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A classification of invertible subsets of affine root systems

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Abstract

All subsets P of an irreducible affine root system R such that P and $R\backslash P$ are closed under addition of roots are classified. It is shown that if $\theta : R \to R'$ is a bijection of root systems such that θ and θ^{-1} preserve closed sets and the irreducible components of R and R' are affine or finite with at most one irreducible component of type A_1 then θ is an isomorphism of root systems. © 1998 Elsevier Science B.V. All rights reserved.

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1. Introduction and notation

A subset P of a root system R is said to be *closed* if $\alpha, \beta \in P$ and $\alpha + \beta \in R$ imply that $\alpha + \beta \in P$. If P is closed and satisfies the property $P \cup -P = R$ then P is said to be *parabolic*. If both P and $R\backslash P$ are closed then P is said to be *invertible*. Note that complements of parabolic sets are closed and thus parabolic sets are invertible.

The parabolic subsets of affine root systems (i.e. root systems associated to affine Lie algebras) were classified by Futorny in [3], and the invertible subsets of finite root systems (i.e. root systems in the sense of Bourbaki) were classified in [2]. The goal

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of this paper is to build on these two results in order to obtain a classification of the invertible subsets of these affine root systems. Invertible subsets arise in [5] (in fact the term "invertible" is due to Malyshev) in relation to the problem of decomposing a root system into a union of two closed subsets. This problem is related to the classification problem for complex homogeneous spaces.

We now review some notation. Let R be an arbitrary root system (finite or affine) with base Π. For any subset $\Delta \subset \Pi$, let R_Δ be the set of all $\alpha \in R$ that can be written as linear combinations of elements of Δ. Then R_Δ is a root system with base Δ. Let R_Δ^+ (resp. R_Δ^-) be the set of positive (resp. negative) roots with respect to Δ. Since the intersection of an arbitrary number of closed sets is closed, we can define the *closure* of a set $M \subset R$ to be the smallest closed set containing M. We denote the closure of M by $\langle M \rangle$.

For a closed set $P \subset R$ let $P_s := P \cap (-P)$ and $P_u := P \backslash P_s$. Now suppose that P is invertible and $Q = R \backslash P$. We define $\bar{P} := P \cup Q_s$. By [2, Lemma 2(b)] the set $\bar{P}$ is parabolic. Note that $\bar{P}_s := (\bar{P})_s = P_s \cup Q_s$, and $\bar{P}_u := (\bar{P})_u = P_u$. For a finite root system R, let $P(\Pi', \Pi'') := R_{\Pi''} \cup (R^+ \backslash R_{\Pi'})$, where $\Pi'' \subset \Pi' \subset \Pi$ and Π'' is orthogonal to $\Pi' \backslash \Pi''$.

2. Parabolic subsets of affine root systems

For a review of affine root systems see [4]. For Sections 2 and 3 let R be an irreducible affine root system with base $\Pi = \{\alpha_0, \ldots, \alpha_l\}$ labelled as in [4, Chap. 4]. Let R^{re} and R^{im} denote the real and imaginary roots of R, respectively. There exist positive integers k_i, with greatest common divisor 1, such that $\delta = \sum_{i=0}^{l} k_i \alpha_i$ is an imaginary root and $R^{\mathrm{im}} = \{k\delta \mid k \in \mathbb{Z} \backslash \{0\}\}$.

We now recall a characterization of the real roots in R. Let $\mathring{R}$ be the finite root system obtained by removing α_0, except in the case of $A_{2l}^{(2)}$, where we remove α_l. In the appropriate cases let $\mathring{R}_{\mathrm{s}}$ and $\mathring{R}_{\mathrm{l}}$ denote the sets of short and long roots in $\mathring{R}$ respectively. The following theorem is based on [4, Theorem 6.3]:

Theorem 1. *The real roots in an irreducible affine root system are as follows*:

1. $R^{\mathrm{re}} = \{\alpha + n\delta \mid \alpha \in \mathring{R},\ n \in \mathbb{Z}\}$ *when R is of type* $X_r^{(1)}$;
2. $R^{\mathrm{re}} = \{\alpha + n\delta \mid \alpha \in \mathring{R}_{\mathrm{s}},\ n \in \mathbb{Z}\} \cup \{\alpha + kn\delta \mid \alpha \in \mathring{R}_{\mathrm{l}},\ n \in \mathbb{Z}\}$ *when R is of type* $X_r^{(k)}$, $k = 2$ *or* 3, *but not of type* $A_{2l}^{(2)}$; *and*
3. $R^{\mathrm{re}} = \{\alpha + n\delta \mid \alpha \in \mathring{R},\ n \in \mathbb{Z}\} \cup \{2\alpha + (2n+1)\delta \mid \alpha \in \mathring{R}_{\mathrm{s}},\ n \in \mathbb{Z}\}$ *when R is of type* $A_{2l}^{(2)}$ *and* $l \geq 1$.

Note: In the case of $R = A_2^{(2)}$ we have $\mathring{R}_{\mathrm{s}} = \mathring{R} = \{\pm\alpha_0\}$.

We also recall some definitions from [3] to be used below. A root $\alpha_i \in \Pi$ is called a *pointed root* (relative to Π) if

1. $k_i = 1$, and
2. $\frac{1}{2}(\delta - \alpha_i) \in R$ if R is of type $A_{2l}^{(2)}$ and $\delta - \alpha_i \in R$ otherwise.

Let $\alpha \in \Pi$ and $\Pi' \subset \Pi \backslash \{\alpha\}$ with Π' nonempty, and define

$$P(\Pi, \alpha, \Pi') := \langle \{\beta + k\delta \in R \mid \beta \in R^+_{\Pi\backslash\{\alpha\}} \backslash R^+_{\Pi'},\ k \in \mathbb{Z}\} \cup (\langle R_{\Pi'} \cup \{\delta\}\rangle \backslash R^-_{\Pi'})\rangle,$$

$$P(\Pi, \alpha, \emptyset) := \langle (\Pi \backslash \{\alpha\}) \cup \{\pm\delta\}\rangle \backslash \{k\delta \mid k < 0\}.$$

The sets $P(\Pi, \alpha, \Pi')$, when α is a pointed root, play an important role in the classification of the parabolic subsets of R as seen in the following theorem based on [3, Theorems 2.6–2.8]:

Theorem 2. *P is a parabolic subset of R with $\delta \in P$ if and only if there exists a base Π of R, a pointed root α (relative to Π), and a subset Π' of $\Pi \backslash \{\alpha\}$, such that one of the following holds:*
1. *$P = P(\Pi, \alpha, \Pi') \cup R^-_{\Pi''}$, where $\Pi'' \subset \Pi'$.*
2. *$P = R^+_{\Pi} \cup R^-_{\Pi' \cup \{\alpha\}}$, where $0 < |\Pi'| < l$.*
3. *$P = \langle P(\Pi, \alpha, \emptyset) \cup \{-\delta\} \cup -\Pi'\rangle$.*

3. Invertible subsets of affine root systems

Let P be an invertible subset of R and let $Q = R \backslash P$. Using the parabolic set $\bar{P}$ associated to the invertible set P and the classification of parabolic subsets of affine root systems, it is possible to classify all invertible subsets of R. Without loss of generality we may assume that $\delta \in P$.

Lemma 3. *If $\bar{P} \cap -\bar{P} = \emptyset$ then there exists a base $\Pi = \{\alpha_0, \ldots, \alpha_l\}$ of R, a set $\Pi_M \subset \{\alpha_0, \ldots, \alpha_l\}$, and a pointed root $\alpha \in \Pi \backslash \Pi_M$ such that $P = P(\Pi, \alpha, \Pi_M)$.*

Proof. By [3, Theorem 2.6] we have $\bar{P} = P(\Pi, \alpha, \Pi_M)$ for some Π, α, Π_M as stated above. By the hypothesis and [3, Lemma 2.4] $\bar{P}_s = \emptyset$, and since $\bar{P}_s = P_s \cup Q_s$ we have $P_s = \emptyset$. Therefore $P = P_u = \bar{P}_u = \bar{P} = P(\Pi, \alpha, \Pi_M)$, as required. □

Lemma 4. *If $\bar{P} \cap -\bar{P} \neq \emptyset$ and $\delta \in P \backslash (-P)$ then there exists a base $\Pi = \{\alpha_0, \ldots, \alpha_l\}$ of R, a set $\Pi_M \subset \{\alpha_0, \ldots, \alpha_l\}$, a pointed root $\alpha \in \Pi \backslash \Pi_M$, and a nonempty subset Π' of Π_M such that one of the following holds:*
1. *$P = (P(\Pi, \alpha, \Pi_M) \backslash R^+_{\Pi'}) \cup R_{\Pi''}$, where Π'' is a subset of Π' such that $\Pi'' \perp \Pi' \backslash \Pi''$.*
2. *$P = (R^+_{\Pi} \backslash R^+_{\Pi' \cup \{\alpha\}}) \cup R_{\Pi''}$, where $|\Pi'| < l$ and Π'' is a subset of $\Pi' \cup \{\alpha\}$ such that $\Pi'' \perp (\Pi' \cup \{\alpha\}) \backslash \Pi''$.*

Proof. By [3, Theorem 2.7], there exist Π, Π', α, Π_M as above such that one of the following conditions are satisfied:

1. $\bar{P} = P(\Pi, \alpha, \Pi_M) \cup R_{\Pi'}$.

2. $\bar{P} = R^+_{\Pi} \cup R_{\Pi' \cup \{\alpha\}}$, $|\Pi'| < l$. Using the facts that $P_u = \bar{P}_u$, $P_s \cup Q_s = \bar{P}_s$, and that $P \cap R_{\Pi \setminus \{\alpha\}}$ is invertible in the finite root system $R_{\Pi \setminus \{\alpha\}}$, we can analyze the structure of P, first for case 1.

Note that by [3, Lemma 2.4], $P(\Pi, \alpha, \Pi_M) \cap -P(\Pi, \alpha, \Pi_M) = \emptyset$, and thus

$$\bar{P}_s = \bar{P} \cap -\bar{P} = (P(\Pi, \alpha, \Pi_M) \cap R_{\Pi'}) \cup (-P(\Pi, \alpha, \Pi_M) \cap R_{\Pi'}) \cup R_{\Pi'} = R_{\Pi'}.$$

This means that $P_u = \bar{P}_u = P(\Pi, \alpha, \Pi_M) \setminus R^+_{\Pi'}$ and that $P_s \subset R_{\Pi \setminus \{\alpha\}}$.

Now consider the invertible set $P \cap R_{\Pi \setminus \{\alpha\}}$, which decomposes as $(P \cap R_{\Pi \setminus \{\alpha\}})_s \cup (P \cap R_{\Pi \setminus \{\alpha\}})_u$. The decomposition of its associated parabolic $\overline{P \cap R_{\Pi \setminus \{\alpha\}}}$ with respect to the root system $R_{\Pi \setminus \{\alpha\}}$ is as follows:

$$\begin{aligned}
\overline{P \cap R_{\Pi \setminus \{\alpha\}}} &= (P \cap R_{\Pi \setminus \{\alpha\}}) \cup (R_{\Pi \setminus \{\alpha\}} \setminus (P \cap R_{\Pi \setminus \{\alpha\}}))_s \\
&= (P \cap R_{\Pi \setminus \{\alpha\}}) \cup ((R \setminus P) \cap R_{\Pi \setminus \{\alpha\}})_s \\
&= (P \cap R_{\Pi \setminus \{\alpha\}}) \cup ((R \setminus P)_s \cap R_{\Pi \setminus \{\alpha\}}) \\
&= \bar{P} \cap R_{\Pi \setminus \{\alpha\}} \\
&= (\bar{P}_s \cap R_{\Pi \setminus \{\alpha\}}) \cup (\bar{P}_u \cap R_{\Pi \setminus \{\alpha\}}).
\end{aligned}$$

Observe that $\overline{P \cap R_{\Pi \setminus \{\alpha\}}}$ has the form $R^+_{\Pi \setminus \{\alpha\}} \cup R_{\Pi'}$ since $\bar{P}_s \cap R_{\Pi \setminus \{\alpha\}} = R_{\Pi'}$ and

$$\begin{aligned}
\bar{P}_u \cap R_{\Pi \setminus \{\alpha\}} &= (P(\Pi, \alpha, \Pi_M) \setminus R^+_{\Pi'}) \cap R_{\Pi \setminus \{\alpha\}} \\
&= (P(\Pi, \alpha, \Pi_M) \cap R_{\Pi \setminus \{\alpha\}}) \setminus R^+_{\Pi'} \\
&= R^+_{\Pi \setminus \{\alpha\}} \setminus R^+_{\Pi'}.
\end{aligned}$$

Since the associated parabolic has this form, it is possible to describe the invertible subset $P \cap R_{\Pi \setminus \{\alpha\}}$ of the finite root system as in [2, Theorem 4]: the sets $\Pi'' := (\Pi \setminus \{\alpha\}) \cap (P \cap R_{\Pi \setminus \{\alpha\}})_s$ and $(\Pi \setminus \{\alpha\}) \cap (Q \cap R_{\Pi \setminus \{\alpha\}})_s$ partition Π' and can be used to form $P \cap R_{\Pi \setminus \{\alpha\}} = P(\Pi', \Pi'')$. This yields that $P_s = P_s \cap R_{\Pi \setminus \{\alpha\}} = R_{\Pi''}$ and that P has the form stated above, where the orthogonality condition arises as in [2, Theorem 4].

The proof of case 2 is entirely analogous, except that in order to consider restriction to a finite root system we use $R_{\Pi \setminus \{\beta\}}$ where β is chosen from $\Pi \setminus (\{\alpha\} \cup \Pi')$, which is nonempty by $|\Pi'| < l$. □

Lemma 5. *If $\bar{P} \cap -\bar{P} \neq \emptyset$ and $\delta \in P \cap (-P)$ then there exists a base $\Pi = \{\alpha_0, \ldots, \alpha_l\}$ of R, a pointed root $\alpha \in \Pi$ and a subset $\Pi' \subset \Pi \setminus \{\alpha\}$ such that $P = \langle P(\Pi, \alpha, \emptyset) \cup \{-\delta\} \cup -\Pi' \rangle$.*

Proof. Since δ and $-\delta$ are in P, they are also in P_s. We will show that in fact $\bar{P}_s = P_s$.

Consider an arbitrary real root $\varphi + m\delta$ for some $\varphi \in \overset{\circ}{R}$ and $m \in \mathbb{Z}$, as in Theorem 1. If $\varphi + m\delta \in P_s$ then any root of the form $\varphi + n\delta$ is in P_s. Hence, if some $\psi + m\delta$ were in Q_s then all roots of the form $\psi + n\delta$ would be in Q_s. But then $-(\psi + p\delta) \in Q_s$ for some $p \neq m$ and hence we would have $(m - p)\delta \in Q_s$, which is a contradiction. Thus $Q_s = \emptyset$ and $\bar{P}_s = P_s$. So $P = P_u \cup P_s = \bar{P}_u \cup \bar{P}_s = \bar{P}$ and by [3, Theorem 2.8], $\bar{P}$ has the form $\langle P(\Pi, \alpha, \emptyset) \cup \{-\delta\} \cup -\Pi' \rangle$. □

Since these three lemmas cover all possibilities, they can be combined to yield the following classification.

Theorem 6. *P is an invertible subset of an irreducible affine root system R if and only if P is one of the following sets:*
1. $P(\Pi, \alpha, \Pi_M)$,
2. $(P(\Pi, \alpha, \Pi_M) \backslash R^+_{\Pi'}) \cup R_{\Pi''}$,
3. $(R^+_{\Pi} \backslash R^+_{\Pi' \cup \{\alpha\}}) \cup R_{\Pi''}$,
4. $\langle P(\Pi, \alpha, \emptyset) \cup \{-\delta\} \cup -\Pi' \rangle$, *or*
5. $R \backslash P'$, *with P' one of 1–4 above,*

where Π, α, Π_M, Π', and Π'' have the appropriate forms as described in the corresponding lemmas.

Proof. Lemmas 3–5 show that 1–5 are the only possibilities for invertible subsets, and direct computation shows that each of the sets in 1–5 is in fact invertible. □

4. Bijections preserving closed sets

The term *closed map* will denote a function $\theta : R \to R'$ between root systems which has the property that $\theta(C)$ is closed whenever $C \subset R$ is closed. We use the term *pseudo-isomorphism* to denote the case where θ is a bijection and both θ and θ^{-1} are closed maps. By an isomorphism of root systems we mean a bijection which can be extended to an isomorphism of the corresponding vector spaces. Then [2, Theorem 7(d)] can be stated as:

Theorem 7. *Suppose R, R' are finite root systems and R has at most one irreducible component of type A_1. If $\theta : R \to R'$ is a pseudo-isomorphism then θ is an isomorphism.*

In this section we will extend Theorem 7 to the case where R may also have affine components. The proof of this fact is similar to that found in [2], but requires special treatment of the imaginary roots (Proposition 10) and the rank two case (Lemmas 13–15).

The following is an extension of a classical result from Bourbaki.

Lemma 8. *Let R be an irreducible affine root system with base $\Pi = \{\alpha_0, \ldots, \alpha_n\}$. For any root $\beta \in R^+$, there exists an expression of the form $\beta = \alpha_{i_1} + \cdots + \alpha_{i_m}$, such that $\alpha_{i_j} \in \Pi$ and $\alpha_{i_1} + \cdots + \alpha_{i_k}$ is a root for all $k \leq m$.*

Proof. The proof of this lemma is similar to that in the finite case, but requires [4, Propositions 5.1(c) and (e)]. □

This allows us to prove that a bijection of affine root systems which behaves linearly with respect to sums and negatives of roots is an isomorphism of root systems. From this point on, we assume unless otherwise indicated that R and R' are root systems whose irreducible components are either finite or affine.

Lemma 9. *Let $\theta : R \to R'$ be a bijection such that*
1. $\theta(-\alpha) = -\theta(\alpha)$ *for all* $\alpha \in R$;
2. α, β, $\alpha+\beta \in R$ *implies* $\theta(\alpha+\beta) = \theta(\alpha)+\theta(\beta)$;
3. α', β', $\alpha'+\beta' \in R'$ *implies* $\theta^{-1}(\alpha'+\beta') = \theta^{-1}(\alpha')+\theta^{-1}(\beta')$.

Then θ is an isomorphism.

Proof. The result follows easily from Lemma 8, analogous to [2, Lemma 6]. □

It should now be clear that we need only show that a pseudo-isomorphism satisfies conditions 1–3 in Lemma 9.

Proposition 10. *Let $\theta : R \to R'$ be a pseudo-isomorphism. Then*
(a) *for $S \subset R$, $\theta(\langle S\rangle) = \langle \theta(S)\rangle$,*
(b) *if R and R' are irreducible and affine then $\theta(m\delta) = m\theta(\delta)$.*

Proof. (a) The argument given in [2, Theorem 7(a)] is purely set-theoretic and is thus still valid.

(b) Since $\langle m\delta\rangle = \{km\delta \mid k>0\}$, while $\langle\alpha\rangle = \{\alpha\}$ for all $\alpha \in R^{\mathrm{rc}}$, we know by part (a) that $\theta|_{R^{\mathrm{im}}}$ is a bijection to $(R')^{\mathrm{im}}$. Since $(R')^{\mathrm{im}} \simeq R^{\mathrm{im}}$, we identify them and suppose $\theta(\delta) = m\delta$ and $\theta(-\delta) = n\delta$ for $m,n \in \mathbb{Z}$. Since θ is a bijection, it is clear that $mn<0$. Since $\theta(\langle\delta\rangle) = \theta((R^{\mathrm{im}})^+) = \{km\delta \mid k>0\} = \langle\theta(\delta)\rangle$ and $\theta(\langle-\delta\rangle) = \theta((R^{\mathrm{im}})^-) = \{kn\delta \mid k>0\} = \langle\theta(-\delta)\rangle$ by part (a), and since θ is a bijection, we get that $mn=-1$ so $\theta(\delta) = \pm\delta$ and $\theta(-\delta) = \mp\delta$.

Now assume that $\theta(\delta) = \delta$, with the other case being similar. Notice that the δ is superfluous since $(R^{\mathrm{im}})^+$ just has the structure of the positive integers, so for the remainder of the proof we will just consider $\theta : \mathbb{Z}^+ \to \mathbb{Z}^+$ with $\theta(1) = 1$ and $\langle m\rangle$ defined in the obvious way.

Let $S_k = \{2^k,\ldots,2^{k+1}-1\}$. Then we claim $\theta(S_k) = S_k$. This follows by induction with the case of $k=0$ being obvious. Suppose it is true for $j<k$. Then $\theta(\langle 2^k,\ldots,2^{k+1}-1\rangle) = \theta(\{2^k, 2^k+1,\ldots,2^{k+1}-1, 2^{k+1},\ldots\}) = \{2^k, 2^k+1,\ldots\}$. This should equal $\langle\theta(2^k),\ldots,\theta(2^{k+1}-1)\rangle$, but if any of $\theta(2^k),\ldots,\theta(2^{k+1}-1)$ are larger than $2^{k+1}-1$ then one of $2^k,\ldots,2^{k+1}-1$ will not appear in this set. Hence the claim holds.

Thus $\theta(\langle m\rangle \cap S_k) = \langle\theta(m)\rangle \cap S_k$, so $|\langle m\rangle \cap S_k| = |\langle\theta(m)\rangle \cap S_k|$. Now set $n_k = |\langle m\rangle \cap S_k|$. Since $|(2^k/m) - n_k| < 1$, we see[4] that $|(1/m) - (n_k/2^k)| < 1/2^k$, and $m = \lim_{k\to\infty} 2^k/n_k$.

[4] We would like to thank A. Blass for this final elegant step.

Similarly,

$$\theta(m) = \lim_{k\to\infty} \frac{2^k}{|\langle\theta(m)\rangle \cap S_k|}.$$

Hence

$$m = \lim_{k\to\infty} \frac{2^k}{|\langle m\rangle \cap S_k|} = \lim_{k\to\infty} \frac{2^k}{|\langle\theta(m)\rangle \cap S_k|} = \theta(m). \qquad \square$$

The proof of Proposition 12 requires the following lemma.

Lemma 11. *If S is an irreducible rank two affine root system and $\{\lambda,\mu\}$ is any linearly independent subset of S^{rc} then there exists a base of S with respect to which λ and μ are both positive roots.*

Proof. We have checked this in the two cases and omit the proof. $\square$

Proposition 12. *Let $\theta : R \to R'$ be a pseudo-isomorphism. If W is a two-dimensional subspace of the vector space spanned by R and $R\cap W$ is an irreducible root system of rank two then $\theta|_{R\cap W}$ can be extended to a linear map on W.*

Proof. Let S be the root system $R\cap W$ with base $\{\alpha,\beta\}$. We show that $\theta(\alpha)$ and $\theta(\beta)$ are linearly independent. This follows as in [2, Theorem 7(c)] if α and β are in a finite component of R so we assume that α and β are in an affine component. Since $\alpha,\beta\in R^{\mathrm{rc}}$ the sets $\langle\alpha\rangle$ and $\langle\beta\rangle$ have cardinality at most 2. Thus the same is true for $\langle\theta(\alpha)\rangle$ and $\langle\theta(\beta)\rangle$, showing that $\theta(\alpha),\theta(\beta)\in R'^{\mathrm{rc}}$. Since S is irreducible, the set $\langle\alpha,\beta\rangle$ has cardinality strictly greater than two. Thus the same is true for the set $\theta\langle\alpha,\beta\rangle = \langle\theta(\alpha),\theta(\beta)\rangle$. This forces $\theta(\alpha)$ and $\theta(\beta)$ to be linearly independent.

Let W' be the 2-dimensional vector space spanned by $\{\theta(\alpha),\theta(\beta)\}$ and let S' be the rank two root system $R'\cap W'$. We wish to show that S' is irreducible. Note that since S is irreducible there exists a root $\gamma\in\langle\alpha,\beta\rangle$ with $\gamma = a\alpha + b\beta$ and $a,b>0$. We have $\theta(\gamma)\in\theta(\langle\alpha,\beta\rangle)$ and therefore by Proposition 10(a), $\theta(\gamma)\in\langle\theta(\alpha),\theta(\beta)\rangle\subset S'$. The same proposition gives that $\theta(\gamma)\notin\langle\theta(\alpha)\rangle\cup\langle\theta(\beta)\rangle$ and thus S' is irreducible.

Let S^+ be the set of positive roots of S with respect to the base $\{\alpha,\beta\}$. Proposition 10(a) gives $\theta(S^+)\subset S'$. By direct inspection of the various possibilities for S it is easy to see that, given $\gamma\in\{\alpha,\beta\}$, there exist two linearly independent roots $\gamma_1,\gamma_2\in S^+$ such that $\gamma+\gamma_1=\gamma_2$. Thus $\gamma_1\in\langle\gamma_2,-\gamma\rangle$ and so by Proposition 10(a) we have $\theta(\gamma_1)\in\langle\theta(\gamma_2),\theta(-\gamma)\rangle$. Since $\gamma_1,\gamma_2\in S^+$ we have $\theta(\gamma_1),\theta(\gamma_2)\in S'$ and it follows that $\theta(-\gamma)\in S'$. Thus $\theta(S^-)=\theta\langle-\alpha,-\beta\rangle=\langle\theta(-\alpha),\theta(-\beta)\rangle\subset S'$. This shows that $\theta(S)\subseteq S'$.

Now we show that $\theta^{-1}(S')\subseteq S$. Let $\{\alpha',\beta'\}$ be a base of S' and let W'' be the span of $\{\theta^{-1}(\alpha'),\theta^{-1}(\beta')\}$. Set $S''=R\cap W''$. As above, we see that W'' is 2-dimensional and that $\theta^{-1}(S')\subseteq S''$. Since $\alpha,\beta\in W''$ we have $W''=W$ and hence $S''=S$. This shows that $\theta^{-1}(S')\subseteq S$. We have shown that θ maps S to S' bijectively, and that θ maps a base of S to a base of S'.

We now show that $\theta(-\lambda) = -\theta(\lambda)$ for all $\lambda \in S$. This is true when $|S| = |S'| < \infty$ by [2, Theorem 7(c)] so we may assume S and S' are affine. By Proposition 10(b) the claim is true for $\lambda \in S^{\mathrm{im}}$. Suppose $\lambda \in S^{\mathrm{re}}$ and that $\theta(\lambda)$ and $\theta(-\lambda)$ are linearly independent. Then by Lemma 11 there exists a base $\{\gamma_1, \gamma_2\}$ of S' with respect to which both $\theta(\lambda)$ and $\theta(-\lambda)$ are positive. But then $\{\theta^{-1}(\gamma_1), \theta^{-1}(\gamma_2)\}$ is a base of S with respect to which both λ and $-\lambda$ are positive, a contradiction. Thus $\theta(\lambda)$ and $\theta(-\lambda)$ are linearly dependent and since these are real roots in an affine root system the only possibility is $\theta(-\lambda) = -\theta(\lambda)$.

Now if $|S| = |S'|$ is finite the rest of the proof of [2, Theorem 7(c)] remains valid and $\theta|_{R \cap W}$ extends as desired. If $|S| = |S'|$ is infinite then we first show in Lemma 13 that S and S' are isomorphic and next show in Lemmas 14 and 15 that θ satisfies Property 2 in Lemma 9 for all elements of S. It is then possible to define a linear map on all of W which agrees with θ on S. $\square$

Lemma 13. *If $\theta : S \to S'$ is a pseudo-isomorphism of irreducible rank two affine root systems then S is isomorphic to S'.*

Proof. Investigation of the tables of irreducible affine root systems indicates that S and S' must be of type $A_1^{(1)}$ or $A_2^{(2)}$. Assume without loss of generality that S is of type $A_1^{(1)}$ and S' is of type $A_2^{(2)}$.

For β, γ distinct roots in S we have that $\langle \beta, \gamma \rangle$ has cardinality two or is infinite. To see this, write $\beta = \beta_0 + k\delta$, $\gamma = \gamma_0 + l\delta$, where $\beta_0, \gamma_0 \in \mathring{S} \cup \{0\} = \{\pm\alpha_1, 0\}$. If $\beta_0 = 0$ or $\gamma_0 = 0$ then $\langle \beta, \gamma \rangle$ is infinite. Otherwise $\beta = -\gamma + m\delta$, $m \in \mathbb{Z}$ or $\beta = \gamma + m\delta$, $m \in \mathbb{Z} \setminus \{0\}$. If $\beta = -\gamma + m\delta$ and $m \neq 0$ then $m\delta = \beta + \gamma \in \langle \beta, \gamma \rangle$ and $\langle \beta, \gamma \rangle$ is infinite. If $\beta = -\gamma$ or $\beta = \gamma + m\delta$, $m \neq 0$ then $\langle \beta, \gamma \rangle = \{\beta, \gamma\}$ and $\langle \beta, \gamma \rangle$ has cardinality two.

Recall that $\mathring{S}' = \{\pm\alpha_0'\}$. Now consider

$$\begin{aligned} \theta^{-1}(\langle \alpha_0', \alpha_0' + \delta' \rangle) &= \theta^{-1}(\{\alpha_0', \alpha_0' + \delta', 2\alpha_0' + \delta'\}) \\ &= \{\theta^{-1}(\alpha_0'), \theta^{-1}(\alpha_0' + \delta'), \theta^{-1}(2\alpha_0' + \delta')\}. \end{aligned}$$

Applying Proposition 10(a) to θ^{-1} shows that this set should be the same as $\langle \theta^{-1}(\alpha_0'), \theta^{-1}(\alpha_0' + \delta') \rangle$, which has cardinality two or is infinite. This contradiction implies that S and S' are both of type $A_1^{(1)}$ or $A_2^{(2)}$. $\square$

Lemma 14. *If $\theta : S \to S$ is a pseudo-isomorphism and S is of type $A_1^{(1)}$ then θ satisfies Property 2 in Lemma 9 for elements of S.*

Proof. It was already shown in Proposition 10(b) that this is true for imaginary roots in S. By the structure of $A_1^{(1)}$ (cf. Theorem 1) it suffices to show that $\theta(\pm\alpha + k\delta) = \pm\theta(\alpha) + k\theta(\delta)$, where $\alpha = \alpha_1$ and $k \in \mathbb{Z}$. We will show $\theta(\alpha + k\delta) = \theta(\alpha) + k\delta$ for $k \in \mathbb{Z}$ assuming that $\theta(\delta) = \delta$, with the cases of $-\alpha$ or $\theta(\delta) = -\delta$ being similar.

Proceed by induction on $k \in \mathbb{Z}^+$. To see that $\theta(\alpha+\delta)=\theta(\alpha)+\delta$ compare

$$\begin{aligned}\theta(\langle\delta,\alpha+\delta\rangle) &= \{\delta,2\delta,\ldots,\theta(\alpha+\delta),\theta(\alpha+2\delta),\ldots\}\\ &= \theta(\langle\delta,\alpha\rangle)\backslash\{\theta(\alpha)\} = \langle\theta(\delta),\theta(\alpha)\rangle\backslash\{\theta(\alpha)\}\\ &= \{\delta,2\delta,\ldots,\theta(\alpha)+\delta,\theta(\alpha)+2\delta,\ldots\}\end{aligned}$$

with

$$\langle\theta(\delta),\theta(\alpha+\delta)\rangle = \langle\delta,2\delta,\ldots,\theta(\alpha+\delta)\rangle.$$

Hence we must have $\theta(\alpha+\delta)=\theta(\alpha)+\delta$. Now assume that $\theta(\alpha+n\delta)=\theta(\alpha)+n\delta$ for $n<k$ and compare

$$\begin{aligned}\theta(\langle\delta,\alpha+k\delta\rangle) &= \{\delta,2\delta,\ldots,\theta(\alpha+k\delta),\theta(\alpha+(k+1)\delta),\ldots\}\\ &= \theta(\langle\delta,\alpha\rangle)\backslash\{\theta(\alpha),\theta(\alpha+\delta),\ldots,\theta(\alpha+(k-1)\delta)\}\\ &= \langle\theta(\delta),\theta(\alpha)\rangle\backslash\{\theta(\alpha),\theta(\alpha)+\delta,\ldots,\theta(\alpha)+(k-1)\delta\}\\ &= \{\delta,2\delta,\ldots,\theta(\alpha)+k\delta,\theta(\alpha)+(k+1)\delta,\ldots\}\end{aligned}$$

with

$$\langle\theta(\delta),\theta(\alpha+k\delta)\rangle = \langle\delta,2\delta,\ldots,\theta(\alpha+k\delta)\rangle.$$

Again we must have $\theta(\alpha+k\delta)=\theta(\alpha)+k\delta$ and the induction follows.

Finally, we show that $\theta(\alpha-k\delta)=\theta(\alpha)-k\delta$ for all $k\in\mathbb{Z}^+$. Assume there exists $k\in\mathbb{Z}^+$ such that $\theta(\alpha-k\delta)=\theta(\alpha)-l\delta$ with $l\neq k$ and chose k to be the minimal positive integer with this property. Since θ is a bijection the only possibility is that $l>k$. Now the set

$$\langle\delta,\alpha-k\delta\rangle = \{\delta,2\delta,\ldots,\alpha-k\delta,\alpha-(k-1)\delta,\ldots,\alpha,\alpha+\delta,\alpha+2\delta,\ldots\}$$

is closed. Since θ is a pseudo-isomorphism this implies that the set

$$\begin{aligned}\theta(\langle\delta,\alpha-k\delta\rangle) = \{&\delta,2\delta,\ldots,\theta(\alpha)-l\delta,\theta(\alpha)-(k-1)\delta,\ldots,\theta(\alpha),\\ &\theta(\alpha)+\delta,\theta(\alpha)+2\delta,\ldots\}\end{aligned}$$

is also closed. But this contradicts $l>k$, and hence the result follows. □

Lemma 15. *If $\theta: S\to S$ is a pseudo-isomorphism and S is of type $A_2^{(2)}$ then θ satisfies Property 2 of Lemma 9 for elements of S.*

Proof. Similar to Proposition 14 it suffices to show that $\theta(\pm c\alpha+k\delta)=\pm c\theta(\alpha)+k\theta(\delta)$, where $\alpha=\alpha_0$ and $c=1$ when k is even or $c=1,2$ when k is odd. Again we will treat only the cases of $\theta(\alpha+k\delta)$ and $\theta(2\alpha+k\delta)$ and assume that $\theta(\delta)=\delta$.

One can show that $\theta(\alpha+k\delta)=\theta(\alpha)+k\delta$ for all $k\in\mathbb{Z}$ (the proof is identical to the proof that $\theta(\alpha+k\delta)=\theta(\alpha)+k\delta$ in Lemma 14 except we replace $\langle\delta,\alpha+k\delta\rangle$ with $\langle 2\delta,\alpha+k\delta\rangle$).

Now let $l\in 2\mathbb{Z}+1$. We have

$$\begin{aligned}\{\theta(\alpha),\theta(\alpha+l\delta),\theta(2\alpha+l\delta)\} &= \theta(\{\alpha,\alpha+l\delta,2\alpha+l\delta\})=\theta(\langle\alpha,\alpha+l\delta\rangle)\\ &= \langle\theta(\alpha),\theta(\alpha+l\delta)\rangle=\langle\theta(\alpha),\theta(\alpha)+l\delta\rangle\end{aligned}$$

and the only possibility is that $\theta(2\alpha+l\delta)=2\theta(\alpha)+l\delta$. □

This completes the proof of Proposition 12 and we can now use the preceding propositions to prove the main result.

Theorem 16. *If $\theta:R\to R'$ is a pseudo-isomorphism of root systems with the finite or affine components and R has at most one irreducible component of type A_1 then θ is an isomorphism.*

Proof. The proof can now proceed as in [2, Theorem 7(d)] by verifying parts 1–3 of Lemma 9. We claim that $\theta(-\alpha)=-\theta(\alpha)$ for all $\alpha\in R$. There are two cases to consider: (i) α is a root in an irreducible component of R of type A_1, or (ii) there exists a two-dimensional subspace W of the vector space spanned by R such that $\alpha\in W$ and $S=R\cap W$ is an irreducible rank two subsystem of R. The claim follows in case (i) since θ is a bijection. In case (ii) the claim follows from Proposition 12.

Finally, we show that $\alpha,\beta,\alpha+\beta\in R$ implies that $\theta(\alpha+\beta)=\theta(\alpha)+\theta(\beta)$. Propositions 10(b) and [2, Theorem 7(b)] yield this result when α and β are linearly dependent, while Proposition 12 does the same when $\alpha+\beta\in R$ and α and β are linearly independent (by taking W to be the span of $\{\alpha,\beta\}$).

By symmetry we have that $\theta^{-1}(\alpha+\beta)=\theta^{-1}(\alpha)+\theta^{-1}(\beta)$ for $\alpha,\beta,\alpha+\beta\in R'$ as well. □

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