ELSEVIER Journal of Pure and Applied Algebra 114 (1997) 209–215

JOURNAL OF PURE AND APPLIED ALGEBRA

# A class of triangular derivations having a slice

Christian Eggermont, Arno van den Essen*

*Catholic University Nijmegen, Toernooiveld 6525 ED Nijmegen, The Netherlands*

Communicated by C.A. Weibel; received 11 July 1994; revised 7 November 1995

**Abstract**

Consider a $G_a$-action on $\mathbb{C}^n (n \geq 3)$ given by the derivation $D = p(X_2,\dots,X_n)\partial_1 + q(X_3,\dots,X_n)\partial_2$. It is shown that $(\mathbb{C}^n)^{G_a} \simeq \mathbb{C}^{n-1}$ if $\gcd(p,q) = 1$ and that $\mathbb{C}^n$ is equivariantly isomorphic to $G_a \times \mathbb{C}^{n-1}$ if the $G_a$-action is fixed point free.

## 0. Introduction

In [4] it is shown by Snow that all free triangular $G_a$-actions on $\mathbb{C}^3$ are equivariantly isomorphic to $G_a \times \mathbb{C}^2$. In this paper we generalise this result to a special class of $G_a$-actions on $\mathbb{C}^n (n \geq 3)$, namely the free $G_a$-actions given by the locally nilpotent derivations of the form $D = p(X_2,\dots,X_n)\partial_1 + q(X_3,\dots,X_n)\partial_2$ (the freeness of the $G_a$-action is equivalent to the fact that $p$ and $q$ have no common zero in $\mathbb{C}^n$. We also say that $D$ is free). In fact only assuming that $\gcd(p,q) = 1$ we show that $\mathbb{C}[X]^{G_a}$ is generated by $n-1$ algebraically independent elements over $\mathbb{C}$, which can be described explicitly (Theorem I). Using this result we show that if $p$ and $q$ have no common zero in $\mathbb{C}^n$, then $D$ has a slice $s$ in $\mathbb{C}[X_1,\dots,X_n]$, i.e. $D(s) = 1$ (Theorem II). So combining both theorems we obtain that $D$ is conjugate to the derivation $\frac{\mathrm{d}}{\mathrm{d}X_n}$ if $D$ is free. In other words, the corresponding $G_a$-action on $\mathbb{C}^n$ is equivariantly isomorphic to $G_a \times \mathbb{C}^{n-1}$. The proof given below is purely algebraic (with $\mathbb{C}$ replaced by a field $k$ of characteristic zero) and is based on an algorithm, given in [3], to compute the kernel of a locally nilpotent derivation.

The context of this paper is arranged as follows. In Section 1 we state the main results (Theorems I and II) and describe some preliminaries; in particular, we recall some essential facts from the algorithm in [3]. In Sections 2 and 3 we give the proofs of

* Corresponding author. E-mail: essen@sci.kun.nl.


*SSDI* 0022-4049(95)00168-9

Theorem I and Theorem II, respectively. Finally, in Section 4 we make some remarks on possible extensions of the main results.

## 1. Preliminaries and statement of the main results

### *1.1. Some notations*

Throughout this paper we have the following notations: $k$ is a field of characteristic zero, $\bar{k}$ an algebraic closure of $k$ and $k[X] := k[X_1,\dots,X_n]$ the polynomial ring in $n$ variables over $k$. If $D$ is a derivation on $k[X]$, then the kernel of $D$ will be denoted by $k[X]^D$ and an element $s \in k[X]$ satisfying $D(s) = 1$ is called a slice. If $D = \Sigma a_i\partial_i$ has a slice $s \in k[X]$, then obviously the $a_i$ have no common zero in $\bar{k}^n$ i.e. $V(a_1,\dots,a_n) = \emptyset$ ($V(a_1,\dots,a_n)$ = the set of common zeros of the $a_i$ in $\bar{k}^n$). Such a derivation is called free or fixed point free. From now on assume that $D \neq 0$ and that $D$ is locally nilpotent (i.e. for every $g \in k[X]$ there exists an $m \in \mathbb{N}$ with $D^m(g) = 0$). Then the map exp $TD : k[X] \to k[X][T]$, given by

$$\exp TD(g) = \sum_{i\geq 0} T^i \frac{1}{i!} D^i(g) \quad \text{for all } g \in k[X]$$

is a ringhomomorphism. In [3] this map is used to give an algorithm which computes generators of the $k$-algebra $R := k[X]^D$. Since the proof of the main results of this paper rest heavely on this algorithm, we briefly recall some of the main ideas of [3] in the next subsection.

### *1.2. The algorithm of [3]*

First choose a non-zero element $a \in k[X]$ such that $D^2(a) = 0$ and $D(a) \neq 0$. Put $d := D(a)$, $s_0 := d^{-1}a \in k(X)$ and $b_i := \exp TD(X_i)_{|T=-s_0}$. So their exist $e_i \in \mathbb{N}$ such that $r_i := d^{e_i}b_i \in k[X]$. It is shown in [3] that the $k$-subalgebra $R_0 := k[r_1,\dots,r_n,d]$ of $k[X]$ satisfies

$$R_0 \subset R \subset R_0[d^{-1}]. \tag{1}$$

Next define inductively the $k$-subalgebras

$$R_m = \{g \in k[X] \mid dg \in R_{m-1}\}, \quad m \geq 1.$$

It is shown that each $R_m$ is a finitely generated $k$-algebra (and an algorithm is given to compute $k$-generators of $R_m$). Furthermore the $R_m$ form an ascending chain $R_0 \subset R_1 \subset R_2 \subset \cdots \subset R$ such that $R = \bigcup R_m$ and if $R$ is a finite $k$-algebra, then $R = R_r$ for some $r \in \mathbb{N}$. However, even if we do not know a priori that $R$ is a finitely generated $k$-algebra, we can conclude that $R$ is finitely generated over $k$ if $R_{m-1} = R_m$. In fact we obtain $R = R_{m-1}$. (To see this let $g \in R$, so $g \in R_p$ for some $p \geq 0$. If $p \leq m$ we are done, so let $p > m$. Then $dg \in R_{p-1}$. By induction on $p$ it follows that $R_{p-1} \subset R_{m-1}$ so $dg \in R_{m-1}$, whence $g \in R_m = R_{m-1}$ by definition of $R_m$.) In this paper we will use this observation only for $m = 1$, i.e. if for all $g \in k[X]$ we have that $dg \in R_0$ implies that $g \in R_0$, then $R = R_0$. In other words:

If for all $h \in R_0$ divisible by $d$ we can conclude that $h/d \in R_0$, then $R = R_0$. (2)

### 1.3. Formulation of the main results

From now on we assume that $n \geq 3$ and put $A := k[X_3, \ldots, X_n]$. We fix the following notations: $p \in A[X_2]$, $q \in A$ and $D = p\partial_1 + q\partial_2$. So $D$ is a triangular derivation on $k[X]$ and hence locally nilpotent. Our first result (Theorem I below) describes $R = k[X]^D$. To compute $R$ we first compute $R_0$, following the construction described in Section 1.2. We take $a = X_2$ and hence $d = D(a) = q$. Observe that $\exp TD(X_i)_{|T=-X_2/q} = X_i$ if $i \geq 3$ (since $D(X_i) = 0$) and that $\exp TD(X_2)_{|T=-X_2/q} = 0$. Finally, put

$$c(D) := q \exp TD(X_1)_{|T=-X_2/q}.$$

Then one readily verifies that $c(D) \in k[X]$. In fact we have

$$c(D) = qX_1 + \sum_{i=1}^{\infty} \frac{1}{i!}(-X_2)^i \partial_2^{i-1}(p). \tag{3}$$

So by (1) we get

$$R_0 = k[c(D), X_3, \ldots, X_n] \subset R = k[X]^D. \tag{4}$$

Of course, to compute $k[X]^D$ we may assume that $\gcd(p, q) = 1$. The first main result asserts that under this assumption we have equality in (4), i.e.

**Theorem I.** *Let $D = p\partial_1 + q\partial_2$ with $p \in k[X_2, \ldots, X_n]$, $q \in k[X_3, \ldots, X_n]$, $n \geq 3$ and $\gcd(p, q) = 1$. Then $k[X]^D = k[X_3, \ldots, X_n, c(D)]$.*

Using this result we consider the question: under what conditions on $D$ there exists a slice in $k[X]$?

As observed in Section 1.1, if $D$ has a slice in $k[X]$, then $D$ is fixed point free. The second main result shows that the converse holds, i.e.

**Theorem II.** *Let $D = p\partial_1 + q\partial_2$ with $p \in k[X_2, \ldots, X_n]$, $q \in k[X_3, \ldots, X_n]$, $n \geq 3$. If $D$ is fixed point free, then $D$ has a slice in $k[X]$.*

## 2. The proof of Theorem I

The proof of Theorem I is based on the following lemma

**Lemma 2.1.** *Assume* $gcd(p,q) = 1$. *Let* $h(T) = \sum_{i=0}^{N} h_i T^i \in A[T]$ *with* $N \geq 1$, $h_N \neq 0$ *and* $q \notin k$. *If* $h(c(D)) \equiv 0 \mod q$ *in* $k[X]$, *then* $h_i \equiv 0 \mod q$ *in* $A$, *for all* $i \geq 0$.

**Proof.** Let $q = q_1^{e_1} \ldots q_r^{e_r}$ be the prime decomposition of $q$. We use induction on $e := \Sigma e_i$. The case $e = 1$, i.e. $q$ irreducible will be proved in Lemma 2.2 below. So assume $e > 1$ and write $q = q_1 \tilde{q}$. Since $h(c(D)) \equiv 0 \mod q$, certainly $h(c(D)) \equiv 0 \mod q_1$. Now define $D_1 := p\partial_1 + q_1\partial_2$ and notice that $c(D_1) \equiv c(D) \bmod q_1$ (use (3)). So $h(c(D_1)) \equiv h(c(D)) \equiv 0 \bmod q_1$. Consequently, by Lemma 2.2 below $h_i \equiv 0 \bmod q_1$ in $A$ for all $i \geq 0$, i.e. $h_i = q_1 \tilde{h}_i$ for some $\tilde{h}_i \in A$. So $h = q_1 \tilde{h}$ where $\tilde{h} = \Sigma \tilde{h}_i T^i$. Since $h(c(D)) \equiv 0 \bmod q$, we deduce that $\tilde{h}(c(D)) \equiv 0 \bmod \tilde{q}$. Finally define $\tilde{D} := p\partial_1 + \tilde{q}\partial_2$. Then arguing as above it follows that $\tilde{h}(c(\tilde{D})) \equiv \tilde{h}(c(D)) \equiv 0 \bmod \tilde{q}$. So from the induction hypothesis, applied to $\tilde{D}$ and $\tilde{h}$, we deduce that $\tilde{h}_i \equiv 0 \bmod \tilde{q}$ for all $i \geq 0$. Consequently, $h_i \equiv 0 \bmod q$ for all $i \geq 0$, as desired. □

**Lemma 2.2.** *Assume* $gcd(p,q) = 1$. *Let* $h(T) = \sum_{i=0}^{N} h_i T^i \in A[T]$ *with* $N \geq 1$ *and* $h_N \neq 0$ *and* $q$ *irreducible. If* $h(c(D)) \equiv 0(mod\ q)$ *in* $k[X]$, *then* $h_i \equiv 0(mod\ q)$ *in* $A$ *for all* $i \geq 0$.

**Proof.** Write $p = \sum_{i=0}^{r} p_i X_2^i$ with $p_i \in A$. Since $q$ is irreducible and $\gcd(p,q) = 1$, there exists $i$ with $p_i \not\equiv 0(\bmod q)$. Let $t$ be minimal with this property, i.e. $p \equiv p_t X_2^t + \text{h.o.t. in } X_2(\bmod q)$ (here "h.o.t. in $X_2$" means higher order terms in $X_2$) with $p_t \not\equiv 0(\bmod q)$. So

$$X_2^{j+1}\partial_2^j(p) \equiv p_t X_2^{j+1}\partial_2^j(X_2^t) + \text{h.o.t. in } X_2(\bmod q).$$

So

$$X_2^{j+1}\partial_2^j(p) \equiv p_t t(t-1)\ldots(t-(j-1))X_2^{t+1} + \text{h.o.t. in } X_2(\bmod q).$$

Using that $\partial_2^{t+1}(X_2^t) = 0$, it follows from (3) that

$$c(D) \equiv p_t c(t) X_2^{t+1} + \text{h.o.t. in } X_2(\bmod q),$$

where

$$c(t) := -1 + \sum_{i=2}^{t+1} \frac{(-1)^i}{i}\binom{t}{i-1}.$$

Now it is an easy excercise to show that $c(t) = -1/(t+1)$ for all $t \geq 0$ (just observe that $(t+1)c(t) = -(t+1) + \sum_{j=1}^{t}(-1)^{j+1}\binom{t+1}{j+1}$ and use that $(1+(-1))^{t+1} = 0$). So in particular $c(t) \neq 0$ for all $t \geq 0$.

Finally, suppose that $h(T) = h_s T^s + h_{s+1}T^{s+1} + \cdots + h_N T^N$ with $h_s \not\equiv 0(\bmod q)$.

Then $h(c(D)) \equiv h_s p_t^s c(t)^s (X_2^{t+1})^s + \text{h.o.t. in } X_2(\bmod q)$.

Since $h(c(D)) \equiv 0(\bmod q)$, it follows that either $h_s$ or $p_t$ are divisible by $q$, a contradiction. So $h_i \equiv 0(\bmod q)$ for all $i$, as desired. □

**Proof of Theorem I.** By (4) we know that $R_0 = A[c(D)]$. Let $h(T) = \sum_{i=0}^{N} h_i T^i \in A[T]$.

By (2) it suffices to show that if $h\big(c(D)\big) \equiv 0 (\text{mod } q)$ in $k[X]$, then $h_i \equiv 0 (\text{mod } q)$ for all $i$. For $N = 0$ this is obvious and for $N \geq 1$ this is exactly the content of Lemma 2.1. □

## 3. The proof of Theorem II

First we prove the theorem when $q$ is irreducible. Therefore we need the following lemma.

**Lemma 3.1.** *Assume $q$ is irreducible and $V(p,q) = \emptyset$. Then $p = q\widetilde{p} + p_0$ with $p_0 \in A$ and $\widetilde{p} \in A[X_2]$.*

**Proof.** Develop $p$ after powers of $X_2$. So $p = \sum_{i=0}^{N} p_i X_2^i$ for some $N \in \mathbb{N}$ and $p_i \in A$. If $N = 0$, we are done, so assume $N \geq 1$. Suppose there exists $i \geq 1$ such that $V(q) \not\subset V(p_i)$. Then there exists $\alpha = (\alpha_3, \ldots, \alpha_n) \in \overline{k}^{n-2}$ such that $q(\alpha) = 0$ and $p_i(\alpha) \neq 0$. Let $i_0$ be maximal with this property. Then $p(\alpha, X_2) = p_{i_0}(\alpha) X_2^{i_0} + \cdots + p_0(\alpha)$. So we can choose $x_2 \in \overline{k}$ such that $p(\alpha, x_2) = 0$. Consequently, $V(p,q) \neq \emptyset$, a contradiction. So $V(q) \subset V(p_i)$ for all $i \geq 1$. From the Nullstellensatz and the fact that $q$ is irreducible we deduce that $q$ divides $p_i$ for all $i \geq 1$, say $p_i = q\widetilde{p}_i$. Hence $p = q\widetilde{p} + p_0$, where $\widetilde{p} = \sum_{i=1}^{N} \widetilde{p}_i X_2^i$, as desired. □

**Corollary 3.2.** *Notations as in Lemma 3.1. Then $D$ has a slice (in $k[X]$).*

**Proof.** Write $p = q\widetilde{p} + p_0$ according Lemma 3.1. Then

$$D = (q\widetilde{p} + p_0)\partial_1 + q\partial_2 = q(\widetilde{p}\partial_1 + \partial_2) + p_0\partial_1.$$

Since $V(p,q) = \emptyset$ it follows that $V(p_0, q) = \emptyset$ (using $p = q\widetilde{p} + p_0$). So by Nullstellensatz there exist $a, b \in A$ with $aq + bp_0 = 1$. Finally, put $s := aX_2 + b(X_1 - \widetilde{p}X_2)$. Then $D(s) = qa + qb(\widetilde{p}\partial_1 + \partial_2)(X_1 - \widetilde{p}X_2) + p_0 b = qa + p_0 b = 1$. □

**Proof of Theorem II.** If $q = 0$, $X_1$ is a slice and if $q \in k^*$, then $q^{-1}X_2$ is a slice, so assume that $q \notin k$ and let $q = q_1^{e_1} \ldots q_r^{e_r}$ be the prime decomposition of $q$. We use induction on $e := \Sigma e_i$. If $e = 1$, the result follows from Corollary 3.2. So let $e > 1$.

(i) Write $q = q_1\widetilde{q}$. Then by Corollary 3.2 and the induction hypothesis the derivations $D_1 = p\partial_1 + q_1\partial_2$ and $\widetilde{D} = p\partial_1 + \widetilde{q}\partial_2$ have a slice, $s_1$, respectively, $\widetilde{s}$ in $k[X]$.

From $D_1(s_1) = 1$ we deduce that $D_1(q_1 s_1 - X_2) = 0$ and hence

$$X_2 - q_1 s_1 = h_1\big(c(D_1)\big) \quad \text{for some } h_1 \in A[T], \tag{5}$$

by Theorem I. Similarly,

$$X_2 - \widetilde{q}\widetilde{s} = \widetilde{h}\big(c(\widetilde{D})\big) \quad \text{for some } \widetilde{h} \in A[T]. \tag{6}$$

As observed in the proof of Theorem I we have

$$c(D) \equiv c(D_1)(\text{mod } q_1) \quad \text{and} \quad c(D) \equiv c(\widetilde{D})(\text{mod } \widetilde{q}) \tag{7}$$

and hence

$$X_2 \equiv \widetilde{h}\big(c(D)\big)(\text{mod } \widetilde{q}) \quad \text{and} \quad X_2 \equiv h_1\big(c(D)\big)(\text{mod } q_1). \tag{8}$$

So $X_2 = \widetilde{h}\big(c(D)\big) + \widetilde{q}b$ for some $b \in k[X]$. Applying $D$ gives $D(X_2) = \widetilde{q}Db$ and hence

$$D(b) = q_1 \tag{9}$$

(since $D(X_2) = q = \widetilde{q}q_1$). In other words $p\partial_1(b) + q\partial_2(b) = q_1$. So $p\partial_1(b) \equiv 0(\text{mod } q_1)$. However $\gcd(p, q_1) = 1$, so $\partial_1(b) \equiv 0(\text{mod } q_1)$. Now write $b = \sum_{i=0}^{N} b_i X_1^i$, with $b_i \in A[X_2]$. Then $\partial_1(b) \equiv 0(\text{mod } q_1)$ implies

$$b_i \equiv 0(\text{mod } q_1) \quad \text{for all } i \geq 1. \tag{10}$$

(ii) Finally, we need to find $s \in k[X]$ with $D(s) = 1$ and hence with $D(q_1 s) = q_1$ i.e. with $D(q_1 s - b) = 0$ (since $D(b) = q_1$ by (9)). So by Theorem I we need to find $s$ with $q_1 s - b = g\big(c(D)\big)$ for some $g \in A[T]$. On the other hand, we know that by (10) $b \equiv b_0(\text{mod } q_1)$ and that $X_2 \equiv h_1\big(c(D)\big)(\text{mod } q_1)$ (by (8)). So if we write $b_0 = \sum_{i=0}^{r} b_{oi} X_2^i$, for some $b_{oi} \in A$, then $b \equiv b_0 \equiv \sum_i b_{oi}\Big(h_1\big(c(D)\big)\Big)^i(\text{mod } q_1)$. So $b = q_1 c + \sum_i b_{oi}\big(c(D)\big)^i$ for some $c \in k[X]$. Applying $D$ gives $q_1 = D(b) = q_1 D(c)$, so $D(c) = 1$, as desired. □

**Corollary 3.9.** *Notations as in Theorem II. Then there exists a polynomial automorphism $\varphi$ such that $\varphi^{-1} D \varphi = \partial_1$.*

**Proof.** Since $D$ has a slice $s$ (by Theorem II), it is well known that $k[X] = k[X]^D[s]$ (cf. [5, Proposition 2.1]). So, by Theorem I, $k[X_1, \dots, X_n] = k[X_3, \dots, X_n, c(D), s]$. So the ringhomomorphism $\varphi$ of $k[X]$ to $k[X]$ defined by $\varphi(X_i) = X_i$ if $i \geq 3$, $\varphi(X_1) = s$ and $\varphi(X_2) = c(D)$ is a polynomial automorphism of $k[X]$. Since both derivations $\varphi^{-1} D \varphi$ and $\partial_1$ coincide on each $X_j$, $1 \leq j \leq n$, it follows that they are equal. □

**Remark 3.10.** It was pointed out to us by the referee that when $n = 3$ another algebraic proof of Theorem II was given by Martha Smith and David Wright in 1989 (unpublished).

**Remark 3.11.** In [4] Snow gives a geometric proof of Theorem II for the case $n = 3$. One easily verifies that the element $h$ he defines is equal to our element $c(D)$. This allows one to give a geometric proof of Theorem II along the lines of Snow's argument: just define the map $\pi : \mathbb{C}^n \to \mathbb{C}^{n-1}$ by $\pi(x_1, \dots, x_n) = (c(D), x_3, \dots, x_n)$ and replace everywhere in Snow's proof $x_3$ by the $(n-2)$-tuple $(x_3, \dots, x_n)$.

## 4. Final remarks

One can wonder if Theorems I and II can be extended to triangular derivations on $k[X]$, with $n \geq 4$, having 3 terms, i.e. to derivations of the form $p\partial_1 + q\partial_2 + r\partial_3$ with $p \in k[X_2,\dots,X_n]$, $q \in k[X_3,\dots,X_n]$ and $r \in k[X_4,\dots,X_n]$. The following example, which can be found in [1], shows that for such derivations in general similar statements as in Theorems I and II do not hold.

**Example 4.1.** Consider the derivation $D = (1+X_3^2)\partial_1+X_3\partial_2+X_4\partial_3$ on $\mathbb{C}[X_1,X_2,X_3,X_4]$. Obviously $D$ is fixed point free. However, $D$ has no slice (cf. [1]). Furthermore $\mathbb{C}[X]^D$ is not a polynomial ring in 3 variables: in fact using the algorithm of [3] one easily finds 4 generators of $\mathbb{C}[X]^D$. Using then a simple Gröbner basis computation shows that $\mathbb{C}[X]^D \simeq \mathbb{C}[Y_1,Y_2,Y_3,Y_4]/(f)$, where $f = 18Y_2 - 24Y_2^2 + 8Y_2^3 + 9Y_3^2 - Y_1Y_4$. Since $f$ has an isolated singularity at $(0, \frac{3}{2},0,0)$ it follows that $\mathbb{C}[X]^D$ is not a polynomial ring in 3 variables.

So in a certain sense the results obtained in this paper are optimal, i.e. extending the main results to triangular derivations with 3 or more terms, in case $n \geq 4$, cannot be done in the "natural" way.

Finally, as we saw in Example 4.1 the kernel of a triangular (locally nilpotent) derivation need not be a polynomial ring in $n-1$ variables. In fact it can even be worse: in [2] Deveney and Finston showed that the triangular derivation $X^3\partial_S + Y^3\partial_T + Z^3\partial_U + (XYZ)^2\partial_Y$ on the polynomial ring $\mathbb{C}[X,Y,Z,S,T,U,V]$ has a kernel which is not even finitely generated over $\mathbb{C}$! So in this light the results obtained in Theorems I and II are rather strong.

## References

[1] J.K. Deveney, D.R. Finston and M. Gehrke, $G_a$ actions on $\mathbb{C}^n$, Comm. Algebra 22(12) (1994) 4977–4988.

[2] J.K. Deveney and D.R. Finston, $G_a$ actions on $\mathbb{C}^3$ and $\mathbb{C}^7$, Comm. Algebra 22(15) (1994) 6295–6302.

[3] A. van den Essen, An algorithm to compute the invariant ring of a $G_a$-action on an affine variety, J. Symbol. Comput. 16 (1993) 551–555.

[4] D. Snow, Triangular actions on $\mathbb{C}^3$, Manuscripta Math. 60 (1988) 407–415.

[5] D. Wright, On the Jacobian conjecture, Illinois J. Math. 25 (1981) 423–440.