathbf{u}(F)(W):=F(u(W)),\qquad \mathbf{u}(F)(f):=F(u(f)),$$

for all objects W and all morphisms $f:W\to W'$ in mod Ω.

Next, for any morphism $\phi:F\to F'$ in Mmod Λ, we define a morphism $\mathbf{u}(\phi):\mathbf{u}(F)\to \mathbf{u}(F')$ by setting

$$\text{(2.5)}\qquad \mathbf{u}(\phi)(W):=\phi(u(W)):F(u(W))\to F'(u(W)),$$

for all $W\in \text{mod}\,\Omega$. It is routine to verify that (2.5) is natural in W, and that (2.4), (2.5) define a functor $\mathbf{u}:\text{Mmod}\,\Lambda\to\text{Mmod}\,\Omega$ which is k-linear and covariant. Finally $\mathbf{u}$ is exact (even if u is not exact!).

Definition. If H is a subgroup of G, and g is an element of G, we define the following functors

$$\text{(2.6)}\qquad \text{Ind}_H^G=\mathbf{res}_H^G:\text{Mmod}\,kH\to\text{Mmod}\,kG,$$

$$\text{(2.7)}\qquad \text{Res}_H^G=\mathbf{ind}_H^G:\text{Mmod}\,kG\to\text{Mmod}\,kH,$$

$$\text{(2.8)}\qquad \text{Cnj}_{H,g}=\mathbf{cnj}_{{}^gH,g^{-1}}:\text{Mmod}\,kH\to\text{Mmod}\,k({}^gH).$$

Here $\mathbf{res}_H^G$, for example, is the functor $\mathbf{u}:\text{Mmod}\,kH\to\text{Mmod}\,kG$ obtained from the functor $u=\text{res}_H^G:\text{mod}\,kG\to\text{mod}\,kH$. Notice that all the functors (2.6), (2.7), (2.8) are k-linear, covariant and exact.

Notation. We shall use notations similar to those employed in representation theory for modules: for example if B is an object, and $\beta:B\to B'$ a morphism in Mmod kH, we write B^G,β^G for $\text{Ind}_H^G(B),\text{Ind}_H^G(\beta)$, respectively. Thus B^G is the object, and β^G is the morphism in Mmod kG, defined according to the general prescriptions (2.4) and (2.5) by

$$\text{(2.9)}\qquad B^G(X):=B(X_H),\qquad B^G(f):=B(f_H)\quad\text{and}\quad \beta^G(X):=\beta(X_H),$$

for all objects X and morphisms $f:X\to X'$ in mod kG.

Similarly if A is an object, and $\alpha:A\to A'$ is a morphism in Mmod kG, we write A_H,α_H for $\text{Res}_H^G(A),\text{Res}_H^G(\alpha)$ respectively. Thus A_H and α_H are the object and morphism in Mmod kH defined by

$$\text{(2.10)}\qquad A_H(Y):=A(Y^G),\qquad A_H(h):=A(h^G)\quad\text{and}\quad \alpha_H(Y):=\alpha(Y^G),$$

for all objects Y and morphisms $h:Y\to Y'$ in mod kH.

Finally if B,β are in Mmod kH, we write ${}^gB,{}^g\beta$ for $\text{Cnj}_{H,g}(B),\text{Cnj}_{H,g}(\beta)$ respectively; these are in Mmod $k({}^gH)$ and are defined by

$$\text{(2.11)}\qquad {}^gB(Z):=B({}^{g^{-1}}Z),\qquad {}^gB(j):=B({}^{g^{-1}}j)\quad\text{and}\quad {}^g\beta(Z):=\beta({}^{g^{-1}}Z),$$

for all objects Z and morphisms $j:Z\to Z'$ in mod $k({}^gH)$.

The next proposition shows that our definitions and notations are compatible with the Yoneda embeddings Y (see Section 1).

(2.12) **Proposition.** *Let $W \in \operatorname{mod} kH$, $V \in \operatorname{mod} kG$. Then there are isomorphisms $(\cdot, W)^G \cong (\cdot, W^G)$, $(\cdot, V)_H \cong (\cdot, V_H)$, ${}^g(\cdot, W) \cong (\cdot, {}^gW)$ in the categories* Mmod kG, Mmod kH, Mmod $k({}^gH)$, *respectively.*

Proof. By (2.9), $(\cdot, W)^G(X) = (\cdot, W)(X_H) = (X_H, W)$, for any X in mod kG. But the Frobenius reciprocity theorem (see for example [9, p. 232]) gives an isomorphism of k-spaces $(X_H, W) \cong (X, W^G)$ which is natural in X, hence determines an isomorphism $(\cdot, W)^G \cong (\cdot, W^G)$ in Mmod kG. The other two isomorphisms in (2.12) are similarly derived.

It follows from (2.12) that each of the functors $\operatorname{Ind}_H^G, \operatorname{Res}_H^G, \operatorname{Cnj}_{H,g}$ takes finitely presented objects to finitely presented objects (and the same is true, with 'finitely generated' replacing 'finitely presented'). For example if $A \in \operatorname{Mmod} kG$ is finitely presented, there exist modules $U, U' \in \operatorname{mod} kG$ and an exact sequence $(\cdot, U) \to (\cdot, U') \to A \to 0$ in Mmod kG. If we apply the exact functor Res_H^G to this, we get the exact sequence $(\cdot, U)_H \to (\cdot, U')_H \to A_H \to 0$ in Mmod kH. And since $(\cdot, U)_H \cong (\cdot, U_H)$, $(\cdot, U')_H \cong (\cdot, U'_H)$, we may construct an exact sequence $(\cdot, U_H) \to (\cdot, U'_H) \to A_H \to 0$, which shows that $A_H = \operatorname{Res}_H^G(A)$ is finitely presented.

Restriction and induction of simple functors. Let $V \in \operatorname{indec} kG$, and let $SV = (\cdot, V)/\mathbf{r}(\cdot, V)$ be the corresponding simple functor. It can happen that $(SV)_H = 0$, for a subgroup H of G. In fact $(SV)_H$ is *non*-zero if and only if there exists $Y \in \operatorname{mod} kH$ such that $(SV)_H(Y) \neq 0$, which by (2.10) is to say $(SV)(Y^G) \neq 0$. So by (1.5) we have: $(SV)_H \neq 0$ if and only if there is some $Y \in \operatorname{mod} kH$ such that $V \mid Y^G$, i.e. if and only if V is *H-projective* (for the theory of relatively projective kG-modules, see [9, §19] or [13, II, §2]). This proves the first part of the

(2.13) **Proposition.** *Let $V \in \operatorname{indec} kG$, and H be a subgroup of G. Then* (i) *$(SV)_H = 0$ if and only if V is not H-projective, and* (ii) *If $\mathscr{E}: 0 \to V_2 \xrightarrow{f} V_1 \xrightarrow{g} V \to 0$ is an almost split sequence in* mod kG, *then the restricted sequence $\mathscr{E}_H: 0 \to V_{2H} \xrightarrow{f_H} V_{1H} \xrightarrow{g_H} V_H \to 0$ is split if and only if V is not H-projective.*

(2.13, ii) is due to Gabriel and Riedtmann (see [16, Lemma 3.1]).

To prove it, apply Res_H^G to the exact sequence (1.6) (with $F = SV$), and use (2.12). We get an exact sequence

$$0 \to (\cdot, V_{2H}) \xrightarrow{(\cdot, f_H)} (\cdot, V_{1H}) \xrightarrow{(\cdot, g_H)} (\cdot, V_H) \to (SV)_H \to 0$$

in mmod kH. Now (2.13, ii) follows from (2.13, i) and the fact that $\mathscr{E}_H$ is split if and only if $(\cdot, g_H): (\cdot, V_{1H}) \to (\cdot, V_H)$ is an epimorphism.

Let $W \in \operatorname{indec} kH$, so that $SW = (\cdot, W)/\mathbf{r}(\cdot, W)$ is a simple object of mmod kH. In contrast to (2.13), we find that $(SW)^G$ is never zero. For there is always some $X \in \operatorname{mod} kG$ such that $W \mid X_H$ (for example, $X = W^G$), hence $(SW)^G(X) = (SW)(X_H)$ is not zero.

In the following proposition, we denote $F/\mathbf{r}F$ by $\operatorname{Hd} F$, for any $F \in \operatorname{mmod} \Lambda$ (Λ being, for the moment, any finite-dimensional k-algebra). Thus $\operatorname{Hd}(\cdot, V) = SV$, for $V \in \operatorname{indec} \Lambda$. In general, $\operatorname{Hd} F$ is *semisimple*, i.e. is isomorphic to a finite direct sum of simple functors [2, p. 321]. In particular, if $X \in \operatorname{mod} \Lambda$ and if $X \cong \coprod_{i \in I} X_i$ (I finite, $X_i \in \operatorname{indec} \Lambda$) we have

$$\operatorname{Hd}(\cdot, X) \cong \coprod_i \operatorname{Hd}(\cdot, X_i) = \coprod_i SX_i \qquad [2, \text{ p. } 321].$$

(2.14) **Proposition.** (i) *If* $V \in \operatorname{indec} kG$, *then* $\operatorname{Hd}(SV)_H \mid \operatorname{Hd}(\cdot, V_H)$, *hence*

$$[SW \mid \operatorname{hd}(SV)_H] \leq [W \mid V_H],$$

for all $W \in \operatorname{indec} kH$.

(ii) *If* $W \in \operatorname{indec} kH$, *then* $\operatorname{Hd}(SW)^G \mid \operatorname{Hd}(\cdot, W^G)$, *hence* $[SV \mid \operatorname{Hd}(SW)^G] \leq [V \mid W^G]$, *for all* $V \in \operatorname{indec} kG$.

(iii) *If* $W \in \operatorname{indec} kH$ *and* $g \in G$, *then* ${}^g(SW) \cong S({}^gW)$.

Proof. (i) Apply Res_H^G to the natural epimorphism $(\cdot, V) \to SV$, and use (2.12). We get an epimorphism $(\cdot, V_H) \to (SV)_H$, hence an epimorphism $\operatorname{Hd}(\cdot, V_H) \to \operatorname{Hd}((SV)_H)$. Now (i) follows by the remarks above. The proof of (ii) is similar, and is left to the reader. To prove (iii), apply $\operatorname{Cnj}_{H,g}$ and (2.12) to $(\cdot, W) \to SW$. We get an epimorphism $(\cdot, {}^gW) \to {}^g(SW)$. But it is clear that ${}^g(SW)$ is simple and ${}^gW$ is indecomposable, hence ${}^g(SW) \cong (\cdot, {}^gW)/\mathbf{r}(\cdot, {}^gW) = S({}^gW)$, as required.

3. Identities involving Ind, Res, Cnj

There are several standard formulae or 'identities' in group representation theory which involve the functors ind, res, cnj mentioned at the beginning of the last section. We shall see that these give rise, by an automatic general procedure, to formulae involving Ind, Res, Cnj. For this purpose, the original formula must be presented as a natural transformation between suitable combinations of ind, res and cnj functors. For example, the 'transitivity of induction' formula gives, when D, H are subgroups of G such that $D \leq H$, a kG-isomorphism

$$(3.1) \qquad J(Z) : (Z^H)^G \to Z^G$$

for each $Z \in \operatorname{mod} kD$; moreover this is natural in Z, so that we have a natural isomorphism $J : u_1 \to u_2$ between two functors $u_1, u_2 : \operatorname{mod} kD \to \operatorname{mod} kG$, namely $u_1 = \operatorname{ind}_H^G \cdot \operatorname{ind}_D^H$ and $u_2 = \operatorname{ind}_D^G$. A more sophisticated example is provided by Mackey's 'subgroup formula' (see [9, p. 237]). Here we have subgroups H, K of G, and the formula gives an isomorphism of kK-modules

(3.2) $$J(X):(X^G)_K \to \coprod_d (({}^dX)_{{}^dH\cap K})^K,$$

for each $X \in \text{mod}\,kH$. Here d runs over a set of representatives of the double cosets KgH, $g \in G$. We have again a natural isomorphism $J:u_1 \to u_2$, the functors $u_1, u_2 : \text{mod}\,kH \to \text{mod}\,k$ being

$$u_1 = \text{res}^G_K \cdot \text{ind}^G_H, \qquad u_2 = \coprod_d \text{ind}^K_{{}^dH\cap K} \cdot \text{res}^{{}^dH}_{{}^dH\cap K} \cdot \text{cnj}_{H,d}.$$

The direct sum in u_2 is to be understood in the following sense. Let $u, u' : \text{mod}\,\Omega \to \text{mod}\,\Lambda$ be covariant functors (Ω, Λ any k-algebras). We define their direct sum $v = u \amalg u' : \text{mod}\,\Omega \to \text{mod}\,\Lambda$ by setting $v(W) := u(W) \amalg u'(W)$ and $v(f) := u(f) \amalg u'(f)$, for all objects W and all morphisms $f: W \to W'$ in $\text{mod}\,\Omega$. It is clear that $u \amalg u'$ is a covariant functor, which is k-linear if u, u' are.

The translation of formulae like (3.1) and (3.2) into formulae in the Mmod categories, requires a functor which we now describe. Let Ω, Λ be any k-algebras, and let $(\text{mod}\,\Omega, \text{mod}\,\Lambda)$ denote the k-linear category whose objects are all k-linear, covariant functors $u : \text{mod}\,\Omega \to \text{mod}\,\Lambda$, and for which the morphisms are all natural transformations (natural morphisms) $J: u_1 \to u_2$, for objects $u_1, u_2 \in (\text{mod}\,\Omega, \text{mod}\,\Lambda)$ (see [1, p. 183]). In the same way we may define a k-linear category $(\text{Mmod}\,\Lambda, \text{Mmod}\,\Omega)$, whose objects and morphisms are, respectively, all k-linear covariant functors $\mathscr{U} : \text{Mmod}\,\Lambda \to \text{Mmod}\,\Omega$, and all natural morphisms $\mathscr{J} : \mathscr{U}_1 \to \mathscr{U}_2$ between such objects.

(3.3) **Proposition.** *Let $J: u_1 \to u_2$ be any natural morphism between k-linear covariant functors $u_1, u_2 : \text{mod}\,\Omega \to \text{mod}\,\Lambda$. Then we may define a natural morphism $\mathbf{J} : \mathbf{u}_2 \to \mathbf{u}_1$ between the k-linear functors $\mathbf{u}_2, \mathbf{u}_1 : \text{Mmod}\,\Lambda \to \text{Mmod}\,\Omega$ (see Section 2) as follows: if $F \in \text{Mmod}\,\Lambda$ we define $\mathbf{J}(F) : \mathbf{u}_2(F) \to \mathbf{u}_1(F)$ by the rule $\mathbf{J}(F)(W) := F(J(W))$. for all $W \in \text{mod}\,\Omega$.*

Moreover the correspondence $u \to \mathbf{u}$, $J \to \mathbf{J}$ defines a k-linear contravariant functor $(\text{mod}\,\Omega, \text{mod}\,\Lambda) \to (\text{Mmod}\,\Lambda, \text{Mmod}\,\Omega)$.

Proof. We must prove that $\mathbf{J}$ as defined above is natural in F, in other words, that the diagram (i) below commutes, for any morphism $\phi : F \to F'$ in $\text{Mmod}\,\Lambda$. The objects and morphisms in this diagram are in $\text{Mmod}\,\Omega$, hence its commutativity is equivalent to that of diagram (ii), for arbitrary $W \in \text{mod}\,\Omega$.

(i) $$\begin{array}{ccc} \mathbf{u}_2(F) & \xrightarrow{\mathbf{J}(F)} & \mathbf{u}_1(F) \\ {\scriptstyle \mathbf{u}_2(\phi)}\downarrow & & \downarrow{\scriptstyle \mathbf{u}_1(\phi)} \\ \mathbf{u}_2(F') & \xrightarrow[\mathbf{J}(F')]{} & \mathbf{u}_1(F') \end{array}$$

$$\text{(ii)} \qquad \begin{array}{ccc} \mathbf{u}_2(F)(W) & \xrightarrow{\mathbf{J}(F)(W)} & \mathbf{u}_1(F)(W) \\ {\scriptstyle \mathbf{u}_2(\phi)(W)}\downarrow & & \downarrow{\scriptstyle \mathbf{u}_1(\phi)(W)} \\ \mathbf{u}_2(F')(W) & \xrightarrow[\mathbf{J}(F')(W)]{} & \mathbf{u}_1(F')(W) \end{array}$$

According to the definitions in Section 2 and in the statement of our proposition, (ii) is the same as (iii) below.

$$\text{(iii)} \qquad \begin{array}{ccc} F(u_2(W)) & \xrightarrow{F(J(W))} & F(u_1(W)) \\ {\scriptstyle \phi(u_2(W))}\downarrow & & \downarrow{\scriptstyle \phi(u_1(W))} \\ F'(u_2(W)) & \xrightarrow[F'(J(W))]{} & F'(u_1(W)) \end{array}$$

But (iii) is commutative, because it represents the naturality of $\phi : F \to F'$ with respect to the morphism $J(W): u_1(W) \to u_2(W)$ in $\operatorname{mod} \Lambda$.

This proves that $\mathbf{J} : \mathbf{u}_2 \to \mathbf{u}_1$ is indeed a morphism in the category ($\operatorname{Mmod} \Lambda$, $\operatorname{Mmod} \Omega$). The rest of (3.3) is easy to prove.

We shall need also the next proposition; its proof is elementary, and we shall omit it.

(3.4) **Proposition.** *Let Ω, Λ, Ψ be k-algebras.*

(i) *Given k-linear covariant functors $u : \operatorname{mod} \Omega \to \operatorname{mod} \Lambda$, $w : \operatorname{mod} \Lambda \to \operatorname{mod} \Psi$, let $v = w \cdot u$. We have then $\mathbf{v} = \mathbf{u} \cdot \mathbf{w}$.*

(ii) *Given k-linear covariant functors $u, u' : \operatorname{mod} \Omega \to \operatorname{mod} \Lambda$, let $v = u \amalg u'$. We have then $\mathbf{v} \cong \mathbf{u} \amalg \mathbf{u}'$.*

To translate (3.1) into a natural isomorphism betwen functors, we simply apply the functor of (3.3) to $J : \operatorname{ind}^G_H \cdot \operatorname{ind}^G_D \cong \operatorname{ind}^G_D$. *We get the isomorphism* $\mathbf{J} : \mathbf{ind}^G_D \cong \mathbf{ind}^H_D \cdot \mathbf{ind}^G_H$, i.e. $\operatorname{Res}^H_D \cdot \operatorname{Res}^G_H \cong \operatorname{Res}^G_D$, which is to say there is an isomorphism $(A_H)_D \cong A_D$, for all $A \in \operatorname{Mmod} kG$. This is the first formula in (3.5b), below.

In the same way, the Mackey isomorphism (3.2) gives us an isomorphism $\mathbf{u}_1 \cong \mathbf{u}_2$, which reads

$$\operatorname{Res}^G_H \cdot \operatorname{Ind}^G_K \cong \coprod_d \operatorname{Cnj}_{{}^dH, d^{-1}} \cdot \operatorname{Ind}^{{}^dH}_{{}^dH \cap K} \cdot \operatorname{Res}^K_{{}^dH \cap K},$$

which is to say that we have a natural isomorphism

$$(F^G)_H \cong \coprod_d {}^{d^{-1}}((F_{{}^dH \cap K})^{{}^dH}),$$

for $F \in \operatorname{Mmod} kK$. If we use (3.5d,e) below, and replace d by d^{-1}, we get the isomorphism (3.5f), which is evidently the exact analogue, for functors, of Mackey's formula for modules. All the isomorphisms in theorem (3.5) are derived in the same way from well-known formulae for modules. We leave the reader to supply the details.

(3.5) **Theorem.** *Let D, H, K be subgroups of a finite group G, with $D \leq H$. Let g, g' be elements of G. In the following, A, B, C, F denote arbitrary objects of the categories* Mmod kG, Mmod kH, Mmod kD, Mmod kK, *respectively. Then there hold natural isomorphisms as follows:*

(a) $A \cong A_G$, $A \cong A^G$, $A \cong {}^gA$.

(b) $(A_H)_D \cong A_D$, $(C^H)^G \cong C^G$.

(c) ${}^{g'}({}^gB) \cong {}^{g'g}B$.

(d) ${}^g(B_D) \cong ({}^gB)_{{}^gD}$.

(e) ${}^g(C^H) \cong ({}^gC)^{{}^gH}$.

(f) $(F^G)_H \cong \coprod_d (({}^dF)_{H \cap {}^dK})^H$, *where d traverses a set of representatives of the double cosets HgK, $g \in G$.*

4. Relative projectivity for functors

Let H be a subgroup of our finite group G. In this section we examine the idea of *(relative) H-projectivity* of objects of Mmod kG, and develop a theory very like that of relative projectivity of kG-modules.

Before we do this, we must be sure that the Krull–Schmidt theorem holds for the objects F which we study. In fact if Λ is any k-algebra and F any object of Mmod Λ, we denote by End F $(=(F, F) = (F, F)_{\text{Mmod}\,\Lambda})$ the endomorphism algebra of F, and observe that the finite direct sum decompositions

$$(4.1) \qquad F = \bigoplus_{i \in I} F_i$$

(I is a finite index set, and the F_i are subfunctors of F) correspond one-to-one with the orthogonal idempotent decompositions $1_F = \sum_{i \in I} e_i$ in End F. Then *provided* End F *is finite-dimensional as k-space,* the Krull–Schmidt theorem will hold for F, that is: there exists a finite decomposition (4.1) with all the F_i indecomposable, and if $F = \bigoplus_{j \in J} F'_j$ is another such, there exists a bijection $b: I \to J$ and an automorphism α of F such that $\alpha(F_i) = F_{b(i)}$, for all $i \in I$. For proof, we may simply take over the proof of the Krull–Schmidt theorem for kG-modules, as given for example in [13, Theorems 3.12, 5.2], since this proof works entirely within the endomorphism algebra. Now the endomorphism algebra End F is certainly finite-dimensional if F is finitely generated. For in this case there exist $W \in \text{mod}\,\Lambda$ and an epimorphism $\beta: (\cdot, W) \to F$. Since $(\cdot, W)$ is a projective object by (1.3), every endomorphism θ of F can be 'lifted' to an endomorphism ϕ of $(\cdot, W)$, so that $\beta\phi = \theta\beta$. It is easy to see that the correspondence $\phi \to \theta$ defines an epimorphism onto End F

from a subalgebra of $\mathrm{End}(\cdot, W)$. But $\mathrm{End}(\cdot, W) \cong \mathrm{End}\, W$ by the Yoneda isomorphism (1.2), hence $\mathrm{End}\, F$ is finite-dimensional. We have then

(4.2) **Proposition.** *The Krull–Schmidt theorem holds for any finitely generated* $F \in \mathrm{Mmod}\, \Lambda$. *In particular, it holds for any object* $F \in \mathrm{mmod}\, \Lambda$.

Notation. Let $E, F \in \mathrm{Mmod}\, \Lambda$. We write $E \mid F$ (and say E is a *summand* of F) if $F \cong E \amalg E'$ for some $E' \in \mathrm{Mmod}\, \Lambda$. It is not hard to see that every summand of a finitely presented F, is finitely presented. One should beware that subfunctors of finitely presented F are not necessarily even finitely generated (in fact a finitely generated subfunctor F' of a finitely presented F is automatically finitely presented). And in general neither ascending nor descending chain conditions hold for the subfunctors of a finitely presented F, or even for the finitely presented subfunctors of F (see [3, Theorem 3.1], and [2, p. 323, lines 9,10]. In both references one may take $\mathfrak{C} = \mathrm{mod}\, \Lambda$.) We end these general remarks with a useful corollary of (4.2):

(4.3) *Let* E, F_1, E_2 *be objects of* $\mathrm{mmod}\, \Lambda$, *with* E *indecomposable. Then* $E \mid F_1 \amalg F_2$ *implies* $E \mid F_1$, *or* $E \mid F_2$ *(or both).*

In the rest of this section, H is a fixed subgroup of a finite group G, and F is an object of $\mathrm{mmod}\, kG$.

Definition. $F \in \mathrm{mmod}\, kG$ is *H-projective* is there is some $E \in \mathrm{mmod}\, kH$ such that $F \mid E^G$.

By means of the formulae in (3.5), and the fact that the functors Ind, Res, Cnj all commute with direct sums, we can copy the argument in [11, p. 432] almost without change, and so prove the next two statements. Recall that $H \leq_G K$ (H, K being subgroups of G) means that $H \leq {}^g K$ for some $g \in G$.

(4.4) *If* F *is H-projective and* $H \leq_G K$, *then* F *is K-projective.*

(4.5) *Suppose that* $F \mid E^G$, *for some D-projective object* E *of* $\mathrm{mmod}\, kH$ $(D \leq H)$. *Then* F *is D-projective.*

The next theorem is the analogue of [11, Theorem 3], and it can be proved using the Mackey formula (3.5f).

(4.6) **Theorem.** *Let* D, H *be any subgroups of* G, *and let* $S \in \mathrm{mmod}\, kD$ *and* F *a summand of* S^G. *Suppose that*

(4.6a) $\qquad F_H = B_1 \amalg \cdots \amalg B_t,$

where $B_1, \ldots, B_t$ *are indecomposable objects of* $\mathrm{mmod}\, kH$. *Then for each*

$i \in \{1, \dots, t\}$ there exists an element $x = x_i \in G$ such that $B_i \mid (({}^xS)_{xDx^{-1} \cap H})^H$. Hence B_i is $(x_iDx_i^{-1} \cap H)$-projective.

To go further, we must anticipate some results from Section 5. From Theorem (5.11) it will follow that for any object $F \in \text{mmod}\, kG$ and subgroup H of G, *F is H-projective if and only if $F \mid (F_H)^G$.* And from Lemma (5.14, ii) we shall see that F is H-projective, for any $F \in \text{mmod}\, kG$ and any Sylow p-subgroup H of G. Armed with these facts, we may take over the argument of [11, pp. 434, 435] and prove the following.

(4.7) Theorem. *Let F be an indecomposable object of* mmod kG. *Then there exists a subgroup D of G with the properties*

(1) *F is D-projective, and*

(2) *If $H \leq G$, then F is H-projective if and only if $D \leq_G H$.*

D is determined uniquely up to G-conjugacy by these properties. D is a p-subgroup of G.

Definitions. Any subgroup D of G with properties (1), (2) is called a *vertex* of F. In this case there must exist an indecomposable $S \in \text{mmod}\, kD$ such that $F \mid S^G$; any such S is called a *source* of F.

These definitions extend the old definitions of vertex and source for modules, as the next proposition shows.

(4.8) Proposition. *Let $V \in \text{mod}\, kG$, $H \leq G$. Then V is H-projective if and only if $(\cdot, V)$ is H-projective. Hence if V is indecomposable, the vertices of V coincide with those of $(\cdot, V)$; moreover if S is a source of V, then $(\cdot, S)$ is a source of $(\cdot, V)$.*

Proof. By (5.11), $(\cdot, V)$ is H-projective if and only if $(\cdot, V) \mid ((\cdot, V)_H)^G$, and by (2.12) $((\cdot, V)_H)^G \cong (\cdot, V_H^G)$. But it is easy to see that $(\cdot, V) \mid (\cdot, V_H^G)$ if and only if $V \mid V_H^G$, that is, if and only if V is H-projective. The rest of (4.8) is very easy to prove, and we leave this to the reader. Notice that V is indecomposable if and only if $(\cdot, V)$ is indecomposable, since the Yoneda isomorphism gives End $V \cong \text{End}(\cdot, V)$.

Clearly many theorems about vertices and sources, together with their proofs, hold just as well in the category mmod kG. We shall not attempt to list such theorems, but give as an example a proposition which we shall need later.

(4.9) Proposition. *Let $H \leq G$, and let F, E be indecomposable objects of* mmod kG, mmod kH *respectively, such that $F \mid E^G$ and $E \mid F_H$. Then F, E have a vertex and source in common.*

Proof. Let D_F, D_E be vertices of F, E. Since $F \mid E^G$, F is D_E-projective by (4.5), hence $D_F \leq_G D_E$. Take $D = D_F$ and S to be a source of F in (4.6). We may also take $B_i = E$, since $E \mid F_H$. So there exists $x \in G$ such that E is $(xD_Fx^{-1} \cap H)$-projective. Hence $D_E \leq_H xD_Fx^{-1} \leq_G D_F$. Taking this with $D_F \leq_G D_F$ we have $D_E =_G D_F$. Let $T \in \operatorname{mmod} kD_E$ be a source of E. Then $E \mid T^H$, hence $F \mid E^G \mid (T^H)^G \cong T^G$, which shows that T is a source of F, and concludes the proof of (4.9).

5. The maps R_H^G and T_H^G

Throughout this section H is a given subgroup of the finite group G, and S a transversal of the set of cosets $\{gH \mid g \in G\}$. We assume that S contains the identity element of G.

Let $M, N \in \operatorname{mod} kG$. Two k-maps

(5.1) $$r_H^G : (M,N) \to (M_H, N_H), \qquad t_H^G : (M_H, N_H) \to (M,N)$$

are fundamental in the theory of relatively projective kG-modules: r_H^G is simply the inclusion $(M,N) \leq (M_H, N_H)$, and t_H^G is the ('relative' or 'interior') *trace map*, given by $t_H^G(f) = \sum_{s \in S} sfs^{-1}$ for all $f \in (M_H, N_H)$ (see, for example, [13, Chapter II]. In Landrock's notation, t_H^G is written $\overline{\operatorname{Tr}}_H^G$.) We shall next define, for any objects $A, B \in \operatorname{Mmod} kG$, k-maps

(5.2) $$R_H^G : (A,B) \to (A_H, B_H), \qquad T_H^G : (A_H, B_H) \to (A,B)$$

which behave very like r_H^G, t_H^G. The main theorem in this section, Theorem 5.11, includes the analogue for functors of D.G. Higman's theorem that a kG-module M is H-projective if and only if there is an element $\eta \in \operatorname{End}(M_H) = (M_H, M_H)$ such that $t_H^G(\eta) = 1_M$ ([12, p. 371]).

It is easy to define R_H^G, it is simply the map induced by the functor Res_H^G. So $R_H^G(\alpha) = \alpha_H$, for all $\alpha \in (A,B)$. Notice that, in general, R_H^G is not injective. It is harder to define T_H^G, because there seems to be no analogue for the sum $\sum sfs^{-1}$. However there is a 'functorial' description (5.4) of t_H^G, and we shall adapt this to define T_H^G.

Let $X \in \operatorname{mod} kG$. One has well-known k-maps

(5.3) $$m_H^G(X) : X \to (X_H)^G, \qquad n_H^G(X) : (X_H)^G \to X,$$

defined by $m_H^G(X)(x) = \sum s \otimes s^{-1}x$, $n_H^G(X)(\sum s \otimes x_s) = \sum sx_s$, for $x, x_s \in X$ (the sums are over all s in the transversal S of $\{gH \mid g \in G\}$). Let $M, N \in \operatorname{mod} kG$ and $f \in (M_H, N_H)$, then by an easy calculation we find

(5.4) $$t_H^G(f) = n_H^G(N) \cdot f^G \cdot m_H^G(M).$$

This will provide the model for our definition (5.6) of T_H^G.

The maps (5.3) are natural in X, and so give morphisms $m_H^G : u_1 \to u_2$, $n_H^G : u_2 \to u_1$ between the functors $u_1 = 1_{\operatorname{mod} kG}$ and $u_2 = \operatorname{ind}_H^G \cdot \operatorname{res}_H^G$. Apply the functor of Proposition (3.3) (with $\Lambda = \Omega = kG$). This gives us morphisms $\mathbf{m}_H^G : \mathbf{u}_2 \to \mathbf{u}_1$, $\mathbf{n}_H^G : \mathbf{u}_1 \to \mathbf{u}_2$

between the functors $\mathbf{u}_1 = 1_{\mathrm{Mmod}\,kG}$, $\mathbf{u}_2 = \mathrm{Ind}_H^G \cdot \mathrm{Res}_H^G$. These functors both take Mmod kG to itself. Thus we have, for each $A \in$ Mmod kG, the following morphisms in Mmod kG:

$$(5.5) \qquad \mathbf{m}_H^G(A) : (A_H)^G \to A, \qquad \mathbf{n}_H^G(A) : A \to (A_H)^G.$$

To calculate these at an arbitrary $X \in$ mod kG, we use the definition given in (3.3), namely $\mathbf{m}_H^G(A)(X) = A(m_H^G(X))$, $\mathbf{n}_H^G(A)(X) = A(n_H^G(X))$. Since $\mathbf{u}_1, \mathbf{u}_2$ both map finitely presented objects of Mmod kG to finitely presented objects, we may regard $\mathbf{u}_1, \mathbf{u}_2$ as functors of mmod kG into itself.

Definition. If $A, B \in$ mmod kG, we define $T_H^G : (A_H, B_H) \to (A, B)$ by the rule

$$(5.6) \qquad T_H^G(\eta) = \mathbf{m}_H^G(B) \cdot \eta^G \cdot \mathbf{n}_H^G(A),$$

for all $\eta \in (A_H, B_H)$. In other words, $T_H^G(\eta)$ is defined by requiring the diagram below to commute.

$$(5.7) \qquad \begin{array}{ccc} (A_H)^G & \xrightarrow{\eta^G} & (B_H)^G \\ {\scriptstyle \mathbf{n}_H^G(A)} \uparrow & & \downarrow {\scriptstyle \mathbf{m}_H^G(B)} \\ A & \xrightarrow[T_H^G(\eta)]{} & B \end{array}$$

Here $\eta^G = \mathrm{Ind}_H^G(\eta)$ of course. Since Ind_H^G is a k-linear functor, it follows from (5.6) that T_H^G is a k-linear map. The maps R_H^G, T_H^G satisfy many identities analogous to identities satisfied by r_H^G, t_H^G. For our present purposes, we need the following.

(5.8) Proposition. *Let Z, A, B, C be objects of* mmod kG. *Then*

(i) $R_H^G(\beta\alpha) = R_H^G(\beta) R_H^G(\alpha)$, *for any* $\alpha \in (A, B)$, $\beta \in (B, C)$.

(ii) $T_H^G(\beta_H \eta) = \beta T_H^G(\eta)$ *and* $T_H^G(\eta \zeta_H) = T_H^G(\eta)\zeta$, *for any* $\eta \in (A_H, B_H)$, $\beta \in (B, C)$ *and* $\zeta \in (Z, A)$.

Proof. (i) This holds because Res_H^G is a covariant functor.

(ii) By definition (5.6) we have

$$T_H^G(\beta_H) = \mathbf{m}_H^G(C) \cdot (\beta_H \eta)^G \cdot \mathbf{n}_H^G(A) = \mathbf{m}_H^G(C) \cdot \beta_H^G \cdot \eta^G \cdot \mathbf{n}_H^G(A).$$

The fact that $\mathbf{m}_H^G(C) = \mathbf{m}(C)$ is natural in C (Proposition (3.3)) gives the commutative diagram

$$\begin{array}{ccc} (B_H)^G & \xrightarrow{\mathbf{m}(B)} & B \\ {\scriptstyle (\beta_H)^G} \downarrow & & \downarrow {\scriptstyle \beta} \\ (C_H)^G & \xrightarrow[\mathbf{m}(C)]{} & C \end{array}$$

Hence $T_H^G(\beta_H)=\beta\cdot\mathbf{m}_H^G(B)\cdot\eta^G\cdot\mathbf{n}_H^G(A)=\beta T_H^G(\eta)$. The proof of the second equation in (ii) is similar. This completes the proof of (5.8).

Let $Y\in \operatorname{mod} kH$. Define the kH-map $c(Y):(Y^G)_H\to(Y^G)_H$ by $c(Y)(\sum s\otimes y_s)=1\otimes y_1$ for all $\sum s\otimes y_s$ in Y^G. An easy calculation shows that

(5.9) $\qquad 1_{X_H}=n(X)_H\cdot c(X_H)\cdot m(X)_H,\quad$ *for all* $X\in\operatorname{mod} kG$.

(In this formula, and in the rest of this section, we have omitted the affixes G, H in n_H^G, m_H^G in the interests of greater legibility – we shall often do the same for $\mathbf{n}_H^G$, $\mathbf{m}_H^G$, res_H^G, etc.)

The map $c(Y)$ is natural in Y, and so it provides a morphism $c:\operatorname{res}\cdot\operatorname{ind}\to\operatorname{res}\cdot\operatorname{ind}$. By Proposition (3.3) we deduce a morphism $\mathbf{c}:\operatorname{Res}\cdot\operatorname{Ind}\to\operatorname{Res}\cdot\operatorname{Ind}$ as follows: if $C\in\operatorname{mmod} kH$, then $\mathbf{c}(C):(C^G)_H\to(C^G)_H$ is the morphism in mmod kH given by $\mathbf{c}(C)(Y)=C(c(Y))$, for all $Y\in\operatorname{mod} kH$. We prove next

(5.10) $\qquad T_H^G(\mathbf{c}(C))=1_{C^G},\quad$ *for all* $C\in\operatorname{mmod} kH$.

Proof. By definition (5.6), $T_H^G(\mathbf{c}(C))=\mathbf{m}(C^G)\cdot\mathbf{c}(C)^G\cdot\mathbf{n}(C^G)$. Each side of this equation is a morphism from C^G to itself. Then for an arbitrary $X\in\operatorname{mod} kG$ we have

$$\begin{aligned}T_H^G(\mathbf{c}(C))(X)&=\mathbf{m}(C^G)(X)\cdot\mathbf{c}(C)^G(X)\cdot\mathbf{n}(C^G)(X)\\&=C^G(m(X))\cdot\mathbf{c}(C)(X_H)\cdot C^G(n(X))\\&=C(m(X)_H)\cdot C(c(X_H))\cdot C(n(X)_H)\\&=C(n(X)_H\cdot c(X_H)\cdot m(X)_H).\end{aligned}$$

Now apply (5.9) to the last term in this equation, which is thereby shown to equal

$$C(1_{X_H})=C((1_X)_H)=C^G(1_X)=(1_{C^G})(X).$$

We have now $T_H^G(\mathbf{c}(C))(X)=(1_{C^G})(X)$, for all $X\in\operatorname{mod} kG$, and this proves (5.10).

(5.11) **Theorem.** *Let* $F\in\operatorname{mmod} kG$, *and let* H *be a subgroup of* G. *Then each of the following six conditions on* F *implies all the others:*

(1) *F is H-projective.*
(2) *There exists* $\eta\in\operatorname{End}F_H=(F_H,F_H)$ *such that* $T_H^G(\eta)=1_F$.
(3) *The map* $T_H^G:\operatorname{End}F_H\to\operatorname{End}F$ *is surjective.*
(4) *The morphism* $\mathbf{m}(F):(F_H)^G\to F$ *has right inverse.*
(5) *The morphism* $\mathbf{n}(F):F\to(F_H)^G$ *has left inverse.*
(6) $F\,|\,(F_H)^G$.

Proof. (1)$\Rightarrow$(2). If F is H-projective, there exists some $C\in\operatorname{mmod} kH$ such that $F\,|\,C^G$. Therefore there are morphisms $\mu:F\to C^G$, $\pi:C^G\to F$ such that $1_F=\pi\mu$. Hence by (5.10), (5.8) we have

$$1_F = \pi \cdot 1_{C^G} \cdot \mu = \pi \cdot T_H^G(\mathbf{c}(C)) \cdot \mu = T_H^G(\pi_H \mathbf{c}(C) \mu_H),$$

which shows that (2) holds with $\eta = \pi_H \mathbf{c}(C) \mu_H$.

(2)$\Leftrightarrow$(3), since by (5.8), Im T_H^G is an ideal of End F.

(2)$\Rightarrow$(4),(5). By (5.6), $T_H^G(\eta) = 1_F$ implies that $\mathbf{m}(F) \cdot \eta^G \cdot \mathbf{n}(F) = 1_F$, which gives both (4) and (5). Finally (4) or (5) implies (6), which clearly implies (1).

Remarks. The implication (1)$\Rightarrow$(6) fills the gap in our proof of the existence of vertices of an indecomposable $F \in$ mmod kG. The equivalence (1)$\Leftrightarrow$(2) is the analogue of D.G. Higman's theorem [12, p. 371]. The analogue for kG-modules of (5.11) is well-known, see for example [9, §19] or [13, p. 94, Corollary 2.4]. But there are important differences between the module and functor categories; for example, while $n(X): (X_H)^G \to X$ (see (5.3)) is epimorphic for all $X \in$ mod kG, the corresponding morphism $\mathbf{m}(F): (F_H)^G \to F$ may fail to be epimorphic for $F \in$ mmod kG – for example, if F is simple, so that $F = SV$ for some $V \in$ indec kG, we saw (2.13) that $F_H = 0$ whenever V is not H-projective, and so in this case $\mathbf{m}(F)$ is not epimorphic. However by (5.11)(4) $\mathbf{m}(F)$ is epimorphic for any H-projective $F \in$ mmod kG. Taking these two remarks together we have the following result.

(5.12) **Theorem.** *Let $V \in$ indec kG, and let H be any subgroup of G. Then V is H-projective (in* mod kG*) if SV is H-projective (in* mmod kG*). Consequently any vertex Q of SV contains some vertex P of V.* (See also (8.11).)

We shall see in Section 8 that it can well happen that $Q > P$.

Another place where mod kG and mmod kG behave differently with regard to relative projectivity is this: a module $M \in$ mod kG is projective (in mod kG) if and only if it is $\{1\}$-projective, whereas a projective object of mmod kG need not be $\{1\}$-projective. In fact $F \in$ mmod kG is projective in mmod kG if $F = (\cdot, V)$ for some $V \in$ mod kG (see (1.3)). But we saw in (4.8) that if V is indecomposable, then the vertices of $(\cdot, V)$ are the same as those of V. Therefore if V is not projective, then $(\cdot, V)$ is not $\{1\}$-projective.

We still have to prove the assertion made at the end of Section 4, that a defect group of any indecomposable $F \in$ mmod kG is always a p-group. This follows from (ii) in the lemma below.

(5.14) **Lemma.** (i) *Let $A, B \in$ mmod kG and let $\xi \in (A, B)$. Then $T_H^G(\xi_H) = (G:H)\xi$, for any subgroup H of G. Here $(G:H)$ denotes the index of H in G.*

(ii) *If $F \in$ mmod kG, then F is H-projective for any $H \in \mathrm{Syl}_p(G)$.*

Proof. (i) By (5.6), $T_H^G(\xi_H) = \mathbf{m}(B) \cdot \xi_H^G \cdot \mathbf{n}(A)$. Since $\mathbf{n}(A): A \to (A_H)^G$ is natural in A, we deduce $\xi_H^G \cdot \mathbf{n}(A) = \mathbf{n}(B) \cdot \xi$, hence $T_H^G(\xi_H) = \mathbf{m}(B)\mathbf{n}(B)\xi$. So (i) will follow if we prove $\mathbf{m}(B)\mathbf{n}(B) = (G:H)1_B$. Take any $X \in$ mod kG. Then

$$(\mathbf{m}(B)\mathbf{n}(B))(X) = \mathbf{m}(B)(X) \cdot \mathbf{n}(B)(X)$$

$$= B(m(X)) \cdot B(n(X)) = B(n(X)m(X)).$$

But it is easily checked that $n(X)m(X) = (G:H)1_X$, and our result follows.

(ii) Take $A = B = F$ in (i), and take $\xi = (G:H)^{-1}1_F$. We get $T_H^G(\xi_H) = 1_F$, hence F is H-projective by (5.11).

6. Some useful formulae

In this section are collected some formulae which will be used in later calculations. We begin with some well-known maps (see [13, II, §1] or [9, p. 232]). Let $X \in \mathrm{mod}\, kG$ and $Y \in \mathrm{mod}\, kH$; S is a transversal (containing the element 1) of the set of cosets $\{gH \mid g \in G\}$ of the subgroup H of G. Define

$$a(Y,X):(Y,X_H) \to (Y^G,X), \qquad b(X,Y):(X_H,Y) \to (X,Y^G),$$

$$e(Y):Y \to (Y^G)_H \quad \text{and} \quad d(Y):(Y^G)_H \to Y$$

as follows: $a(Y,X)(\theta)$ takes $\sum_s s \otimes y_s \to \sum_s s\theta(y_s)$, for any $\theta \in (Y,X_H)$; $b(X,Y)(\phi)$ takes $x \to \sum_s s \otimes \phi(s^{-1}x)$, for any $\phi \in (X_H,Y)$; $e(Y)$ takes $y \to 1 \otimes y$; $d(Y)$ takes $\sum_s s \otimes y_s \to y_1$.

(6.1) **Proposition.** *$a(Y,X)$, $b(X,Y)$ are both k-isomorphisms, natural in X and Y. The inverse of $a(Y,X)$ takes $\Theta \to \Theta \cdot e(Y)$, for any $\Theta \in (Y^G,X)$. The inverse of $b(X,Y)$ takes $\Phi \to d(Y) \cdot \Phi$, for any $\Phi \in (X,Y^G)$.*

These standard facts are easily verified by direct calculation. We have already used implicitly the isomorphisms $a(\cdot,X):(\cdot,X_H) \to (\cdot,X)_H$ and $b(\cdot,Y):(\cdot,Y)^G \to (\cdot,Y^G)$, see (2.12). The maps $a(Y,X)$ and $b(X,Y)$ also give isomorphisms in the categories of *covariant* k-linear functors, namely $a(Y,\cdot):(Y,\cdot)^G \to (Y^G,\cdot)$ and $b(X,\cdot):(X_H,\cdot) \to (X,\cdot)_H$ (induction and restriction for covariant functors are defined by the appropriate adaptation of the definitions in Section 2).

(6.2) **Proposition.** *Let $M,N \in \mathrm{mod}\, kG$. Define $r_H^G, t_H^G, m_H^G, n_H^G$ as in* (5.1), (5.3). *There hold the following equations:*

(i) $t_H^G = (m_H^G(M),N) \cdot a(M_H,N) = (M,n_H^G(N)) \cdot b(M,N_H)$, *and*
(ii) $r_H^G = a(M_H,N)^{-1} \cdot (n_H^G(M),N) = b(M,N_H)^{-1} \cdot (M,m_H^G(N))$.

These, too, are easily verified by direct calculations which we leave to the reader. We want next to make an explicit connexion between the maps T_H^G and t_H^G of section 5. Let $N,N' \in \mathrm{mod}\, kG$, and let $\xi:(\cdot,N)_H \to (\cdot,N')_H$ be a morphism in mmod kH. It is clear that there is a unique morphism $(\cdot,N_H) \to (\cdot,N'_H)$ which makes the diagram (6.3) commute; by (1.2) this morphism has the form $(\cdot,h)$ for a unique $h \in (N_H,N'_H)$.

$$(6.3)\qquad \begin{array}{ccc} (\cdot,N_H) & \xrightarrow{a(\cdot,N)} & (\cdot,N)_H \\ {\scriptstyle (\cdot,h)}\downarrow & & \downarrow{\scriptstyle \xi} \\ (\cdot,N'_H) & \xrightarrow[a(\cdot,N')]{} & (\cdot,N')_H \end{array}$$

(6.4) Proposition. *Suppose that ξ, h are such that the diagram* (6.3) *commutes. Then* $T_H^G(\xi)=(\cdot,t_H^G(h)):(\cdot,N)\to(\cdot,N')$.

Proof. Take any $M\in \operatorname{mod} kG$. By (5.6) we have

$$T_H^G(\xi)=(\cdot,t_H^G(h)):(\cdot,N)\to(\cdot,N').$$

(omitting some affixes G,H for clarity), where $A=(\cdot,N)$ and $B=(\cdot,N')$. From the definitions of $\mathbf{m}=\mathbf{m}_H^G$ and $\mathbf{n}=\mathbf{n}_H^G$ (see (5.5)), $\mathbf{m}(B)(M)=B(m(M))=(m(M),N')$, and $\mathbf{n}(A)(M)=A(n(M))=(n(M),N)$. And $\xi^G(M)=\xi(M_H)$, which by the commutativity of (6.3) equals $a(M_H,N')\cdot(M_H,h)\cdot a(M_H,N)^{-1}$. Therefore $T_H^G(\xi)(M)$ is equal to the product

$$(m(M),N')\cdot a(M_H,N')\cdot(M_H,h)\cdot a(M_H,N)^{-1}\cdot(n(M),N),$$

hence by (6.2) to $t_H^G\cdot(M_H,h)\cdot r_H^G$, where $t_H^G:(M_H,N'_H)\to(M,N')$ and $r_H^G:(M,N)\to(M_H,N_H)$. That is, $T_H^G(\xi)(M):(M,N)\to(M,N')$ takes an element $z\in(M,N)$ to $t_H^G(r_H^G(z)h)=t_H^G(zh)=zt_H^G(h)$, and is therefore equal to $(M,t_H^G(h))$. This completes the proof of (6.4).

We saw in Section 1 that each object F of mmod kG can be specified (in many ways) by a single element $T_\alpha\in D(X,V)$, for suitable $X,V\in\operatorname{mod} kG$. For by Auslander–Reiten's theorem (1.7) there exist modules $X,V\in\operatorname{mod} kG$ and a morphism $\alpha:(\cdot,V)\to D(X,\cdot)$ such that $F\cong\operatorname{Im}\alpha$, and by Yoneda's lemma α is completely specified by the element $T_\alpha=\alpha(V)(1_V)$ of $D(X,V)$. To reconstruct α from T_α we have the formula

$$(6.5)\qquad \alpha(M)(f)=D(X,f)(T_\alpha),\quad \text{for all } M\in\operatorname{mod} kG,\ f\in(M,V).$$

We want an analogous specification of F_H. It is clear that $F_H\cong\operatorname{Im}\alpha_H$, hence that $F_H\cong\operatorname{Im}\beta$ where β is the unique morphism $(\cdot,V_H)\to D(X_H,\cdot)$ which makes the diagram

$$(6.6)\qquad \begin{array}{ccc} (\cdot,V_H) & \xrightarrow{a(\cdot,V)} & (\cdot,V)_H \\ {\scriptstyle \beta}\downarrow & & \downarrow{\scriptstyle \alpha_H} \\ D(X_H,\cdot) & \xleftarrow[Db(X,\cdot)]{} & (D(X,\cdot))_H \end{array}$$

commute. The next proposition shows how to calculate T_β.

(6.7) **Proposition.** *Let $X, V \in \operatorname{mod} kG$. Given morphisms $\alpha: (\cdot, V) \to D(X, \cdot)$ and $\beta: (\cdot, V_H) \to D(X_H, \cdot)$ such that* (6.6) *commutes, then $T_\beta = T_\alpha \cdot t_H^G$. That is, T_β is the composite map*

$$(X_H, V_H) \xrightarrow{t_H^G} (X, V) \xrightarrow{T_\alpha} k.$$

Proof. By definition $T_\beta = \beta(V_H)(1_{V_H})$, so by (6.6)

$$T_\beta = (Db(X, V_H) \cdot \alpha_H(V_H) \cdot a(V_H, V))(1_{V_H}).$$

We find $a(V_H, V)(1_{V_H}) = n_H^G(V)$ by direct calculation (or by putting $M = N = V$ in (6.2, (ii)), and $\alpha_H(V_H) = \alpha(V_H^G)$ by definition of α_H. Thus

$$T_\beta = Db(X, V_H)(\alpha(V_H^G)(n(V)) = (Db(X, V_H) \cdot D(X, n(V)))(T_\alpha)$$

by (6.5), that is $T_\beta = D((X, n(V)) \cdot b(X, V_H))(T_\alpha) = D(t_H^G)(T_\alpha)$ by (6.2, i). But this means $T_\beta = T_\alpha \cdot t_H^G$, and the proof of (6.7) is complete.

There is a companion piece to (6.7), which can be used to calculate E^G, where E is any object of $\operatorname{mmod} kH$. There exist $Y, W \in \operatorname{mod} kH$ and a morphism $\omega: (\cdot, W) \to D(Y, \cdot)$ such that $E \cong \operatorname{Im} \omega$; therefore E is specified by the element $T_\omega = \omega(W)(1_W) \in D(Y, W)$. Then $E^G \cong \operatorname{Im} \omega^G \cong \operatorname{Im} \xi$, where ξ is the unique morphism $(\cdot, W^G) \to D(Y^G, \cdot)$ which makes the diagram

$$\text{(6.8)} \qquad \begin{array}{ccc} (\cdot, W^G) & \xrightarrow{b(\cdot, W)} & (\cdot, W)^G \\ \Big\downarrow \xi & & \Big\downarrow \psi^G \\ D(Y^G, \cdot) & \xleftarrow[Da(Y, \cdot)]{} & (D(Y, \cdot))^G \end{array}$$

commute. The next proposition shows how to calculate T_ξ. It uses, in place of t_H^G, the map $u_H^G: (Y^G, W^G) \to (Y, W)$ given by

$$\text{(6.9)} \qquad u_H^G(f) = d(W) \cdot f \cdot e(Y), \quad \text{for all } f \in (Y^G, W^G).$$

(6.10) **Proposition.** *Let $Y, W \in \operatorname{mod} kH$. Given morphisms $\omega: (\cdot, W) \to D(Y, \cdot)$ and $\xi: (\cdot, W^G) \to D(Y^G, \cdot)$ such that* (6.8) *commutes, then $T_\xi = T_\omega \cdot u_H^G$.*

Proof. By definition $T_\xi = \xi(W^G)(1_{W^G})$, so by (6.8)

$$T_\xi = (Da(Y, W^G)^{-1} \cdot \omega^G(W^G) \cdot b(W^G, W)^{-1})(1_{W^G}).$$

By (6.1), $b(W^G, W)^{-1}(1_{W^G}) = d(W)$. Hence $T_\xi = (Da(Y, W^G)^{-1}(\omega(W_H^G)(d(W)))$. Thus

$$T_\xi = (Da(Y, W^G)^{-1} \cdot D(Y, d(W)))(T_\omega) = D((Y, d(W)) \cdot a(Y, W^G)^{-1})(T_\omega).$$

Take any $f\in(Y^G, W^G)$. We have $T_\xi(f)=(T_\omega\cdot(Y,d(W))\cdot a(Y,W^G)^{-1})(f)$ by what has just been proved. But $a(Y,W^G)^{-1}(f)=f\cdot e(Y)$, by (6.1). Thus $((Y,d(W))\cdot a(Y,W^G)^{-1})(f)=d(W)\cdot f\cdot e(Y)=u_H^G(f)$. We have now $T_\xi(f)=T_\omega(u_H^G(f))$, and the proof of (6.10) is complete.

We shall need the following lemma in the next section. The proof is straightforward, and is left to the reader.

(6.11) **Lemma.** *Let $Z, Y, W\in \operatorname{mod} kH$. Then $u_H^G(fx^G)=u_H^G(f)x$, for all $f\in(Y^G, W^G)$ and all $x\in(Z,Y)$.*

7. Applications to module theory

Throughout this section H is a subgroup of G, $W\in \operatorname{indec} kH$ and $V\in \operatorname{indec} kG$. We are interested in cases where the functors $(SW)^G$ or $(SV)_H$ are semisimple, or even simple. If H is normal in G, then $(SV)_H$ is always semisimple - this is the analogue of Clifford's theorem [8, Theorem 1]. More surprisingly, $(SW)^G\cong SV$ whenever W, V are related by the module correspondence mentioned in the Introduction. It turns out that this is a reformulation of Burry–Carlson's 'strong correspondence theorem' [7, Theorems 5, 6].

We give a criterion for a finitely presented functor of a certain type to be semisimple.

(7.1) **Lemma.** *Let Λ be a k-algebra of finite k-rank, and let $M\in \operatorname{mod}\Lambda$. Let $\alpha:(\cdot,M)\to D(M,\cdot)$ be a morphism in* $\operatorname{mmod}\Lambda$, *and let $T_\alpha=\alpha(M)(1_M)$. Then $F=\operatorname{Im}\alpha$ is semisimple if and only if $T_\alpha(\operatorname{rad}\operatorname{End} M)=0$.*

Proof. Since $F\cong(\cdot,M)/\operatorname{Ker}\alpha$, F is semisimple if and only if $\mathbf{r}(\cdot,M)\leq\operatorname{Ker}\alpha$. That is, F is semisimple if and only if

$$(*)\qquad f\in\mathbf{r}(X,M)\ \Rightarrow\ \alpha(X)(f)=0$$

for all $X\in\operatorname{mod}\Lambda$ and all $f\in(X,M)$. By (6.5), $\alpha(X)(f)=D(M,f)(T_\alpha)$, hence $\alpha(X)(f)=0$ if and only if $T_\alpha(fh)=0$ for all $h\in(M,X)$.

Suppose first that F is semisimple. Putting $M=X$ in $(*)$ we find: $f\in\operatorname{rad}\operatorname{End}M\Rightarrow T_\alpha(fh)=0$ all $h\in\operatorname{End}M\Rightarrow T_\alpha(f)=0$. In other words, $T_\alpha(\operatorname{rad}\operatorname{End}M)=0$, as required. Next assume that $T_\alpha(\operatorname{rad}\operatorname{End}M)=0$. Suppose $X\in\operatorname{mod}\Lambda$, $f\in(X,M)$ and that $f\in\mathbf{r}(X,M)$. By (1.4) we have $fh\in\operatorname{rad}\operatorname{End}M$, for all $h\in(M,X)$, hence $T_\alpha(fh)=0$ for all $h\in(M,X)$. But this implies that $\alpha(X)(f)=0$. Hence $(*)$ holds, therefore F is semisimple. The proof of (7.1) is now complete.

We next apply (7.1), to give criteria for $(SW)^G$ or $(SV)_H$ to be semisimple.

(7.2) **Theorem.** (i) *If* $W \in \operatorname{indec} kH$, *then* $(SW)^G$ *is semisimple if and only if* $u_H^G(\operatorname{rad} \operatorname{End} W^G) \leq \operatorname{rad} \operatorname{End} W$.

(ii) *If* $V \in \operatorname{indec} kG$, *then* $(SV)_H$ *is semisimple if and only if* $t_H^G(\operatorname{rad} \operatorname{End} V_H) \leq \operatorname{rad} \operatorname{End} V$.

Proof. (i) From (1.8), $SW \cong \operatorname{Im} \omega$, where $\omega: (\cdot, W) \to D(W, \cdot)$ is the morphism is $\operatorname{mmod} kH$ specified by an element $T_\omega = \omega(W)(1_W)$ of $D(\operatorname{End} W)$ which satisfies $T_\omega \neq 0$, $T_\omega(\operatorname{rad} \operatorname{End} W) = 0$. By (6.10), $(SW)^G = \operatorname{Im} \xi$, where $\xi: (\cdot, W^G) \to D(W^G, \cdot)$ is specified by the element $T_\xi = T_\omega \cdot u_H^G$. By (7.1), then, $(SW)^G$ is semisimple if and only if $u_H^G(\operatorname{rad} \operatorname{End} W^G) \leq \operatorname{Ker} T_\omega$. But (6.11) tells us that $R = u_H^G(\operatorname{rad} \operatorname{End} W^G)$ is a right ideal of End W. It is then clear, since rad End W is the unique maximal right ideal of End W, that $R \leq \operatorname{Ker} T_\omega$ if and only if $R \leq \operatorname{rad} \operatorname{End} W$. This completes the proof of (7.2, i). We leave the reader to give the, exactly parallel, proof of (7.2, ii). (In place of (6.11), we need the well-known equation $t_H^G(fx_H) = t_H^G(f)x$, which holds for all $X, M, N \in \operatorname{mod} kG$ and $f \in (M_H, N_H)$, $x \in (X, M)$.)

The next theorem describes some criteria for semisimplicity of $(SW)^G$ and $(SV)_H$, which do not require explicit calculation of endomorphism rings.

(7.3) **Theorem.** (i) *If* $W \in \operatorname{indec} kH$ *and if* $[W \mid W_H^G] = 1$, *then* $(SW)_G$ *is semisimple.*

(ii) *If* $V \in \operatorname{indec} kG$ *and if* H *is normal in* G, *then* $(SV)_H$ *is semisimple.*

Proof. (i) Our hypothesis says that there is an isomorphism $\theta: W_H^G \to W_1 \amalg \cdots \amalg W_r$, where the $W_i \in \operatorname{indec} kH$ and $W = W_1$, while $W_i \not\cong W$ for $i = 2, \ldots, r$. Let $\pi_i: W_H^G \to W_i$ and $\mu_i: W_i \to W_H^G$ be the projections and injections which result from θ. We may (and shall) arrange that π_1 and μ_1 are the maps $d(W)$ and $e(W)$ defined at the beginning of Section 6, because $e(W)d(W)$ is an idempotent of End W_H^G whose image is isomorphic to W.

Every element $f \in \operatorname{End} W_H^G$ has a *matrix* (f_{ij}), whose coefficient $f_{ij} = \pi_i \cdot f \cdot \mu_j \in (W_j, W_i)$, for all $i, j = 1, \ldots, r$ (see for example [9, p. 462]). Define a map $\Phi: \operatorname{End} W_H^G \to (\operatorname{End} W)/R$, where $R = \operatorname{rad} \operatorname{End} W$, by the rule $\Phi(f) = f_{11} + R$. Then Φ is a k-algebra map; the only difficulty in proving this, is to show that $\Phi(fg) = \Phi(f)\Phi(g)$, for any $f, g \in \operatorname{End} W_H^G$. But $(fg)_{11} = f_{11}g_{11} + f_{12}g_{21} + \cdots + f_{1r}g_{r1}$, and for each $i \neq 1$, $W_i \not\cong W$, which shows that $f_{1i}g_{i1}$ is not an automorphism of W, and hence by Fitting's lemma $f_{1i}g_{i1} \in R$. Thus $(fg)_{11} \equiv f_{11}g_{11} \bmod R$, which is what we need.

Now we apply this k-algebra map Φ to End W^G, which is a subalgebra of End W_H^G. If $f \in \operatorname{End} W^G$, we have $f_{11} = \pi_1 \cdot f \cdot \mu_1 = d(W) \cdot f \cdot e(W) = u_H^G(f)$, so that $\Phi(f) = u_H^G(f) + R$. Let $S = \operatorname{rad} \operatorname{End} W^G$. By (6.11), $u_H^G(S)$ is a right ideal of End W. Therefore $\Phi(S)$ is a right ideal of (End $W)/R$; but $\Phi(S)$ is also nilpotent, and (End $W)/R$ is a division ring. It follows $\Phi(S) = 0$, that is, $u_H^G(S) \leq R$, and so by (7.2, i) $(SW)_G$ is semisimple.

(ii) Since H is normal in G, each element $s\in G$ induces on End V_H a k-algebra automorphism $f\to sfs^{-1}$ ($f\in$ End V_H). If $R=$ rad End V_H, then we have $sRs^{-1}=R$, for all $s\in G$. Therefore $t_H^G(R)\leq R$. But $t_H^G(R)$ is a right ideal of End V (see the remark at the end of the proof of (7.2)), and is nilpotent because it is contained in R. Hence $t_H^G(R)\leq$ rad End V, and so by (7.2, ii), $(SV)_H$ is semisimple. This completes the proof of Theorem (7.3).

The next lemma has the corollary (see (7.5, i)) that in (7.3, i), $(SW)^G$ is in fact *simple*.

(7.4) **Lemma.** *Let $W\in$ indec kH be such that $(SW)^G$ is semisimple, so that there exist $t\geq 1$ mutually non-isomorphic modules $V_1,\dots,V_t\in$ indec kG, and positive integers $r_1,\dots,r_t$ such that*

$$(7.4a)\qquad (SW)^G\cong \coprod_{j=1}^{t} r_jSV_j.$$

Then for any $M\in$ indec kG there hold the following:

(i) $\delta(W)[W|M_H]=\sum r_j\delta(V_j)[V_j|M]$ *(sum over $j=1,\dots,t$),*

(ii) $W|M_H$ *if and only if* $M\cong V_j$ *for some* $j\in\{1,\dots,t\}$,

(iii) $[V_j|W^G]\geq r_j$, *for all* $j=1,\dots,t$, *and*

(iv) $[W|W_H^G]\geq \sum r_j[W|(V_j)_H]$ *(sum over $j=1,\dots,t$).*

Proof. (i) Evaluate both sides of (7.4a) at M, take dimensions, and use (1.5a). Notice that $(SW)^G(M)=(SW)(M_H)$, by (2.9).

(ii) From (i) it is clear that $[W|M_H]$ is positive if and only if $[V_j|M]$ is positive for some j. This proves (ii).

(iii) Take any $j\in\{1,\dots,t\}$. Then $r_j=[SV_j|(SW)^G]=[SV_j|Hd(SW)^G]\leq[V_j|W^G]$ by (2.14, ii).

(iv) By (iii), $\amalg r_jV_j$ (sum over $j=1,\dots,t$) is a summand of W^G. Therefore $\amalg r_j(V_j)_H$ is a summand of W_H^G, whence (iv).

(7.5) **Theorem.** *Let $W\in$ indec kH be such that $[W|W_H^G]=1$. Then there is a module $V\in$ indec kG such that:*

(i) $(SW)^G\cong SV$.

(ii) $[W|V_H]=1=[V|W^G]$.

(iii) *If $M\in$ indec kG, then $W|M_H$ only if $M\cong V$.*

(iv) $\Delta(W)\cong\Delta(V)$.

(v) *W, V have a vertex and source in common.*

(vi) *SW, SV have a vertex and source in common.*

Proof. By (7.3, i), $(SW)^G$ is semisimple. Therefore $(SW)^G$ can be written as (7.4a), and all the statements of (7.4) apply. In particular by (7.4, ii), $[W|(V_j)_H]\geq 1$ for all j.

(i) Put $[W|W_H^G]=1$ in (7.4, iv). We must conclude that $t=1$, $r_1=1$ and $[W|(V_1)_H]=1$. So we have $(SW)^G \cong SV$, with $V=V_1$.

(ii) We have just proved that $[W|V_H]=1$, and from (7.3, iii) $[V|W^G]\geq 1$. In fact $[V|W^G]=1$, since if $[V|W^G]\geq 2$, then W_H^G would have at least two summands W, against our hypothesis.

(iii) Equation (7.4, i) now reads $\delta(W)[W|M_H]=\delta(V)[V|M]$, from which (iii) follows.

(iv) If we put $M=V$ in the equation above, we get $\delta(W)=\delta(V)$, so that the division algebras $\Delta(W)\cong \operatorname{End}(SW)$, $\Delta(V)\cong \operatorname{End}(SV)$ have the same k-dimension. Therefore any (non-zero) k-algebra map $\mu: \Delta(W)\to\Delta(V)$ must be an isomorphism. Such a map μ exists, because the functor Ind_H^G: mmod $kH\to$ mmod kG provides a k-algebra map $(SW, SW)\to((SW)^G, (SW)^G)\cong(SV, SV)$.

(v) This follows at once from the 'module version' of (4.9), because we have from (ii) that $W|V_H$ and $V|W^G$.

(vi) This also follows from (4.9). Clearly $SV|(SW)^G$, since $SV\cong(SW)^G$ by (i). And $SW|(SV)_H$, since Mackey's formula (3.5f) shows that SW is a summand of $((SW)^G)_H$.

Remarks. Theorem 7.5 is largely a statement about modules, since its hypothesis $[W|W_H^G]=1$, and all of its conclusions except (i), (vi) refer to modules. It is easy to prove directly that there exists $V\in$ indec kG satisfying (ii), for if $W^G\cong V^{(1)}\amalg\cdots\amalg V^{(s)}$ ($V^{(j)}\in$ indec kG), the condition $[W|W_H^G]=1$ implies that $W|V_H^{(j)}$ for exactly one $j\in\{1,\ldots,s\}$. But (iii) is less trivial, and is related to the Burry–Carlson theorem (see (7.7, iii)). The following lemma will enable us to make this relation clearer.

(7.6) **Lemma.** *Let H be a subgroup of G, and let $W\in$ indec kH have vertex P and source $S\in$ indec kP. Define the* stabilizer $J(S)$ *of S in G, to be the subgroup consisting of all $g\in N_G(P)$ such that ${}^gS\cong S$. Assume that $H\geq J(S)$. Then $[W|W_H^G]=1$.*

Proof. By Mackey's formula $W_H^G\cong W\amalg\coprod_d(({}^dW)_{{}^dH\cap H})^H$, where d runs over the elements $d\notin H$ in some transversal of $H\backslash G/H$. Therefore if the lemma is false, there must be some $d\in G\setminus H$ such that $W|R^H$, where $R=({}^dW)_K$, $K={}^dH\cap H$. But $S|W_P$ because S is a source of W (see the proof of (7.7, i), below), and so $S|R_P^H\cong\coprod_e(({}^eR)_{{}^eK\cap P})^P$, with e running over a transversal of $P\backslash H/K$. Therefore there is some $e\in H$ such that $S|(({}^eR)_{{}^eK\cap P})^P$. Since P is the vertex of S, we must have ${}^eK\cap P=P$, which means that $P\leq{}^eK$, and we have $S|({}^eR)_P\cong({}^{ed}W)_P$. Now $W|S^H$, because S is a source of W, hence ${}^{ed}W|({}^{ed}S)^L$, where $L={}^{ed}H$. Therefore

$$S|({}^{ed}W)_P|({}^{ed}S)_P^L=\coprod_x(({}^{xed}S)_{{}^{xed}P\cap P})^P,$$

x running over a transversal of ${}^{ed}P\backslash L/P$. So there is some $x\in L$ such that $S|(({}^{xed}S)_{{}^{xed}P\cap P})^P$, and since P is the vertex of S we have ${}^{xed}P\cap P=P$, that is

${}^{xed}P=P$, and $S\,|\,{}^{xed}S$. This can happen only if $S\cong{}^{xed}S$, which means that $xed\in J(S)\le H$. So $ed(ed)^{-1}x(ed)\in H$. But $x\in L={}^{ed}H$ implies $(ed)^{-1}x(ed)$ is in H, as also is e. Hence $d\in H$, a contradiction which proves the lemma.

We collect our conclusions in the following theorem.

(7.7) **Theorem.** *Let $V\in$ indec kG have vertex P and source S, and let H be a subgroup of G.*

(i) *If $H\ge P$, then V_H has a component $W\in$ indec kH with vertex P and source S.*

(ii) *Assume now that $H\ge J(S)$, where $J(S)$ is the stabilizer of S in G (notice that $P\le J(S)\le N_G(P)$, by the definition of $J(S)$). Let W be any component of V_H which has vertex P and source S. Then all the conclusions* (i) *through* (vi) *of* (7.5) *hold. In particular $V\,|\,W^G$, and V is uniquely characterized (up to kG-isomorphism) by the properties $V\in$ indec kG, $W\,|\,V_H$.*

(iii) *(A special case of* (ii).) *If $H\ge N_G(P)$ and if $W=fV$, then all the conclusions* (i) *through* (vi) *of* (7.5) *hold.*

Proof. (i) V_P has a component S_0 with vertex P [11, Theorem 6(2)], and by [11, Theorem 6(3)], S_0 is a source of V. By [11, Theorem 5] ${}^{x}S_0=S$ for some $x\in N_G(P)$. Now $S_0\,|\,V_P$ implies $S\,|\,{}^{x}(V_P)\cong V_P$. Since $S\,|\,V_P$ and $H\ge P$, there must be some component W of V_H such that $S\,|\,W_P$. From $W\,|\,V_H$ and $S\,|\,W_P$ follow that P is a vertex of W, hence S is a source of W by [11, Theorem 6(3)].

(ii) (7.6) shows that $[W\,|\,W^G_H]=1$, hence by (7.5) there is some $V_0\in$ indec kG such that (i) through (vi) of (7.5) hold. But $V\cong V_0$, by (7.5, iii).

(iii) If $H\ge N_G(P)$, then $W=fV\in$ indec kH is determined uniquely up to isomorphism by the two properties (1) $W\,|\,V_H$, (2) W has vertex P. By (i), W must also have source S. Now we apply (ii), and thus complete the proof of (7.7).

Remarks. (1) If V is projective, then $J(S)=G$ and the theorem is vacuous.

(2) In case $H\ge N_G(P)$, the parts of Theorem (7.7) which relate to modules, are due to Burry and Carlson [7].

(3) The module W is not uniquely determined up to isomorphism by V, in general, although this is so if $H\ge N_G(P)$.

We now prove the theorem on almost split sequences which was mentioned in the Introduction.

(7.8) **Theorem.** *Let $V\in$ indec kG be non-projective, with vertex P as source S, and let H be a subgroup of G which contains $J(S)$. Let $W\in$ indec kH be any component of V_H which has vertex P and source S. If $0\to W_2\to W_1\to W\to 0$ is an almost split sequence in* mod kH, *then $0\to W_2^G\to W_1^G\to W^G\to 0$ is an exact sequence in* mod kG, *which is isomorphic to the direct sum of an almost split sequence $0\to V_2\to V_1\to V\to 0$*

with a split exact sequence $0 \to L \to L \oplus M \to M \to 0$, *for some* L, M *in* mod kG.

In particular this holds if $H \geq N_G(P)$ *and* $W \cong fV$.

Proof. Since $0 \to W_2 \xrightarrow{f} W_1 \xrightarrow{g} W \to 0$ is almost split, we have a sequence

$$0 \to (\cdot, W_2) \xrightarrow{(\cdot, f)} (\cdot, W_1) \xrightarrow{(\cdot, g)} (\cdot, W) \to SW \to 0$$

in mmod kH which is a minimal projective resolution of SW (see Section 1). Apply Ind_H^G to this, and then we use the isomorphisms $(\cdot, W_2)^G \cong (\cdot, W_2^G)$, etc. of (2.12), together with the isomorphism $(SW)^G \cong SV$ from (7.7). We get an exact sequence

$$(*) \qquad 0 \to (\cdot, W_2^G) \xrightarrow{(\cdot, f^G)} (\cdot, W_1^G) \xrightarrow{(\cdot, g^G)} (\cdot, W^G) \to SV \to 0$$

in mmod kG, which is therefore a projective resolution, not necessarily minimal, of SV. Standard arguments (see [1, Proposition 4.9(b)]) show that (*) is isomorphic to the direct sum of the minimal projective resolution $0 \to (\cdot, V_2) \to (\cdot, V_1) \to (\cdot, V) \to SV \to 0$ of SV, with a projective resolution of zero. This last is easily shown to have the form $0 \to (\cdot, L) \to (\cdot, L \oplus M) \to (\cdot, M) \to 0 \to 0$, for some $L, M \in \mathrm{mod}\, kG$. Theorem (7.8) now follows.

Remark. This theorem is certainly well-known, at least in case $H \geq N_G(P)$ - see for example Webb [16, Theorem 3.2(ii)]. Benson and Parker's 'atom-copying theorem' [6, Theorem 11.2] is equivalent to (7.7) in case $H \geq N_G(P)$.

We end this section with an elementary proposition on restriction to a normal subgroup.

(7.9) **Proposition.** *Let* $V \in \mathrm{indec}\, kG$, $H \trianglelefteq G$ *and let* $W_1, \ldots, W_n \in \mathrm{indec}\, kH$ *be such that every component of* V_H *is isomorphic to* W_i *for exactly one* $i \in \{1, \ldots, n\}$. *We have then*

(i) $V_H \cong s_1 W_1 \amalg \cdots \amalg s_n W_n$, *where* $s_i = [W_i \mid V_H]$.

(ii) $(SV)_H \cong r_1(SW_1) \amalg \cdots \amalg r_n(SW_n)$, *where* $r_i = (\delta(V)/\delta(W_i))[V \mid W_i^G]$.

(iii) *If* V *is H-projective, then* $\{W_1, \ldots, W_n\}$ *is a single G-conjugacy class, hence by* (2.14, iii), $\{SW_1, \ldots, SW_n\}$ *is a single G-conjugacy class.*

(iv) *If* SV *is H-projective, then* SV *has a vertex and source in common with* SW, *for any* $W \in \{W_1, \ldots, W_n\}$.

Proof. (i) is just the definition of the s_i.

(ii) By (7.3, ii), $(SV)_H$ is semisimple. If $W \in \mathrm{indec}\, kH$ is not isomorphic to any of $W_1, \ldots, W_n$, then by (2.14, i), $[SW \mid (SV)_H] \leq [W \mid V_H] = 0$. The multiplicity r_i of SW_i in $(SV)_H$ is found from

$$r_i \delta(W_i) = \dim(SV)_H(W_i) = \dim(SV)(W_i^G) = \delta(V)[V \mid W_G^i].$$

(iii) In this case $V \mid V_H^G$, hence $V \mid W_i^G$ for some i; result follows.

(iv) Since SV is H-projective, it has vertex and source in common with some indecomposable summand of $(SV)_H$ [11, Theorem 6]. But all these summands are G-conjugate to SW, by (ii).

8. Vertices of simple functors. Examples

This section contains some remarks on the problem: given $V \in \operatorname{indec} kG$ with vertex P and source $S \in \operatorname{indec} kP$, to find a vertex Q and source $T \in \operatorname{mmod} kQ$ for the simple functors $SV \in \operatorname{mmod} kG$. We know from (4.7) that Q is a p-subgroup of G, and from (5.12) that we may assume $Q \geq P$. Also (7.7) shows that there is an indecomposable component W of $V_{J(S)}$ having vertex P and source S, and such that $SV \cong (SW)^G$, and any source and vertex of SW are source and vertex of SV. For this reason we shall assume henceforth that $J(S) = G$, that is, we take $V \in \operatorname{indec} kG$ to be a module with vertex $P \trianglelefteq G$, and source S which is G-stable (i.e. ${}^gS \cong S$ for all $g \in G$).

(8.1) **Example.** Let V, P, S be as just given, and assume also that G is a p-group, and that k is algebraically closed. We shall prove that the vertex of SV is G.

If this is not true, there must be a maximal subgroup H of G such that SV is H-projective. By (5.12), V must be H-projective too, which implies that $H \geq P$.

We have $V \cong S^G$, since $V | S^G$ and S^G is indecomposable (see for example [11, Theorem 8, p. 438]). Then

$$V_H \cong (S^G)_H \cong \coprod_d (({}^dS)_{{}^dP \cap H})^H,$$

where d runs over a transversal of the cosets dH in G. But ${}^dP = P$, ${}^dS \cong S$ for all $d \in G$. So we find $V_H = pW$, where $W = S^H$ and $p = (G:H)$. Now from (7.9, ii) we get $(SV)_H \cong SW$, since $\delta(V) = \delta(W) = [V | W^G] = 1$ (the last equality holds because $V \cong W^G$). Since SV is H-projective, we have $SV | (SV)_H^G \cong (SW)^G$. But $(SW)^G$ is indecomposable, since it is a proper epimorphic image of $(\cdot, W^G) \cong (\cdot, V)$ and hence, like $(\cdot, V)$, has unique maximal subfunctor. Therefore $SV \cong (SW)^G$. But this leads to a contradiction, namely

$$1 = \dim(SV)(V) = \dim(SW)^G(V) = \dim(SW)(V_H) = p$$

(we are using (1.5b) and the fact that $V_H \cong p \cdot W$). This proves that the vertex of SV must be G. If we combine this with the argument at the beginning of this section, we have the following

(8.2) **Theorem.** *Let G be a p-group and k an algebraically closed field of characteristic p. If $V \in \operatorname{indec} kG$ has vertex P and source S, then the simple functor $SV \in \operatorname{mmod} kG$ has vertex $J(S)$ (the stabilizer of S in G), and source SW, for any indecomposable summand W of $V_{J(S)}$ which has vertex P and source S.*

(8.3) **Example.** Let $V \in \operatorname{mod} kG$ be projective and indecomposable. Then V has vertex $P=\{1\}$, and its source $S=k_{\{1\}}$, that is, $S=k$, regarded as trivial $k\{1\}$-module. Clearly $P \trianglelefteq G$, and S is G-stable.

In order to find a vertex of SV, we first consider an arbitrary $F \in \operatorname{mmod} kG$, and an arbitrary subgroup H of G, and give a procedure to decide whether F is H-projective or not. Take any projective resolution of F

$$(8.4) \qquad 0 \to (\cdot, V_2) \xrightarrow{(\cdot, f)} (\cdot, V_1) \xrightarrow{(\cdot, g)} (\cdot, V) \xrightarrow{\alpha} F \to 0,$$

so that V, V_1, V_2, f, g all belong to $\operatorname{mod} kG$. Apply the functor Res_H^G, which gives the exact sequence (8.5) in $\operatorname{mmod} kH$.

$$\begin{array}{lccccccccc}
(8.5) & 0 \longrightarrow & (\cdot, V_2)_H & \xrightarrow{(\cdot, f)_H} & (\cdot, V_1)_H & \xrightarrow{(\cdot, g)_H} & (\cdot, V)_H & \xrightarrow{\alpha_H} & F_H & \longrightarrow 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow & \\
(8.6) & 0 \longrightarrow & (\cdot, V_{2H}) & \xrightarrow{(\cdot, f_H)} & (\cdot, V_{1H}) & \xrightarrow{(\cdot, g_H)} & (\cdot, V_H) & \xrightarrow{\alpha'_H} & F_H & \longrightarrow 0
\end{array}$$

If we define $\alpha'_H = \alpha_H \cdot a(\cdot, V)$, where $a(\cdot, V): (\cdot, V_H) \to (\cdot, V)_H$ is the isomorphism defined at the beginning over Section 6, we have another sequence (8.6); moreover the diagram (8.5)–(8.6) whose vertical arrows are $a(\cdot, V_2), a(\cdot, V_1), a(\cdot, V)$ and 1_{F_H}, commutes. This shows that (8.6) is exact, and is, therefore, a projective resolution of F_H in $\operatorname{mmod} kH$. If η is any element of $\operatorname{End} F_H = (F_H, F_H)$, a standard argument for projective resolutions shows that there exist elements $h \in \operatorname{End} V$, $h_1 \in \operatorname{End} V_1$ and $h_2 \in \operatorname{End} V_2$ such that diagram (8.7) commutes.

$$\begin{array}{lccccccccc}
 & 0 \longrightarrow & (\cdot, V_{2H}) & \xrightarrow{(\cdot, f_H)} & (\cdot, V_{1H}) & \xrightarrow{(\cdot, g_H)} & (\cdot, V_H) & \xrightarrow{\alpha'_H} & F_H & \longrightarrow 0 \\
(8.7) & & \downarrow{\scriptstyle (\cdot, h_2)} & & \downarrow{\scriptstyle (\cdot, h_1)} & & \downarrow{\scriptstyle (\cdot, h)} & & \downarrow{\scriptstyle \eta} & \\
 & 0 \longrightarrow & (\cdot, V_{2H}) & \xrightarrow[(\cdot, f_H)]{} & (\cdot, V_{1H}) & \xrightarrow[(\cdot, g_H)]{} & (\cdot, V_H) & \xrightarrow[\alpha'_H]{} & F_H & \longrightarrow 0
\end{array}$$

(8.8) **Lemma.** *If the diagram* (8.7) *commutes, then so does the diagram* (8.9), *below.*

$$\begin{array}{lccccccccc}
 & 0 \longrightarrow & (\cdot, V_2) & \xrightarrow{(\cdot, f)} & (\cdot, V_1) & \xrightarrow{(\cdot, g)} & (\cdot, V) & \xrightarrow{\alpha} & F & \longrightarrow 0 \\
(8.9) & & \downarrow{\scriptstyle (\cdot, th_2)} & & \downarrow{\scriptstyle (\cdot, th_1)} & & \downarrow{\scriptstyle (\cdot, th)} & & \downarrow{\scriptstyle T\eta} & \\
 & 0 \longrightarrow & (\cdot, V_2) & \xrightarrow[(\cdot, f)]{} & (\cdot, V_1) & \xrightarrow[(\cdot, g)]{} & (\cdot, V) & \xrightarrow[\alpha]{} & F & \longrightarrow 0
\end{array}$$

In this diagram, th_2 stands for $t_H^G(h_2)$, th_1 for $t_H^G(h_1)$, th for $t_H^G(h)$ and $T\eta$ for $T_H^G(\eta)$.

Proof. By hypothesis the first 'square' of (8.7) commutes, which implies $h_1 f_H = f_H h_2$. Apply t_H^G to this equation. Since f is a kG-map, we get $t_H^G(h_1)f = f t_H^G(h_2)$, which shows that the first square of (8.9) commutes. Similarly, the second square of (8.9) commutes. For the third square of (8.7), our hypothesis gives $\eta\alpha'_H = \alpha'_H \cdot (\cdot, h)$, which can be rewritten as $\eta\alpha_H = \alpha_H \xi$, where $\xi = a(\cdot, V)\cdot(\cdot, h)\cdot a(\cdot, V)^{-1}$. Apply T_H^G to the first of these equations, and use (5.8, ii). We get $T_H^G(\eta)\alpha = \alpha T_H^G(\xi)$. But it follows at once from (6.4) that $T_H^G(\xi) = (\cdot, t_H^G(h))$, hence the third square of (8.9) commutes, and this completes the proof of (8.8).

Lemma (8.8) leads to the following criterion for F to be H-projective.

(8.10) **Criterion.** *Suppose that $F \in \mathrm{mmod}\, kG$, and that H is a subgroup of G. Let* (8.4) *be a* minimal *projective resolution of F. Then F is H-projective if and only if there exists an element $h \in \mathrm{End}\, V_H$ which satisfies the following two conditions:*

(1) *$(\cdot, h)$ maps $\mathrm{Ker}\, \alpha'_H = \mathrm{Im}(\cdot, g_H)$ into itself – this means that the k-map (Y, h) maps $\mathrm{Ker}\, \alpha'_H(Y) = \mathrm{Im}(Y, g_H)$ into itself, for all $Y \in \mathrm{mod}\, kH$.*

(2) *$t_H^G(h)$ is an automorphism of V.*

Proof. We know from (5.11) that F is H-projective if and only if there exists an element $\eta \in \mathrm{End}\, F_H$ such that $T_H^G(\eta) = 1_F$. Suppose first that we have such an η. We then construct the commutative diagram (8.7), from which it is clear that h satisfies condition (1). But from (8.8) we know that the diagram (8.9) also commutes, and so by the minimality of (8.4) and the fact that $T_H^G(\eta) = 1_F$, it follows that all the vertical arrows in (8.9) are isomorphisms; in particular condition (2) holds.

Conversely suppose we are given $h \in \mathrm{End}\, V_H$ which satisfies (1) and (2). Condition (1) ensures that there is some η in $\mathrm{End}\, F_H$ to make the third square in (8.7) commute; we may then find h_1, h_2 in the usual way, so that (8.7) is a commutative diagram. By Lemma (8.8), (8.9) is also commutative. But condition (2) forces $T_H^G(\eta) = \gamma$ to be an automorphism of F, hence F is H-projective since $T_H^G(\eta\gamma_H^{-1}) = T_H^G(\eta)\gamma^{-1} = 1_F$.

(8.11) **Corollary to this proof.** *If F is H-projective and if* (8.4) *is minimal, then the modules V_1, V_2 and V are all H-projective. In particular if $0 \to V_1 \to V_2 \to V \to 0$ is an almost split sequence in* $\mathrm{mod}\, kG$ *and if Q is a vertex of SV, then V_1, V_2 and V are all Q-projective* (this provides a second proof of (5.12)).

Proof. If F is H-projective, we can find kH-endomorphisms h_1, h_2, h of V_1, V_2, V such that (8.9) commutes, with $T\eta = 1_F$. Then $t_H^G(h_1), t_H^G(h_2), t_H^G(h)$ are all automorphisms by the minimality of (8.4), hence V_1, V_2, V are all H-projective by Higman's theorem. The second part of (8.11) follows by taking $F = SV$.

We return to our example, where $V \in \mathrm{indec}\, kG$ is projective and $F = SV$. By [5, Proposition 3.1a] or [10, Theorem 1.3], the sequence

$$0 \to 0 \to (\cdot, \mathbf{r}V) \xrightarrow{(\cdot, g)} (\cdot, V) \to SV \to 0$$

is a minimal projective resolution of SV, where g is the inclusion of $\mathbf{r}V$ in V.

Let H be a subgroup of G. According to (8.10), SV is H-projective if and only if there is some $h \in \operatorname{End} V_H$ which satisfies conditions (1), (2). Condition (1) requires that for any $Y \in \operatorname{mod} kH$ and any $f \in (Y, V_H)$

$$(*) \qquad f \in \operatorname{Im}(Y, g_H) \Rightarrow hf \in \operatorname{Im}(Y, g_H).$$

It is clear that $f \in \operatorname{Im}(Y, g_H)$ if and only if $\operatorname{Im} f \leq \mathbf{r}V$. Taking $Y = (\mathbf{r}V)_H$, $f = g_H$ in $(*)$, we see that h must map $\mathbf{r}V$ into itself; conversely if h maps $\mathbf{r}V$ into itself, then h satisfies $(*)$ for all Y, f. So $h \in \operatorname{End} V_H$ satisfies (1), if and only if $h(\mathbf{r}V) \leq \mathbf{r}V$.

Suppose that SV is H-projective, and that $h \in \operatorname{End} V_H$ satisfies (1), (2). Since $h(\mathbf{r}V) \leq \mathbf{r}V$, h induces a map $\theta(h)$ on the simple kG-module $M = V/\mathbf{r}V$. It is easy to check that $\theta(h) \in \operatorname{End} M_H$, and that $t_H^G(\theta(h)) = \theta(t_H^G(h))$, hence by (2) $t_H^G(\theta(h))$ is an automorphism of M. Therefore M is H-projective.

Conversely suppose that $M = V/\mathbf{r}V$ is H-projective, and that $\eta \in \mathit{End}\, M_H$ satisfies $t_H^G(\eta) = 1_M$. Because V_H is projective, we may 'lift' any $\eta' \in \operatorname{End} M_H$ to some $h' \in \operatorname{End} V_H$, so that the diagram (8.12) commutes.

$$(8.12) \qquad \begin{array}{ccccc} V_H & \longrightarrow & M_H & \longrightarrow & 0 \\ {\scriptstyle h'} \downarrow & & \downarrow {\scriptstyle \eta' = \theta(h')} & & \\ V_H & \longrightarrow & M_H & \longrightarrow & 0 \end{array}$$

By this diagram it is clear that h' maps $\mathbf{r}V$ into itself, i.e. h' satisfies (1). If now h is the lift of our kH-endomorphism η, we have $\theta(t_H^G(h)) = t_H^G(\eta) = 1_M$. This proves that $t_H^G(h)$ is a non-nilpotent endomorphism of V, hence by Fitting's lemma is an automorphism of V. But this shows that h satisfies both conditions (1) and (2), therefore SV is H-projective by (8.10). We have proved that SV is H-projective if and only if $M = V/\mathbf{r}V$ is H-projective. This gives the theorem below.

(8.13) **Theorem.** *If $V \in \operatorname{mod} kG$ is projective and indecomposable, then the vertices of the simple functor SV coincide with those of the simple module $M = V/\mathbf{r}V$.*

9. Appendix

We sketch here a proof of Theorem 1.4. This proof is essentially that deducible from Auslander [3, p. 281]. (It also works when k is replaced by a complete discrete valuation ring, cf. Roggenkamp–Schmidt [15, pp. 904, 905].)

Given any $M \in \operatorname{mod} \Lambda$, we set up two maps $\alpha: \mathbf{A} \to \mathbf{B}$, $\beta: \mathbf{B} \to \mathbf{A}$, where $\mathbf{A}$ is the set of all subfunctors F of $(\cdot, M)$, and $\mathbf{B}$ is the set of all right ideals R of $\operatorname{End} M$. Namely if $F \in \mathbf{A}$, we set $\alpha(F) = F(M)$, which is a right ideal of $\operatorname{End} M$ (see remark preceding Theorem (1.5)), and if $R \in \mathbf{B}$, we define $\beta(R) \leq (\cdot, M)$ by

$$\beta(R)(X) = \{f \in (X,M) \mid fg \in R \text{ for all } g \in (M,X)\},$$

for all $X \in \operatorname{mod} \Lambda$. (Of course, one must check that this does define a subfunctor $\beta(R)$ of $(\cdot, M)$.) With this notation, Theorem (1.4) reads

(9.1) $\quad \mathbf{r}(\cdot, M) = \beta(\operatorname{rad} \operatorname{End} M)$, *for any* $M \in \operatorname{mod} \Lambda$.

To prove (9.1), first verify the following:

(i) $F \le \beta(\alpha(F))$, *for all* $F \in \mathbf{A}$.
(ii) $R = \alpha(\beta(R))$, *for all* $R \in \mathbf{B}$.
(iii) *If* $\alpha(F) = (M,M)$, *then* $F = (\cdot, M)$.
(iv) *If* F *is maximal in* $(\cdot, M)$, *then* $\beta(\alpha(F)) = F$.
(v) *If* R *is maximal* (*as right ideal*) *in* End M, *then* $\beta(R)$ *is maximal in* $(\cdot, M)$.
(vi) β *commutes with intersections.*

From (i)–(v) one finds that β induces a *bijection* between the sets $\mathbf{B}_{\max}$ of all maximal right ideals of End M, and $\mathbf{A}_{\max}$ of all maximal subfunctors of $(\cdot, M)$. So using (vi) we have

$$\beta(\operatorname{rad} \operatorname{End} M) = \beta\left(\bigcap \mathbf{B}_{\max}\right) = \bigcap \beta(\mathbf{B}_{\max}) = \bigcap \mathbf{A}_{\max} = \mathbf{r}(\cdot, M),$$

and (9.1) is proved.

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