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LENGTHS OF CERTAIN GENERALIZED FRACTIONS

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Dedicated to Jan-Erik Roos on his 50-th birthday

1. Introduction

Throughout, let A be a (commutative Noetherian) local ring (with identity) of dimension $d>0$, having maximal ideal $\mathfrak{m}$. For an $\mathfrak{m}$-primary ideal $\mathfrak{q}$ of A, we shall denote the multiplicity of $\mathfrak{q}$ by $e(\mathfrak{q})$ or $e_A(\mathfrak{q})$; if $x_1, \dots, x_d$ form a system of parameters ('s.o.p.' will be used as an abbreviation for 'system of parameters', and 's.s.o.p.' will stand for 'subset of a system of parameters') for A, we shall use $e(x_1, \dots, x_d)$ to denote $e(\sum_{i=1}^d Ax_i)$. The length of an A-module M will be denoted by $l(M)$ or $l_A(M)$.

It is well known that, for an $\mathfrak{m}$-primary ideal $\mathfrak{q}$, for all large values of the positive integer n, the expression $l(A/\mathfrak{q}^n)$ takes the same values as a rational polynomial in n of degree d having leading coefficient $e(\mathfrak{q})/d!$, so that

$$e(\mathfrak{q}) = \lim_{n\to\infty} \frac{l(A/\mathfrak{q}^n)}{n^d/d!}.$$

In the case when $\mathfrak{q} = \sum_{i=1}^d Ax_i$ (so that the elements $x_1, \dots, x_d$ form a s.o.p. for A), there is also a related limit formula due to C. Lech [2, Theorem 2], which states that

$$e(x_1, \dots, x_d) = \lim_{\min\{n_1, \dots, n_d\}\to\infty} \frac{l(A/(\sum_{i=1}^d Ax_i^{n_i}))}{n_1 \cdots n_d}.$$

This formula therefore raises the following.

Question 1.1. Given $x_1, \dots, x_d$ which form a s.o.p. for A, does there exist a polynomial $g \in \mathbb{Q}[X_1, \dots, X_d]$ (where $X_1, \dots, X_d$ are indeterminates) of total degree d, linear in each X_i and having homogeneous component of degree d equal to $e(x_1, \dots, x_d)X_1 \cdots X_d$, such that, provided $n_1, \dots, n_d$ are all sufficiently large,

$$l\left(A\bigg/\left(\sum_{i=1}^d Ax_i^{n_i}\right)\right) = g(n_1, \dots, n_d)\,?$$

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This question certainly has an affirmative answer in the case when A is Cohen-Macaulay, for then [4, p. 311], for all positive integers $n_1, \ldots, n_d$,

$$l\left(A \Big/ \sum_{i=1}^{d} Ax_i^{n_i}\right) = e(x_1^{n_1}, \ldots, x_d^{n_d}) = e(x_1, \ldots, x_d)n_1 \cdots n_d,$$

and also in the case when A is a generalized Cohen-Macaulay local ring [8, §3], for then [8, (3.3) and (3.7)], provided the integers $n_1, \ldots, n_d$ are all sufficiently large,

$$l\left(A \Big/ \sum_{i=1}^{d} Ax_i^{n_i}\right) = e(x_1, \ldots, x_d)n_1 \cdots n_d + \sum_{i=0}^{d-1} \binom{d-1}{i} l(H_{\mathfrak{m}}^i(A)).$$

(Here, $H_{\mathfrak{m}}^i(A)$ denotes the i-th local cohomology module of A with respect to $\mathfrak{m}$.)

However, Question 1.1 does not always have an affirmative answer: D. Kirby pointed out to the authors that the local ring

$$B = K[[X_1, X_2, X_3]]/(X_1X_3, X_2X_3),$$

where K is a field and X_1, X_2, X_3 are indeterminates, provides a situation where the answer is negative: if we let, for $i = 1, 2, 3$, t_i denote the natural image of X_i in B, then $(t_1 + t_3)$, $(t_2 + t_3)$ form an s.o.p. for B and, for all positive integers n_1, n_2,

$$l_B(B/((t_1 + t_3)^{n_1}, (t_2 + t_3)^{n_2})) = n_1n_2 + \min\{n_1, n_2\}.$$

This paper is concerned with a similar, and sometimes related, question. In [11, 3.5], a description of $H_{\mathfrak{m}}^d(A)$ was given in terms of the modules of generalized fractions introduced in [10]. Write

$$U(A)_{d+1} = \{(x_1, \ldots, x_d, 1) \in A^{d+1}: \text{there exists } j \text{ with } 0 \le j \le d \text{ such that } x_1, \ldots, x_j \text{ form an s.s.o.p. for } A \text{ and } x_{j+1} = \cdots = x_d = 1\}.$$

Then $U(A)_{d+1}$ is a triangular subset [10, 2.1] of A^{d+1}, and $H_{\mathfrak{m}}^d(A) \cong U(A)_{d+1}^{-d-1}A$, the module of generalized fractions of A with respect to $U(A)_{d+1}$. Let $x_1, \ldots, x_d$ form a s.o.p. for A, and let $n_1, \ldots, n_d \in \mathbb{N}$ (we use $\mathbb{N}$ to denote the set of positive integers). Consider the generalized fraction $1/(x_1^{n_1}, \ldots, x_d^{n_d}, 1) \in U(A)_{d+1}^{-d-1}A$: by [10, 3.3(ii)], it is annihilated by $\sum_{i=1}^{d} Ax_i^{n_i}$, and so the cyclic submodule of $U(A)_{d+1}^{-d-1}A$ generated by this generalized fraction has

$$l(A(1/(x_1^{n_1}, \ldots, x_d^{n_d}, 1))) \le l\left(A \Big/ \sum_{i=1}^{d} Ax_i^{n_i}\right) < \infty.$$

(We call $l(A(1/(x_1^{n_1}, \ldots, x_d^{n_d}, 1)))$ the *length* of the generalized fraction $1/(x_1^{n_1}, \ldots, x_d^{n_d}, 1)$.) This paper is concerned with the following question.

Question 1.2. Given $x_1, \ldots, x_d$ which form a s.o.p. for A, does there exist a polynomial $h \in \mathbb{Q}[X_1, \ldots, X_d]$ such that, provided $n_1, \ldots, n_d$ are all sufficiently large,

$$l(A(1/(x_1^{n_1}, \ldots, x_d^{n_d}, 1))) = h(n_1, \ldots, n_d)\ ?$$

It is perhaps worth pointing out the following intriguing link between Questions

1.1 and 1.2. Let $x_1, \dots, x_d$ form a s.o.p. for A. It is well known that $H^d_{\mathfrak{m}}(A)$ may be viewed as a direct limit of the modules $A/(\sum_{i=1}^d Ax_i^{n_i})$ (where $n_1, \dots, n_d \in \mathbb{N}$); on the other hand, $U(A)_{d+1}^{-d-1}A$, which is isomorphic to $H^d_{\mathfrak{m}}(A)$, is, by [11, 3.6], the union of its cyclic submodules $A(1/(x_1^{n_1}, \dots, x_d^{n_d}, 1))$.

Question 1.2 has an affirmative answer in the case when A is Cohen–Macaulay, for in that case it is a consequence of the exactness theorem [12, 3.15] that the annihilator of $1/(x_1^{n_1}, \dots, x_d^{n_d}, 1)$ (where $x_1, \dots, x_d$ form an s.o.p. for A and $n_1, \dots, n_d \in \mathbb{N}$) is exactly $\sum_{i=1}^d Ax_i^{n_i}$, so that

$$l(A(1/(x_1^{n_1}, \dots, x_d^{n_d}, 1))) = l\left(A\Big/\sum_{i=1}^d Ax_i^{n_i}\right) = e(x_1, \dots, x_d)n_1 \cdots n_d.$$

(Incidentally, O'Carroll has provided, in [5, §3], a simpler proof of the exactness theorem which does not require the underlying ring to be Noetherian.)

At the time of writing, we do not know the complete answer to Question 1.2. We do know that the answer is affirmative in the special case when $d=2$ (or 1), so that for 2-dimensional local rings the situation concerning this question is perhaps more satisfactory than that for Question 1.1. We also know that Question 1.2 has an affirmative answer in the special case in which A is a generalized Cohen–Macaulay local ring (of arbitrary (positive) dimension). Below, we present our proofs of these results.

2. Preliminaries

When discussing modules of generalized fractions, we shall use the notation of [10] and [11], except that we shall use slightly different notation concerning matrices, in that round brackets will now be used instead of square ones, we shall agree to use n-tuples $(u_1, \dots, u_n)$ of elements of R (where R is a commutative ring) and $1 \times n$ row matrices $(u_1 \ \cdots \ u_n)$ over R interchangeably, and we shall cease to use boldface letters to denote matrices. We still use $^{\mathrm{T}}$ to denote matrix transpose and, for $i \in \mathbb{N}$, $D_i(R)$ to denote the set of all $i \times i$ lower triangular matrices over R; we shall, in addition, use $M_i(R)$ to denote the set of all $i \times i$ matrices over R.

Our first lemma is concerned with the standard technique of factoring out $H^0_{\mathfrak{m}}(A)$; it will enable us, in some of our proofs, to reduce to the case where $\operatorname{depth} A > 0$.

Lemma 2.1. *Let $\mathfrak{b}$ be the annihilator of an $\mathfrak{m}$-primary ideal of A, let $\bar{A} = A/\mathfrak{b}$, and let $\bar{\ }: A \to \bar{A}$ be the natural ring homomorphism. Let $\bar{\mathfrak{m}} = \mathfrak{m}/\mathfrak{b}$, the maximal ideal of $\bar{A}$. Let $x_1, \dots, x_d \in A$.*

(i) *The local ring $\bar{A}$ again has dimension d, and, as A-modules,*

$$H^i_{\bar{\mathfrak{m}}}(\bar{A}) \cong H^i_{\mathfrak{m}}(A) \quad \textit{for all } i \in \mathbb{N}.$$

(ii) *The elements $x_1, \dots, x_d$ form a s.o.p. for A if and only if $\bar{x}_1, \dots, \bar{x}_d$ form a* *s.o.p. for* $\bar{A}$; *when this is the case,*

$$e_{\bar{A}}(\bar{x}_1, \dots, \bar{x}_d) = e_A(x_1, \dots, x_d).$$

(iii) *There is an isomorphism of A-modules*

$$\theta : U(A)_{d+1}^{-d-1}A \to U(\bar{A})_{d+1}^{-d-1}\bar{A}$$

which is such that, for $a \in A$ *and* $(u_1, \dots, u_d, 1) \in U(A)_{d+1}$,

$$\theta(a/(u_1, \dots, u_d, 1)) = \bar{a}/(\bar{u}_1, \dots, \bar{u}_d, \bar{1}).$$

Proof. (i) That $\dim \bar{A} = d$ comes from the fact that $\mathfrak{b}$ is contained in every prime ideal of A. Since $\mathrm{Supp}(\mathfrak{b}) \subseteq \{\mathfrak{m}\}$, we have $H_{\mathfrak{m}}^i(\mathfrak{b}) = 0$ for all $i \in \mathbb{N}$; hence $H_{\mathfrak{m}}^i(\bar{A}) \cong H_{\mathfrak{m}}^i(A)$ for all $i \in \mathbb{N}$; and the claim now follows from [9, 4.3].

(ii) The first claim is easy, and the second can be proved by, for example, use of [4, Theorem 5 on p. 302, Proposition 5 on p. 307, and (7.4.3) on p. 300].

(iii) In view of the first part of (ii), it is routine to check that there is an A-epimorphism $\theta : U(A)_{d+1}^{-d-1}A \to U(\bar{A})_{d+1}^{-d-1}\bar{A}$ given by the formula in the statement of the lemma, and it remains to show that this θ is injective. So let $a \in A$, $(u_1, \dots, u_d, 1) \in U(A)_{d+1}$ be such that, in $U(\bar{A})_{d+1}^{-d-1}\bar{A}$, $\bar{a}/(\bar{u}_1, \dots, \bar{u}_d, \bar{1}) = 0$. By [10, 3.3(ii)], we may assume that $u_1, \dots, u_d$ form a s.o.p. for A. There exist $(y_1, \dots, y_d, 1) \in U(A)_{d+1}$ and $H^* \in D_{d+1}(\bar{A})$ such that

$$H^*(\bar{u}_1, \dots, \bar{u}_d, \bar{1})^{\mathrm{T}} = (\bar{y}_1, \dots, \bar{y}_d, \bar{1})^{\mathrm{T}} \quad \text{and} \quad |H^*|\, \bar{a} \in \sum_{i=1}^{d} \bar{A}\bar{y}_i.$$

For $P = (p_{ij}) \in M_{d+1}(A)$, we use $\bar{P}$ to denote the matrix $(\bar{p}_{ij})$ of $M_{d+1}(\bar{A})$. There exists $H \in D_{d+1}(A)$ such that $\bar{H} = H^*$ and

$$H(u_1, \dots, u_d, 1)^{\mathrm{T}} = (y_1', \dots, y_d', 1)^{\mathrm{T}}$$

for suitable $y_1', \dots, y_d' \in A$. By part (ii), $y_1', \dots, y_d'$ form a s.o.p. for A. Also, there exists $n \in \mathbb{N}$ such that $(\sum_{i=1}^{d} A y_i'^{\,n})\mathfrak{b} = 0$: let D denote the diagonal matrix $\mathrm{diag}(y_1'^{\,n}, \dots, y_d'^{\,n}, 1) \in D_{d+1}(A)$. Then

$$DH(u_1, \dots, u_d, 1)^{\mathrm{T}} = (y_1'^{\,n+1}, \dots, y_d'^{\,n+1}, 1)^{\mathrm{T}}.$$

Moreover, $|H|\, a \in \sum_{i=1}^{d} A y_i' + \mathfrak{b}$; thus

$$|DH|\, a \in \sum_{i=1}^{d} A y_i'^{\,n+1}$$

and $a/(u_1, \dots, u_d, 1) = 0$ in $U(A)_{d+1}^{-d-1}A$. Thus θ is injective and the proof is complete.

Note. We are grateful to the referee for pointing out the above proof of (iii) to us; our original proof was considerably longer.

In the situation where $\mathrm{depth}\, A > 0$, we shall sometimes wish to factor out by an ideal generated by a non-zerodivisor $x_1 \in \mathfrak{m}$, and, when doing so, we shall make use of the following fact from [13, §2].

Proposition 2.2. (See [13, 2.4 and 2.7].) *Suppose that $d \geq 2$. Let $x_1 \in \mathfrak{m}$ be a non-zerodivisor on A (so that x_1 automatically forms a s.s.o.p. for A). Let $\bar{A} = A/Ax_1$ and let $^- : A \to \bar{A}$ denote the natural ring homomorphism. There is a homomorphism*

$$\eta^{d+1} : U(\bar{A})_d^{-d}\bar{A} \to U(A)_{d+1}^{-d-1}A$$

which is such that, for all $\bar{a} \in \bar{A}$ and $(\bar{y}_2, \dots, \bar{y}_d, \bar{1}) \in U(\bar{A})_d$,

$$\eta^{d+1}(\bar{a}/(\bar{y}_2, \dots, \bar{y}_d, \bar{1})) = a/(x_1, y_2, \dots, y_d, 1).$$

(We adopt the convention here that if some $\bar{y}_j = \bar{1}$, then 1 is to be used for y_j. Of course, the notation $\bar{a}$ has a different meaning from that of 2.1.)

Moreover, $\ker \eta^{d+1} \cong H_{\mathfrak{m}}^{d-1}(A)/x_1 H_{\mathfrak{m}}^{d-1}(A)$.

As the module $H_{\mathfrak{m}}^{d-1}(A)$ is Artinian [3, 2.1], the ideas of [13, §1] will be helpful. We give a brief review of these here, and introduce some new definitions.

Remarks and Definitions 2.3. (See [13, §1] for more details.) Let L be an Artinian A-module. Let

$$L = C_1 + \cdots + C_h$$

($h \geq 0$) be a minimal secondary representation of L, with C_i $\mathfrak{p}_i$-secondary for $i = 1, \dots, h$, and set

$$L_0 = \sum_{\substack{i=1 \\ \mathfrak{p}_i \neq \mathfrak{m}}}^{h} C_i.$$

This submodule, which is independent of the choice of minimal secondary representation for L, will be called the *residuum* of L. Note that, by [13, 1.1], L_0 is the smallest submodule K of L for which L/K has finite length. We shall call $l(L/L_0)$ the *residual length* of L.

We say that an element a of A is *pseudo-L-coregular* if $a \in \mathfrak{m}$ but a does not belong to any of the non-maximal attached primes of L. (The members of $\mathrm{Att}(L)$ are called the *attached primes* of L.) Note that this is equivalent to the condition that $a \in \mathfrak{m}$ and $aL_0 = L_0$. There exists $t \in \mathbb{N}$ such that $\mathfrak{m}^t L \subseteq L_0$; we define the *stability index* $s = s(L)$ of L to be the least integer $i \geq 0$ such that $\mathfrak{m}^i L \subseteq L_0$. Note that s is the least integer $i \geq 0$ such that $\mathfrak{m}^i L = \mathfrak{m}^{i+1} L$, that $\mathfrak{m}^s L = L_0$, and that $a^s L = L_0$ for each pseudo-L-coregular element $a \in \mathfrak{m}$.

We shall need to use a result of A.M. Riley concerning permutations of systems of parameters: it follows from the proof of [7, II, (3.1)] that, if $x_1, \dots, x_d$ form an s.o.p. for A and σ is a permutation of $\{1, \dots, d\}$, then, in $U(A)_{d+1}^{-d-1}A$, for $a \in A$,

$$\frac{a}{(x_{\sigma(1)},\dots,x_{\sigma(d)},1)}=\frac{(\operatorname{sign}\sigma)a}{(x_1,\dots,x_d,1)}.$$

As no proof of this is available in the literature (apart, that is, from Riley's thesis), we show below how the above result can be quickly derived from a result of O'Carroll [6, 3.3].

Lemma 2.4 (G. Gibson and L. O'Carroll). *Let R be a commutative ring (with identity), let X be an R-module, let $n\in\mathbb{N}$, and let V be a triangular subset of R^n. Suppose that $v=(v_1,\dots,v_n)$, $w=(w_1,\dots,w_n)\in V$ and $P\in M_n(R)$ are such that $P(v_1,\dots,v_n)^{\mathrm{T}}=(w_1,\dots,w_n)^{\mathrm{T}}$. Note that $V\times\{1\}$ is a triangular subset of R^{n+1}. Then, in $(V\times\{1\})^{-n-1}M$, for $m\in M$,*

$$m/(v_1,\dots,v_n,1)=|P|\,m/(w_1,\dots,w_n,1).$$

Proof. There exist $x=(x_1,\dots,x_n)\in V$ and $H,K\in D_n(R)$ such that $Hv^{\mathrm{T}}=x^{\mathrm{T}}=Kw^{\mathrm{T}}=KPv^{\mathrm{T}}$. Hence, by [6, 3.3], if we let D be the diagonal matrix $\operatorname{diag}(x_1,\dots,x_n)$, then

$$|D|\,|H|-|D|\,|K|\,|P|\in\sum_{i=1}^{n}Ax_i^2. \qquad (*)$$

For $Q\in M_n(R)$, denote by Q^* the matrix in $M_{n+1}(R)$ which has Q as its top left $n\times n$ submatrix, has 1 as its $(n+1,n+1)$-th entry, and has all remaining entries in its $(n+1)$-st row and $(n+1)$-st column equal to zero. Then

$$D^*H^*(v_1,\dots,v_n,1)^{\mathrm{T}}=(x_1^2,\dots,x_n^2,1)^{\mathrm{T}}=D^*K^*(w_1,\dots,w_n,1)^{\mathrm{T}}.$$

Now $|Q^*|=|Q|$ for all $Q\in M_n(R)$, and so, in $(V\times\{1\})^{-n-1}M$,

$$\frac{m}{(v_1,\dots,v_n,1)}=\frac{|D|\,|H|\,m}{(x_1^2,\dots,x_n^2,1)}=\frac{|D|\,|K|\,|P|\,m}{(x_1^2,\dots,x_n^2,1)}=\frac{|P|\,m}{(w_1,\dots,w_n,1)}$$

in view of [10, 3.3(ii)].

We can now use permutation matrices to obtain the above-mentioned result of Riley.

Corollary 2.5. (See A.M. Riley [7, II, (3.1)].) *Let $x_1,\dots,x_d$ form a s.o.p. for A, let σ be a permutation of $\{1,\dots,d\}$ and let $a\in A$. Then, in $U(A)_{d+1}^{-d-1}A$,*

$$a/(x_1,\dots,x_d,1)=(\operatorname{sign}\sigma)a/(x_{\sigma(1)},\dots,x_{\sigma(d)},1).$$

Next we give a technical lemma which facilitates considerably calculation in certain modules of generalized fractions. Let R be a commutative ring (with identity), let $n\in\mathbb{N}$, let $x=(x_1,\dots,x_n)\in R^n$, and let

$$U(x)=\{(x_1^{\alpha_1},\dots,x_n^{\alpha_n}):(\alpha_1,\dots,\alpha_n)\in\mathbb{N}^n\},$$

a triangular subset of R^n. For $\alpha=(\alpha_1,\dots,\alpha_n)\in\mathbb{N}^n$, it will be convenient to denote

$(x_1^{\alpha_1},\dots,x_n^{\alpha_n})$ by x^α, and we shall use obvious extensions of this notation without further comment. We use $\mathbb{N}_0$ to denote $\mathbb{N}\cup\{0\}$.

Given $\alpha\in\mathbb{N}^n$ and $\beta=(\beta_1,\dots,\beta_n)\in\mathbb{N}_0^n$, $(x^\alpha)^{\mathrm{T}}$ and $(x^{\alpha+\beta})^{\mathrm{T}}$ are related by a diagonal matrix: we have

$$\operatorname{diag}(x_1^{\beta_1},\dots,x_n^{\beta_n})(x^\alpha)^{\mathrm{T}}=(x^{\alpha+\beta})^{\mathrm{T}}.$$

At first sight, it would appear that, when working with the module of generalized fractions $U(x)^{-n}N$ of an R-module N with respect to $U(x)$, we shall need to work with general lower triangular matrices in $D_n(R)$. The next lemma shows that, for some purposes, we only need consider diagonal matrices of the above type when working with $U(x)^{-n}N$.

Lemma 2.6. (See H. Zakeri [14, II, (2.2)(ii)].) *Let the notation be as above. Let $g,h\in N$ and let $\alpha=(\alpha_1,\dots,\alpha_n)$, $\beta=(\beta_1,\dots,\beta_n)\in\mathbb{N}^n$. Then $g/x^\alpha=h/x^\beta$ in $U(x)^{-n}N$ if and only if there exists $\delta=(\delta_1,\dots,\delta_n)\in\mathbb{N}^n$ such that $\delta_i\geq\max\{\alpha_i,\beta_i\}$ for all $i=1,\dots,n$ and*

$$x_1^{\delta_1-\alpha_1}\cdots x_n^{\delta_n-\alpha_n}g-x_1^{\delta_1-\beta_1}\cdots x_n^{\delta_n-\beta_n}h\in\left(\sum_{i=1}^{n-1}Rx_i^{\delta_i}\right)N$$

(so that $Dx^{\alpha\mathrm{T}}=x^{\delta\mathrm{T}}=Ex^{\beta\mathrm{T}}$ (where $D=\operatorname{diag}(x_1^{\delta_1-\alpha_1},\dots,x_n^{\delta_n-\alpha_n})$ and $E=\operatorname{diag}(x_1^{\delta_1-\beta_1},\dots,x_n^{\delta_n-\beta_n})$) and $|D|\,g-|E|\,h\in(\sum_{i=1}^{n-1}Rx_i^{\delta_i})N$).

Proof. ($\Leftarrow$) This is clear.

($\Rightarrow$) Choose $\gamma=(\gamma_1,\dots,\gamma_n)\in\mathbb{N}^n$ such that $\gamma_i\geq\max\{\alpha_i,\beta_i\}$ for all $i=1,\dots,n$. Since

$$0=\frac{g}{x^\alpha}-\frac{h}{x^\beta}=\frac{(x_1^{\gamma_1-\alpha_1}\cdots x_n^{\gamma_n-\alpha_n}g-x_1^{\gamma_1-\beta_1}\cdots x_n^{\gamma_n-\beta_n}h)}{x^\gamma},$$

it is enough to prove the claim in the case when $h=0$. This we do.

There exists $\gamma=(\gamma_1,\dots,\gamma_n)\in\mathbb{N}^n$ and $H\in D_n(R)$ such that $Hx^{\alpha\mathrm{T}}=x^{\gamma\mathrm{T}}$ and $|H|\,g\in(\sum_{i=1}^{n-1}Rx_i^{\gamma_i})N$. We may assume that $\gamma_i\geq\alpha_i$ (and β_i) for each $i=1,\dots,n$. Set $D_1=\operatorname{diag}(x_1^{\gamma_1-\alpha_1},\dots,x_n^{\gamma_n-\alpha_n})$ and $D_2=\operatorname{diag}(x_1^{\gamma_1},\dots,x_n^{\gamma_n})$. Now

$$Hx^{\alpha\mathrm{T}}=x^{\gamma\mathrm{T}}=D_1x^{\alpha\mathrm{T}}.$$

It thus follows from [10, 2.3] that $|D_2H|-|D_2D_1|\in\sum_{i=1}^{n-1}Rx_i^{2\gamma_i}$; since $|D_2H|\,g\in(\sum_{i=1}^{n-1}Rx_i^{2\gamma_i})N$, we have

$$x_1^{2\gamma_1-\alpha_1}\cdots x_n^{2\gamma_n-\alpha_n}g=|D_2D_1|\,g\in\left(\sum_{i=1}^{n-1}Rx_i^{2\gamma_i}\right)N.$$

Take $\delta_i=2\gamma_i$ (for $i=1,\dots,n$) to complete the proof.

Remark 2.7. Note that 2.6 may be used not only for $U(x)^{-n}N$, but also for $\overline{U(x)}^{-n}N$, where $\overline{U(x)}$ denotes the expansion [10, 3.2] of $U(x)$: this is a consequence of the natural isomorphism between $U(x)^{-n}N$ and $\overline{U(x)}^{-n}N$.

Lemma 2.6 has many uses, and we give one now.

Lemma 2.8. *Let R be a commutative ring (with identity) and let $x \in R$. Let $V = U((x,1))$ (in the notation of 2.6). Let $s \in \mathbb{N}$. Then*

$$(0 :_{V^{-2}R} x^s) = \{r/(x^s,1) : r \in R\}.$$

Proof. One inclusion is immediate from [10, 3.3(ii)], and so we suppose that $r/(x^t,1)$, where $t \in \mathbb{N}$ with $t > s$, is an element of $V^{-2}R$ annihilated by x^s. Thus $r/(x^{t-s},1) = 0$ and so, by 2.6, there exists $q \in \mathbb{N}_0$ such that $x^q r \in Rx^{q+t-s}$. Hence there exists $r' \in R$ and $b \in (0 :_R x^q)$ such that $r = r'x^{t-s} + b$. Hence, in $V^{-2}R$,

$$\frac{r}{(x^t,1)} = \frac{r'x^{t-s}+b}{(x^t,1)} = \frac{r'}{(x^s,1)} + \frac{x^q b}{(x^{t+q},1)} = \frac{r'}{(x^s,1)}.$$

The result follows.

Finally in this preliminary section, we recall from [8, §3] the definition of *generalized Cohen–Macaulay local ring*: A is such a ring if and only if $H^i_{\mathfrak{m}}(A)$ has finite length for all $i = 0, \ldots, d-1$. Note that, when this is the case, the residuum of $H^{d-1}_{\mathfrak{m}}(A)$ is 0, and the residual length of $H^{d-1}_{\mathfrak{m}}(A)$ is therefore just the length $l(H^{d-1}_{\mathfrak{m}}(A))$. We draw the reader's attention to the various characterizations of generalized Cohen–Macaulay local rings provided by [8, (3.3)].

3. The results

We begin by answering Question 1.2 in the case when $d = 1$.

Proposition 3.1. *Suppose that $d = \dim A = 1$ and that x_1 forms a s.o.p. for A. Then, for all $n_1 \in \mathbb{N}$, the length of the generalized fraction $1/(x_1^{n_1},1)$ of $U(A)_2^{-2}A$ is given by*

$$l(A(1/(x_1^{n_1},1))) = e(x_1)n_1.$$

Proof. There exists $t \in \mathbb{N}$ such that $(0 : \mathfrak{m}^t) = (0 : \mathfrak{m}^{t+i})$ for all $i \in \mathbb{N}$. Now $A/(0 : \mathfrak{m}^t)$ has positive depth, and it therefore follows from 2.1 that we may assume that depth $A > 0$, so that, as $\dim A = 1$, A is Cohen–Macaulay.

But, as remarked in the Introduction, the result is then easy: by [12, 3.15], $l(A(1/(x_1^{n_1},1))) = l(A/Ax_1^{n_1})$, and this is $e(x_1)n_1$ by [4, p. 311].

Next we answer Question 1.2 for 2-dimensional local rings.

Theorem 3.2. *Suppose that $d = \dim A = 2$. Let l' be the residual length (see 2.3) of the Artinian module $H^1_{\mathfrak{m}}(A)$, and let s be the stability index of $H^1_{\mathfrak{m}}(A)$. Let x_1, x_2*

form an s.o.p. for A. Then, for all $n_1, n_2 \in \mathbb{N}$ *with* $n_1, n_2 \geq s$, *the length of the generalized fraction* $1/(x_1^{n_1}, x_2^{n_2}, 1)$ *of* $U(A)_3^{-3}A$ *is given by*

$$l(A(1/(x_1^{n_1}, x_2^{n_2}, 1))) = e(x_1, x_2)n_1 n_2 - l'.$$

Proof. There exists $t \in \mathbb{N}$ such that $(0 : \mathfrak{m}^t) = (0 : \mathfrak{m}^{t+i})$ for all $i \in \mathbb{N}$. If we change rings from A to $A/(0 : \mathfrak{m}^t)$ then, by 2.1(i), we do not change the values of l' and s. Since $A/(0 : \mathfrak{m}^t)$ has positive depth, it thus follows from 2.1 that we may assume depth $A > 0$.

Choose $n_1, n_2 \in \mathbb{N}$ with $n_1, n_2 \geq s$. Set

$$\mathscr{P} = \operatorname{Ass} A \cup (\operatorname{Att}(H^1_{\mathfrak{m}}(A)) \setminus \{\mathfrak{m}\}),$$

a finite set of non-maximal prime ideals of A. Since

$$Ax_1^{n_1} + Ax_2 \nsubseteq \bigcup_{\mathfrak{p} \in \mathscr{P}} \mathfrak{p},$$

it follows from [1, Theorem 124] that there exists $a_1 \in A$ with

$$y_2 := a_1 x_1^{n_1} + x_2 \notin \bigcup_{\mathfrak{p} \in \mathscr{P}} \mathfrak{p}.$$

Thus y_2, and all positive powers of it, are non-zerodivisors on A and pseudo-$H^1_{\mathfrak{m}}(A)$-coregular. Note that there exists $b_1 \in A$ such that $y_2^{n_2} = b_1 x_1^{n_1} + x_2^{n_2}$. Set

$$H = \begin{pmatrix} 1 & 0 & 0 \\ b_1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in D_3(A);$$

since $H(x_1^{n_1}, x_2^{n_2}, 1)^{\mathrm{T}} = (x_1^{n_1}, y_2^{n_2}, 1)^{\mathrm{T}}$ we see that, in view of 2.5, in $U(A)_3^{-3}A$ we have

$$1/(x_1^{n_1}, x_2^{n_2}, 1) = 1/(x_1^{n_1}, y_2^{n_2}, 1) = -1/(y_2^{n_2}, x_1^{n_1}, 1).$$

Since $Ax_1 + Ax_2 = Ay_2 + Ax_1$, we have $e(x_1, x_2) = e(y_2, x_1)$, and it is enough for us to show that

$$l(A(1/(y_2^{n_2}, x_1^{n_1}, 1))) = e(y_2, x_1)n_1 n_2 - l'.$$

We now use 2.2: note that $y_2^{n_2}$ is a non-zerodivisor on A. So we set $\bar{A} = A/Ay_2^{n_2}$ and use $\bar{\ } : A \to \bar{A}$ to denote the natural ring homomorphism. By 2.2, there is an A-homomorphism

$$\eta^3 : U(\bar{A})_2^{-2}\bar{A} \to U(A)_3^{-3}A$$

such that $\eta^3(\bar{a}/(\bar{z}_2, \bar{1})) = a/(y_2^{n_2}, z_2, 1)$ for each $\bar{a} \in \bar{A}$ and s.o.p. $\{\bar{z}_2\}$ for $\bar{A}$. Note also that $\ker \eta^3 \cong H^1_{\mathfrak{m}}(A)/y_2^{n_2}H^1_{\mathfrak{m}}(A)$ and, by choice of y_2, $y_2^{n_2}H^1_{\mathfrak{m}}(A)$ is the residuum of $H^1_{\mathfrak{m}}(A)$. Thus $\ker \eta^3$ has finite length equal to l'. Note also that it is annihilated by x_1^s, and so, by 2.8, [10, 3.2] and [11, 3.5],

$$\ker \eta^3 \subseteq \bar{A}(\bar{1}/(\bar{x}_1^s, \bar{1})) \subseteq \bar{A}(\bar{1}/(\bar{x}_1^{n_1}, \bar{1})).$$

We therefore have an exact sequence of A-modules

$$0 \to \ker \eta^3 \to \bar{A}(\bar{1}/(\bar{x}_1^{n_1}, \bar{1})) \to A(1/(y_2^{n_2}, x_1^{n_1}, 1)) \to 0.$$

All the modules in this sequence have finite length: we remarked earlier that $l(\ker \eta^3) = l'$. By 3.1,

$$l_A(\bar{A}(\bar{1}/(\bar{x}_1^{n_1}, \bar{1}))) = l_{\bar{A}}(\bar{A}(\bar{1}/(\bar{x}_1^{n_1}, \bar{1}))) = e_{\bar{A}}(\bar{x}_1)n_1,$$

and, by [4, (7.4.2) and (7.4.3)], $e_{\bar{A}}(\bar{x}_1) = e_A(y_2^{n_2}, x_1)$. It therefore follows, in view of [4, p. 311], that

$$l_A(A(1/(y_2^{n_2}, x_1^{n_1}, 1))) = e_A(y_2, x_1)n_1n_2 - l',$$

as required.

In the case when A is a 2-dimensional generalized Cohen–Macaulay local ring, there are simplifications in the statement of 3.2.

Corollary 3.3. *Suppose that A is a 2-dimensional generalized Cohen–Macaulay local ring. Let x_1, x_2 form an s.o.p. for A. Then, for all positive integers $n_1, n_2 \geq s(H_{\mathfrak{m}}^1(A))$, we have*

$$l(A(1/(x_1^{n_1}, x_2^{n_2}, 1))) = e(x_1, x_2)n_1n_2 - l(H_{\mathfrak{m}}^1(A)).$$

All our attempts up to the time of writing to obtain an extension of 3.2 which applies to arbitrary local rings of higher dimensions have failed; we have, however, been able to provide an affirmative answer to Question 1.2 in the special case in which A is a generalized Cohen–Macaulay local ring of arbitrary (positive) dimension. Before we provide the proof of this, some preparatory results are necessary.

Remark 3.4. Let $N \to P \to Q$ be an exact sequence of A-modules and let $\mathfrak{b}, \mathfrak{c}$ be ideals of A such that $\mathfrak{b}N = 0 = \mathfrak{c}Q$. Then $\mathfrak{b}\mathfrak{c}P = 0$. Consequently, if N, P, Q all have finite length, then $s(P) \leq s(N) + s(Q)$.

Corollary 3.5. *Suppose that A is a generalized Cohen–Macaulay local ring and $x_1 \in \mathfrak{m}$ is a non-zerodivisor on A. Then, for all $i = 1, \ldots, d-2$,*

$$s(H_{\mathfrak{m}/Ax_1}^i(A/x_1A)) \leq s(H_{\mathfrak{m}}^i(A)) + s(H_{\mathfrak{m}}^{i+1}(A)).$$

Proof. Use the long exact sequence of local cohomology modules induced by the exact sequence

$$0 \to A \xrightarrow{x_1} A \to A/Ax_1 \to 0,$$

in conjunction with 3.4 and [9, 4.3].

Proposition 3.6. *Suppose that A is a generalized Cohen–Macaulay local ring and let $x_1, \dots, x_d$ form a s.o.p. for A. Set*

$$t = \sum_{i=1}^{d-1} \binom{d-1}{i-1} s(H^i_{\mathfrak{m}}(A)).$$

Let $r \in \mathbb{N}$. Then

$$(0 :_{U(A)^{-d-1}_{d+1}A} \mathfrak{m}^r) \subseteq A(1/(x_1^{r+t}, \dots, x_d^{r+t}, 1)).$$

Proof. We use induction on d; the result in the case in which $d = 1$ follows from 2.8, [11, 3.5] and [10, 3.2]. So we suppose that $d > 1$ and the result has been proved for generalized Cohen–Macaulay local rings of smaller (positive) dimension.

Lemma 2.1 shows that we may, after factoring out by the maximum ideal of A of finite length, assume that depth $A > 0$. Then every $x \in \mathfrak{m}$ which forms a s.s.o.p. for A is automatically a non-zerodivisor on A, since, by [8, (3.3)], for some $n \in \mathbb{N}$,

$$(0 :_A x) \subseteq (0 :_A \mathfrak{m}^n) = 0;$$

it also follows from 3.4 and 3.5 that A/xA is a generalized Cohen–Macaulay local ring (of dimension $d-1$).

So, let $\alpha \in U(A)^{-d-1}_{d+1}A$ be such that $\mathfrak{m}^r\alpha = 0$. By [11, 3.6], there exist $a \in A$ and positive integers $n_1, \dots, n_d$ such that $\alpha = a/(x_1^{n_1}, \dots, x_d^{n_d}, 1)$. Let $\bar{A} = A/x_1^{n_1}A$, a $(d-1)$-dimensional generalized Cohen–Macaulay local ring, and let $\bar{\ }: A \to \bar{A}$ be the natural ring homomorphism. By 2.2, there is an exact sequence of A-modules and A-homomorphisms

$$0 \to H^{d-1}_{\mathfrak{m}}(A)/x_1^{n_1}H^{d-1}_{\mathfrak{m}}(A) \to U(\bar{A})^{-d}_{d}\bar{A} \xrightarrow{\eta^{d+1}} U(A)^{-d-1}_{d+1}A$$

in which $\eta^{d+1}(\bar{b}/(\bar{y}_2, \dots, \bar{y}_d, \bar{1})) = b/(x_1^{n_1}, y_2, \dots, y_d, 1)$ for each $b \in A$ and each s.o.p. $\{\bar{y}_2, \dots, \bar{y}_d\}$ for $\bar{A}$. Set $s = s(H^{d-1}_{\mathfrak{m}}(A))$. Now $\mathfrak{m}^s \ker \eta^{d+1} = 0$ and

$$\mathfrak{m}^{r+s}(\bar{a}/(\bar{x}_2^{n_2}, \dots, \bar{x}_d^{n_d}, \bar{1})) = 0.$$

Hence, by the inductive assumption, there exists $a_1 \in A$ such that

$$\frac{\bar{a}}{(\bar{x}_2^{n_2}, \dots, \bar{x}_d^{n_d}, \bar{1})} = \frac{\bar{a}_1}{(\bar{x}_2^{r+s+t'}, \dots, \bar{x}_d^{r+s+t'}, \bar{1})}$$

where

$$t' = \sum_{i=1}^{d-2} \binom{d-2}{i-1} s(H^i_{\bar{\mathfrak{m}}}(\bar{A}))$$

(and $\bar{\mathfrak{m}}$ is the maximal ideal of $\bar{A}$). By 3.5, $t' + s \le t$. Hence we see that, after application of η^{d+1}, there exists $a_2 \in A$ such that

$$\alpha = a_2/(x_1^{n_1}, x_2^{r+t}, \dots, x_d^{r+t}, 1).$$

We now use Riley's permutation result 2.5 to write

$$\alpha = -a_2/(x_2^{r+t}, x_1^{n_1}, x_3^{r+t}, \dots, x_d^{r+t}, 1),$$

repeat the above argument, and use 2.5 again to complete the inductive step.

An interesting question is whether we can replace, in the conclusion of 3.6, t by a smaller integer and still obtain a valid statement. If so, what is the smallest such replacement?

We are now in a position to settle Question 1.2 affirmatively for generalized Cohen–Macaulay local rings of arbitrary (positive) dimension.

Theorem 3.7. *Suppose that A is a generalized Cohen–Macaulay local ring and let $x_1, \ldots, x_d$ form a s.o.p. for A. Set*

$$t = \sum_{i=1}^{d-1} \binom{d-1}{i-1} s(H_{\mathfrak{m}}^i(A)).$$

Then, for all positive integers $n_1, \ldots, n_d \geq t$, we have

$$l(A(1/(x_1^{n_1}, \ldots, x_d^{n_d}, 1))) = e(x_1, \ldots, x_d)n_1 \cdots n_d - \sum_{i=1}^{d-1} \binom{d-1}{i-1} l(H_{\mathfrak{m}}^i(A)).$$

Proof. We use induction on d. The special cases of the result in which $d=1$ and $d=2$ have been proved in 3.1 and 3.3. (Of course we interpret an empty sum as zero.) So we suppose that $d>2$ and the result has been proved for generalized Cohen–Macaulay local rings of smaller (positive) dimension.

We begin as we began the proof of 3.6: Lemma 2.1 enables us to assume that depth $A>0$; then every $x \in \mathfrak{m}$ which forms an s.s.o.p. for A is automatically a non-zerodivisor on A for which A/xA is a generalized Cohen–Macaulay local ring of dimension $d-1$.

Choose $n_1, \ldots, n_d \in \mathbb{N}$ with $n_i \geq t$ for $i=1, \ldots, d$. Write $\bar{A} = A/x_1^{n_1}A$, let $^-: A \to \bar{A}$ be the natural map, and note that $x_1^{n_1} H_{\mathfrak{m}}^{d-1}(A) = 0$. Use the homomorphism η^{d+1} of 2.2 again: this time we have an exact sequence

$$0 \to H_{\mathfrak{m}}^{d-1}(A) \to U(\bar{A})_d^{-d}\bar{A} \xrightarrow{\eta^{d+1}} U(A)_{d+1}^{-d-1}A$$

and $\mathfrak{m}^s \ker \eta^{d+1} = 0$, where $s = s(H_{\mathfrak{m}}^{d-1}(A))$. Thus, by 3.6,

$$\ker \eta^{d+1} \subseteq \bar{A}(\bar{1}/(\bar{x}_2^{s+t'}, \ldots, \bar{x}_d^{s+t'}, \bar{1})),$$

where

$$t' = \sum_{i=1}^{d-2} \binom{d-2}{i-1} s(H_{\bar{\mathfrak{m}}}^i(\bar{A}))$$

(and $\bar{\mathfrak{m}}$ is the maximal ideal of $\bar{A}$). By 3.5, $t' + s \leq t$. Hence

$$\ker \eta^{d+1} \subseteq \bar{A}(\bar{1}/(\bar{x}_2^{n_2}, \ldots, \bar{x}_d^{n_d}, \bar{1})),$$

and so we have an exact sequence of A-modules of finite length

$$0 \to H_{\mathfrak{m}}^{d-1}(A) \to \bar{A}(\bar{1}/(\bar{x}_2^{n_2}, \ldots, \bar{x}_d^{n_d}, \bar{1})) \to A(1/(x_1^{n_1}, x_2^{n_2}, \ldots, x_d^{n_d}, 1)) \to 0.$$

By the inductive hypothesis, since $n_2, \dots, n_d \geq t'$,

$$l_{\bar{A}}(\bar{A}(\bar{1}/(\bar{x}_2^{n_2}, \dots, \bar{x}_d^{n_d}, \bar{1}))) = e_{\bar{A}}(\bar{x}_2, \dots, \bar{x}_d)n_2 \cdots n_d - \sum_{i=1}^{d-2} \binom{d-2}{i-1} l_{\bar{A}}(H^i_{\mathfrak{m}}(\bar{A})).$$

By [4, (7.4.2), (7.4.3), and Corollary 1 on p. 311],

$$e_{\bar{A}}(\bar{x}_2, \dots, \bar{x}_d) = e_A(x_1^{n_1}, x_2, \dots, x_d) = n_1 e_A(x_1, x_2, \dots, x_d),$$

so that the above exact sequence yields

$$\begin{aligned} &l(A(1/(x_1^{n_1}, x_2^{n_2}, \dots, x_d^{n_d}, 1))) \\ &\quad = e(x_1, x_2, \dots, x_d)n_1 n_2 \cdots n_d - l(H^{d-1}_{\mathfrak{m}}(A)) - \sum_{i=1}^{d-2} \binom{d-2}{i-1} l_A(H^i_{\mathfrak{m}}(\bar{A})) \end{aligned}$$

on use of [9, 4.3].

Next, the exact sequence

$$0 \to A \xrightarrow{x_1^{n_1}} A \to \bar{A} \to 0$$

induces a long exact sequence

$$\begin{aligned} &0 \to H^0_{\mathfrak{m}}(\bar{A}) \\ &\to H^1_{\mathfrak{m}}(A) \xrightarrow{x_1^{n_1}} H^1_{\mathfrak{m}}(A) \to H^1_{\mathfrak{m}}(\bar{A}) \\ &\to H^2_{\mathfrak{m}}(A) \xrightarrow{x_1^{n_1}} H^2_{\mathfrak{m}}(A) \to H^2_{\mathfrak{m}}(\bar{A}) \to \cdots \end{aligned}$$

of local cohomology modules, in which all the endomorphisms of the $H^i_{\mathfrak{m}}(A)$ (for $1 \leq i \leq d-1$) given by multiplication by $x_1^{n_1}$ are zero: this is because $n_1 \geq s(H^i_{\mathfrak{m}}(A))$ for each $i = 1, \dots, d-1$. Thus

$$l(H^i_{\mathfrak{m}}(\bar{A})) = l(H^i_{\mathfrak{m}}(A)) + l(H^{i+1}_{\mathfrak{m}}(A)) \quad \text{for } i = 1, \dots, d-2;$$

thus

$$l(H^{d-1}_{\mathfrak{m}}(A)) + \sum_{i=1}^{d-2} \binom{d-2}{i-1} l(H^i_{\mathfrak{m}}(\bar{A})) = \sum_{i=1}^{d-1} \binom{d-1}{i-1} l(H^i_{\mathfrak{m}}(A)),$$

and the inductive step is complete.

It is perhaps worth pointing out that, in the situation of 3.7, for $n_1, \dots, n_d \geq t$,

$$l(A(1/(x_1^{n_1}, \dots, x_d^{n_d}, 1))) = e(x_1^{n_1}, \dots, x_d^{n_d}) - \sum_{i=1}^{d-1} \binom{d-1}{i-1} l(H^i_{\mathfrak{m}}(A))$$

falls short of $e(x_1^{n_1}, \dots, x_d^{n_d})$ by the integer $\sum_{i=1}^{d-1} \binom{d-1}{i-1} l(H^i_{\mathfrak{m}}(A))$, while, by [8, (3.7)], provided $n_1, \dots, n_d$ are all sufficiently large,

$$l\left(A \Big/ \sum_{i=1}^{d} Ax_i^{n_i}\right) = e(x_1^{n_1}, \dots, x_d^{n_d}) + \sum_{i=0}^{d-1} \binom{d-1}{i} l(H^i_{\mathfrak{m}}(A))$$

exceeds $e(x_1^{n_1}, \dots, x_d^{n_d})$ by $\sum_{i=0}^{d-1} \binom{d-1}{i} l(H^i_{\mathfrak{m}}(A))$.

As in the case of 3.6, there are interesting questions concerning the role of the integer t in 3.7: can we replace t in its conclusion by a smaller integer and still obtain a valid statement, and, if so, what is the smallest such replacement? Thus this research not only leaves open Question 1.2 in its full generality, but also raises further questions! We have spent many hours trying to extend the results of this paper but, as indicated above, have been unsuccessful up to the time of writing.

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