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A COHOMOLOGICAL INTERPRETATION OF THE GRADED BRAUER GROUP II

S. CAENEPEEL
Free University of Brussels, V.U.B., Belgium

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0. Introduction

It is a well-known fact in the theory of *Brauer groups* of commutative rings, that there exists a natural monomorphism $\mathrm{Br}(R)\to H^2(R,U)$, where $H^2(R,U)$ is the *étale Brauer group* of R. As was recently pointed out by O. Gabber, it is even true that $\mathrm{Br}(R)\cong H^2(R,U)_t$. In this paper, we deal with the so-called '*graded Brauer group*' of $\mathbb{Z}$-graded commutative ring (cf. F. Van Oystaeyen [17]), and try to generalise the above properties.

As we encounter some technical difficulties trying to generalise a fundamental theorem due to M. Artin [1], we restrict ourselves to the so-called *quasistrongly graded rings*. It will turn out that some importatnt classes of graded rings fall within this scope, and that $\mathrm{Br}^g(R)$ may be embedded in $\varinjlim H^2(S/R,U_0)$, where the limit is taken over graded étale coverings of R, and $U_0(R)$ is the multiplicative group of units of degree zero of R.

Also, a more general class of rings is considered (*d-quasistrongly graded rings*), and it is shown that their graded Brauer group may embedded in $\varinjlim H^2_{\mathrm{gr}}(S/R,U)$, a cohomology group containing non-homogeneous cocycles. The existence of two different embeddings is not surprising, since we have two corresponding graded versions of the crossed product theorem (cf. [4]).

In the last section, it is pointed out that Gabber's theorem holds for quasistrongly graded rings. To this end, we need a description of the Grothendieck group of the category of graded progenerators (Section 3) and some variants of the Mayer–Vietoris sequences (Section 4).

The theory of Brauer groups can be found in [2], [8], [11].

For the graded Brauer group, we refer to [17], [18], [21].

1. Preliminary results and notations

All rings R considered in this paper will be $\mathbb{Z}$-graded. R_i will be the set of elements of degree i, and $h(R)$ the set of homogeneous elements. For $a\in R_i$, we denote $\deg_R a=i$, and omit the subscript whenever ambiguity is excluded. Similar notations are introduced for graded R-modules, graded R-algebras, ...

A homomorphism $f:M\to N$ of graded R-modules is called *homogeneous of degree p* if $f(M_i)\subset N_{i+p}$ for all i in $\mathbb{Z}$. A homomorphism of degree zero will also be called a *graded homomorphism*. If there exists a graded isomorphism between M and N, then we say that M and N are *graded isomorphic*, or *gr-isomorphic*, denoted $M\cong_g N$.

R-gr is the category of graded R-modules with graded homomorphisms.

There exists a natural forgetful functor $\mathrm{Un}:R\text{-mod}\to R\text{-gr}$, sending a graded R-module M to its underlying ungraded module $\mathrm{Un}(M)=\underline{M}$.

The unadorned symbol $\otimes$ will always mean $\otimes_R$. We denote $S^{(n)}=S\otimes S\otimes\cdots\otimes S$, and $\otimes_n$ will be a shorter notation for $\otimes_{S^{(n)}}$. The natural switch map $A\otimes B\to B\otimes A$ is denoted by τ.

For R-modules $M_{(1)},\dots,M_{(n)}$, we define $\varepsilon_i:M_{(1)}\otimes\cdots\otimes M_{(n)}\to M_{(1)}\otimes\cdots\otimes S\otimes\cdots\otimes M_{(n)}$ by $\varepsilon_i(m_1\otimes\cdots\otimes m_n)=m_1\otimes\cdots\otimes 1\otimes\cdots\otimes m_n$.

For an R-module M, let $M_1=S\otimes M$. $M_2=M\otimes S$, $M_{12}=M_{11}=S\otimes S\otimes M,\dots$. An $S^{(2)}$-homomorphism $g:M_1\to M_2$ then induces three homomorphisms $g_1:M_{11}\to M_{13}$, $g_3:M_{13}\to M_{23}$, $g_2:M_{11}\to M_{23}$.

Let us briefly recall the definition of the graded Brauer group: consider the set of gr-isomorphism classes of graded R-Azumaya algebras, modulo the equivalence relation $\sim$, given by

$$A\sim B \Leftrightarrow A\otimes B^0\cong_g \mathrm{END}_R(P)$$

for some graded R-progenerator P. Recall that $\mathrm{END}_R(P)$ is the graded R-algebra generated by all homogeneous endomorphisms of P, and that $\mathrm{Un}(\mathrm{END}_R(P))=\mathrm{End}_R(P)$ if P is finitely generated. The set thus obtained forms a group under the operation induced by the tensor product, the graded Brauer group of R, denoted $\mathrm{Br}^g(R)$.

The natural mapping $\mathrm{Br}^g(R)\to\mathrm{Br}(R)$ is monomorphic in some situations, e.g. when R is a graded Krull domain (cf. [6], [17]), but not in general (M. Van den Bergh, private communication).

The proofs of the following three propositions are left to the reader, as they are very similar to the corresponding ungraded proofs (cf. e.g. [11] for 1.1 and 1.2, and [7], [22] for 1.3.).

1.1. Proposition. $\mathrm{Br}^g(R)$ *is a torsion group.*

1.2. Proposition. *For any graded R-Azumaya algebra A, there exists a Noetherian subring R′ of R, and a graded R′-Azumaya algebra A′ such that* $R\otimes_{R'}A'\cong_g A$.

1.3. Proposition. *If I is a nilpotent graded ideal of R, then the canonical homomorphism* $\mathrm{Br}^g(R)\to\mathrm{Br}^g(R/I)$ *is an isomorphism.*

1.4. Notations. For a graded commutative ring R, we define the following categories with product by their objects and morphisms.

Category	Objects	Morphisms	Product
$\mathbf{P}(R)$	finitely generated projective R-modules	R-module homomorphisms	$\oplus$
$\mathbf{FP}(R)$	R-progenerators	R-module homomorphisms	$\otimes$
$\mathbf{Pic}(R)$	R-progenerators of rank one	R-module homomorphisms	$\otimes$
$\mathbf{Az}(R)$	R-Azumaya algebras	R-algebra homomorphisms	$\otimes$

Let $\mathbf{P}^g(R)$, $\mathbf{FP}^g(R)$, $\mathbf{Pic}^g(R)$, $\mathbf{Az}^g(R)$ be the respective full subcategories consisting of graded R-modules and graded R-algebras.

$\mathbf{P}_g(R)$, $\mathbf{FP}_g(R)$, $\mathbf{Pic}_g(R)$, $\mathbf{Az}_g(R)$ are the categories with the same respective objects, but with *graded* R-module or R-algebra homomorphisms.

It is then easily seen that $K_0\,\mathbf{Pic}_g(R)=\mathrm{Pic}_g(R)$, $K_0\,\mathbf{Pic}^g(R)=\mathrm{Pic}^g(R)$, respectively the groups of graded isomorphism classes and isomorphism classes of graded R-progenerators of rank one.

Furthermore, $K_1\,\mathbf{Pic}^g(R)=K_1\,\mathbf{Pic}(R)=U(R)$, and $K_1\,\mathbf{Pic}_g(R)=U_0(R)$. Consider the natural functor $F:\mathbf{Pic}_g(R)\to\mathbf{Pic}^g(R)$, and the group $K_1\underline{\phi F}$ (cf. [3]). $K_1\underline{\phi F}$ can then be described as follows: let $\mathrm{gr}(R)$ be the set of graded ring isomorphism classes of graded R-progenerators isomorphic to R as an R-module, and which have a ring structure in this way, i.e. the set of all possible R-module gradations on R. This forms a group under the operation induced by the tensor product. A little computation shows that $a:\mathrm{gr}(R)\to K_1\underline{\phi F}$, given by $a([T])=[(R,\mathrm{id},T)]$ is an isomorphism. We therefore have (cf. [3], [4]):

1.5. Proposition. *We have an exact sequence*

$$1\to U_0(R)\to U(R)\xrightarrow{d}\mathrm{gr}(R)\to\mathrm{Pic}_g(R)\to\mathrm{Pic}^g(R)\to 1$$

where the homomorphism d is given by the following data:

$$\textit{for } u\in U(R):\quad \begin{cases}\mathrm{Un}(d(u))=\mathrm{Un}(R),\\ \deg_{d(u)}(u)=0\end{cases}$$

(i.e. $a(d(u))=[(R,u^{-1},R)]$*).*

Let $I,I'\in\mathbf{Pic}^g(R)$ such that there exists an isomorphism $\phi:I\to I'$. Then, according to [4, 2.8], there exists a (unique) $T\in\mathrm{gr}(R)$ and a graded isomorphism $\chi:I\to I'\otimes T$ such that $\mathrm{Un}(\chi)=\phi$. We denote $\mathrm{gr}(\phi)=T$; We also have that $\mathrm{gr}(\phi\cdot\phi')=\mathrm{gr}(\phi)\otimes\mathrm{gr}(\phi')$.

1.6. Lemma. *Suppose $T, T' \in \mathrm{gr}(R)$ such that $\mathrm{id}: I \otimes T \to I \otimes T'$ is graded, for some $I \in \mathbf{Pic}_g(R)$. Then $T = T'$ in $\mathrm{gr}(R)$.*

Proof. In $K_1\underline{\phi F}$, we have $a(T) = [(R, \mathrm{Id}, T)] = [(I, \mathrm{Id}, I \otimes T)] = [(I, \mathrm{id}, I \otimes T')] = [(R, \mathrm{Id}, T')] = a(T')$, where the third equality holds in view of the following diagram of R-isomorphisms (the vertical ones being graded):

$$\begin{array}{ccc}
I & \xrightarrow{\mathrm{id}} & I \otimes T \\
{\scriptstyle \mathrm{id}}\downarrow & & \downarrow{\scriptstyle \mathrm{id}} \\
I & \longrightarrow & I \otimes T'
\end{array}$$

Hence $T = T'$ in $\mathrm{gr}(R)$.

If R is reduced, then $\mathrm{gr}(R)$ can be calculated, in fact we have (cf. [4, 2.3]):

1.7. Theorem. *If R is reduced (semi-prime), then $\mathrm{gr}(R) \cong H^0(R)$, the set of continuous functions from $\mathrm{Spec}(R)$ to $\mathbb{Z}$.*

The preceding theorem has an interesting corollary: if R is reduced, then all invertible elements of R are 'nearly' homogeneous:

1.8. Corollary. *If R is a reduced commutative graded ring, then every $u \in U(R)$ is of the form $\sum_{i=1}^n u_i e_i$, where $e_i \in \mathrm{Idemp}(R_0)$, $u_i \in h(R)$.*

Proof. Consider $M = d(u)$. According to 1.7, there exist $e_1, \ldots, e_n \in \mathrm{Idemp}(R) = \mathrm{Idemp}(R_0)$ (cf. [5, I.3]) such that $\deg_{M_i}(e_i) = r_i$ for some $r_i \in \mathbb{Z}$, and $\sum_{i=1}^n e_i = 1$. Put $M_i = e_i M$, then the gradation on M_i is given by $\deg_{M_i}(e_i) = r_i$. This means that for any x in Re_i, $\deg_{M_i} x = \deg_{Re_i} x + r_i$. Since $\deg_{M_i}(ue_i) = 0$, we have $\deg_{Re_i}(ue_i) = -r_i$. Now $u = \sum_{i=1}^n (ue_i)e_i$, establishing the result.

Let S be a graded faithfully flat ringextension of R. Then the Amitsurcomplexes $\mathscr{C}(U)$ and $\mathscr{C}(\mathrm{gr})$ can be written as follows (cf. [4]):

$$\begin{array}{ccccccccc}
1 & \longrightarrow & U(S) & \xrightarrow{\Delta_0} & U(S^{(2)}) & \xrightarrow{\Delta_1} & U(S^{(3)}) & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle d_0} & & \downarrow{\scriptstyle d_1} & & \downarrow{\scriptstyle d_2} & & \\
1 & \longrightarrow & \mathrm{gr}(S) & \xrightarrow{D_0} & \mathrm{gr}(S^{(2)}) & \xrightarrow{D_1} & \mathrm{gr}(S^{(3)}) & \longrightarrow & \cdots
\end{array}$$

where $d_i: U(S^{(2)}) \to \mathrm{gr}(S^{(2)})$ is the map described in 1.4. We define a new complex $(C, \nabla) = \mathscr{C}_{\mathrm{gr}}(U)$:

$$C^n = U(S^{(n+1)}) \times \mathrm{gr}(S^{(n)}),$$

$$\nabla_n = (U, I) = (\Delta_n u, D_{n-1} I \cdot (d_n u)^{-1}).$$

The corresponding cohomology group $H^n_{\mathrm{gr}}(S/R, U) = \mathrm{Ker}\, \nabla_n / \mathrm{Im} \nabla_{n-1}$ is called the *n-th graded Amitsur cohomology group* of U. Recall the crossed product theorems from [4]:

1.9. Proposition. *If S is a graded R-progenerator, then*

$$\mathrm{Pic}^{\mathrm{g}}(S) = \mathrm{Pic}^{\mathrm{g}}(S \otimes S) = 1 \;\Rightarrow\; \mathrm{Br}^{\mathrm{g}}(S/R) \cong H^2_{\mathrm{gr}}(S/R, U);$$

$$\mathrm{Pic}_{\mathrm{g}}(S) = \mathrm{Pic}_{\mathrm{g}}(S \otimes S) = 1 \;\Rightarrow\; \mathrm{Br}^{\mathrm{g}}(S/R) \cong H^2(S/R, U_0).$$

An extension S of R is called a graded étale covering of R if it is a graded R-algebra which is an étale covering, i.e. if S is graded, R-separable, finitely presented as an R-algebra and faithfully flat over R. Thanks to the arguments given in [11, Ch. V, §1], we may define for any covariant functor F from the category of graded R-algebras to abelian groups:

$$H^n_{\mathrm{g}}(R, F) = \varinjlim H^n(S/R, F),$$

$$H^n_{\mathrm{ggr}}(R, U) = \varinjlim H^n_{\mathrm{gr}}(S/R, U)$$

where the limits are taken over all graded étale coverings S of R.

If no confusion is possible, we write $H^n_{\mathrm{g}}(F) = H^n_{\mathrm{g}}(R, F)$.

2. Embedding of $\mathrm{Br}^{\mathrm{g}}(R)$ in the étale Brauer group

As before, R is a graded commutative ring. $R_{(n)}$ is the graded ring defined by

$$\mathrm{Un}(R_{(n)}) = \mathrm{Un}(R), \qquad R_i = (R_{(n)})_{ni},$$

i.e. we have $\deg_{R(n)}(a) = n \deg_R(a)$.

2.1. Definition. R is called *strongly graded* if $R_n R_m = R_{n+m}$ for all n, m in $\mathbb{Z}$, or equivalently, if $RR_1 = R$. R is called *quasistrongly graded* if there exists a graded étale covering of R containing an invertible element of degree 1. R is called *d-quasistrongly graded* ($d \in \mathbb{N}_0$) if $R = T_{(d)}$ for some quasistrongly graded ring T.

2.2. Examples. (a) If R contains an invertible element T of degree $d > 0$, and if $d \in U(R)$, then R is quasistrongly graded. Indeed, $S = R[X]/(X^d - T)$ is a graded étale covering of R satisfying the condition.

(b) If R contains a *generalised Rees ring* $A = \bigoplus_{n \in \mathbb{Z}} I^n X^n$, where $\deg(IX) = d$, and $d \in U(A)$, then R is quasistrongly graded. Recall from [19] that a generalised Rees ring is obtained as follows: let A_0 be a domain, and I an invertible fractional ideal of A_0. Then $A=\bigoplus_{n\in\mathbb{Z}} I^nX^n$ is a graded ring. Let S_0 be an étale covering of R_0 such that $S_0\otimes_{R_0} I\cong S_0$. Then $S=S_0\otimes_{A_0}A\cong_g S_0[X,X^{-1}]$, so $S\otimes A$ satisfies the conditions of (a).

Similar arguments show that the *scaled Rees rings* and *Lepidoptorous rings*, studied in [16], [20] fall within this scope.

As the following proposition shows, d-quasistrongly gradedness is a gr-local property. Q_p^g means localisation at $h(R-p)$.

2.3. Proposition. *For a graded commutative ring R, the following conditions are equivalent:*

(i) *R is quasistrongly graded.*

(ii) *For all p in* $\mathrm{Spec}^g(R)$, *$Q_p^g(R)$ is quasistrongly graded.*

(iii) *For all p in* $\mathrm{Spec}^g(R)$, *$Q_p^g(R)$ contains an invertible element of invertible degree.*

Proof. The implications (i)$\Rightarrow$(ii) and (iii)$\Rightarrow$(ii) are obvious. For (ii)$\Rightarrow$(i), suppose that $Q_p^g(R)$ is quasistrongly graded, for all p in $\mathrm{Spec}^g(R)$. Let $S(p)$ be a graded étale covering of $Q_p^g(R)$ containing a unit of degree one. Then there exists $f\in h(R-p)$, and a graded étale covering $S(f)$ of $Q_f(R)$ containing a unit of degree one.

Using the quasicompactness of $\mathrm{Spec}^g(R)$, we therefore get a graded Zarisky covering $\prod_{i=1}^n S(f_i)$ of R, which is a graded étale covering.

For (ii)$\Rightarrow$(iii), consider a gr-local quasistrongly graded ring R. Then R/m is quasistrongly graded, so it is of the form $k[T,T^{-1}]$, where $\deg T=d$ is invertible in k. Let T' be a homogeneous lifting of T, then T' is invertible, $\deg T'=d$, and d is invertible in R.

The next proposition is a graded version of Artin's theorem [1, 4.1]. We tried to prove it for general (Noetherian) graded rings, but encountered some technical difficulties, lying in the fact that not every graded projective module of constant rank over a gr-semilocal ring is graded free.

2.4. Proposition. *Let R be a quasistrongly graded Noetherian ring, and S_i some graded étale R-algebras ($1\le i\le n$). If U is a graded étale covering of $S_1\otimes\cdots\otimes S_n$, then there exist graded étale coverings S_i' of S_i such that we have a factorisation*

$$S_1\otimes\cdots\otimes S_n\to U\to S_1'\otimes\cdots\otimes S_n'$$

where the connecting homomorphisms are graded.

Proof. First, suppose that R contains a unit x of degree one. Then $R=R_0[x,x^{-1}]$, so $\otimes_R S_i=\otimes_{R_0}(S_i)_0[x,x^{-1}]$, and $U=U_0[x,x^{-1}]$, where U_0 is a graded étale covering of $\otimes_{R_0}(S_i)_0$. Hence there exist étale coverings $(S_i')_0$ and a factorisation $\otimes_{R_0}(S_i)_0\to U_0\to\otimes_{R_0}(S_i')_0$, by Artin's theorem. Next let $S_i'=(S_i')_0[x,x^{-1}]$.

For the general case, let $\tilde{R}$ be a graded étale covering of R containing a unit x of degree one; putting $\tilde{S}_i = S_i \otimes \tilde{R}$, $\tilde{U} = U \otimes \tilde{R}$, one finds graded étale coverings S_i' of $\tilde{R}$ and R, and a factorisation

$$\otimes_{\tilde{R}} \tilde{S}_i = (\otimes S_i) \otimes \tilde{R} \to U \otimes \tilde{R} \to \otimes_{\tilde{R}} S_i' = \otimes S_i',$$

hence $\otimes S_i \to U \to \otimes S_i'$, establishing the result.

Let A be a graded R-Azumaya algebra, S a graded faithfully flat extension of R splitting A in the graded sense, i.e. we have a graded isomorphism $\sigma : A \otimes S \to \mathrm{END}_S(P)$, for some graded S-progenerator P. As usual, define the graded isomorphism $\phi : \mathrm{END}_S(2)(P_1) \to \mathrm{END}_S(2)(P_2)$ by

$$\phi = (\sigma \otimes 1) \circ (\tau \otimes 1) \circ (1 \otimes \sigma^{-1}).$$

By the graded version of the Morita theorems (cf. [4]), ϕ is induced by a graded isomorphism $f : P_1 \otimes_2 I \to P_2$, where $I \in \mathrm{Pic}^{\mathrm{g}}(S^{(2)})$.

2.5. Definition. If $[I] = 1$ in $\mathrm{Pic}^{\mathrm{g}}(S^{(2)})$ ($\mathrm{Pic}_{\mathrm{g}}(S^{(2)})$), then we say that (S, P, σ, ϕ, f) is a *(very) good graded splitting datum* for A.

2.6. Theorem. *Let A be a graded R-Azumaya algebra. If R is d-quasistrongly graded, then A admits a good graded splitting datum which is an étale covering. If R is quasistrongly graded, then A admits a very good graded étale covering splitting datum.*

Proof. Let us first state that A admits a graded étale covering splitting. For a gr-local ring R, this can be done in general, but in our situation, it suffices to refer to [6, Proposition 2.4], where the result was proved for a gr-local ring containing a homogeneous unit of degree different from zero.

Hence for all p in $\mathrm{Spec}^{\mathrm{g}}(R)$, there exists a graded étale covering $S(p)$ of $Q_p^{\mathrm{g}}(R)$ splitting $Q_p^{\mathrm{g}}(R)$. But then we can find $f \in h(R-p)$, and a graded étale covering $S(f)$ of R_f splitting A_f. Denote $U_f = \{p \in \mathrm{Spec}^{\mathrm{g}}(R) : f \notin p\}$, then the collection of all obtained U_f cover $\mathrm{Spec}^{\mathrm{g}}(R)$, which is quasicompact, so can be restricted to a finite covering $\{U_{f_1}, \ldots, U_{f_n}\}$ of $\mathrm{Spec}^{\mathrm{g}}(R)$. But then $S = \prod_i S(f_i)$ is a graded étale covering of $\prod_i R_{f_i}$ and R, and a standard computation shows that S splits A. Also note that this construction implies $\deg S = \deg R$. Thanks to Proposition 1.2, we can restrict ourselves to the case where R is Noetherian. Suppose R quasistrongly graded, and S obtained as above. Then we get $I \in \mathbf{Pic}^{\mathrm{g}}(S^{(2)})$. There exists a graded étale covering U of $S^{(2)}$ such that $I \otimes_2 U \cong U$; indeed, one first observes that $Q_p^{\mathrm{g}}(I) \cong Q_p^{\mathrm{g}}(S^{(2)})$, for all $p \in \mathrm{Spec}(S^{(2)})$, and then uses the quasicompactness. If we chose S such that it contains a unit of degree one, then we have $Q_p^{\mathrm{g}}(I) \cong g Q_p^{\mathrm{g}}(S^{(2)})$, hence U can be chosen such that $I \otimes_2 U \cong_{\mathrm{g}} U$. Next, apply 2.4 to get a graded étale covering S' of S factorising $S^{(2)} \to U \to S'^{(2)}$. Then

$$I \otimes_2 S'^{(2)} \cong_{\mathrm{g}} (I \otimes_2 U) \otimes_U S'^{(2)} \cong_{\mathrm{g}} U \otimes_U S'^{(2)} \cong_{\mathrm{g}} S'^{(2)} = 1$$

in $\mathrm{Pic}_g(S'^{(2)})$. This establishes a very good graded splitting $(S', P' = P \otimes_1 S', \sigma' = \sigma \otimes_1 1_{S'}, \phi', f')$.

Finally, for the d-quasistrongly graded case, repeat the above argument, noting that the isomorphism $I \otimes_2 U \cong U$ is not graded in general, so that we do get a good graded splitting which might be not very good.

Also observe that $\deg S = \deg S^{(2)} = \deg U$, so that we can apply 2.1 to $R_{(1/n)}$, obtaining S'.

Note. Not every $A \in \mathbf{Az}_g(R)$ admits a very good étale covering splitting datum, if R is d-quasistrongly graded. For example, consider $K = \mathbb{F}_2[T, T^{-1}, \deg T = 2]$. Then K is 2-quasistrongly graded, but not quasistrongly graded. let l be the Galois extension of k defined by the equation $\alpha^2 + \alpha - 1 = 0$, and $L = l[T, T^{-1}]$.

Let A be the graded K-Azumaya algebra defined as a Galois-crossed product by the cocycle f:

$$f(1,1) = f(1,\sigma) = f(\sigma,1) = 1; \qquad f(\sigma,\sigma) = T.$$

Then A does not admit a very good étale splitting datum (since any graded étale covering of K is $2\mathbb{Z}$-graded). Also cf. [4, Theorem 4.7].

Note however that A admits a very good splitting datum which is faithfully flat but not étale: let $M = k[x, x^{-1}]$ where $x^2 = T$, then A can be written as a *differential* crossed product over M in degree zero: A is generated by $\{1, Z\}$, with multiplication rules $Z^2 = T^{-1}$, $ZX = XZ + 1$ $(\alpha = XZ)$.

2.7. Proposition. *If R is an arbitrary commutative graded ring, then*

(a) $U_0(R) = H^0_g(U_0) = H^0_{ggr}(U)$.

(b) $U(R) = H^0_g(U) = H^0(U)$.

(c) $\mathrm{Pic}^g(R) = \mathrm{Ker}(H^1_g(U) \to H^1_g(gr)) = \mathrm{Im}(H^1_{ggr}(U) \to H^1_g(U))$.

(d) *If R is quasistrongly graded, then* $\mathrm{Pic}_g(R) = H^1_g(U_0)$.

Proof. (a) and (b) follow directly from [4, Propositions 4.1 and 4.4].

(c) follows from [4, 4.3] and the observation that for any $[I] \in \mathrm{Pic}^g(R)$, one can find a graded faithfully flat extension S such that $[I \otimes S] = 1$ in $\mathrm{Pic}^g(S)$ (cf. the proof of 2.6).

(d) follows in a similar way from [4, 4.1b].

2.8. Theorem. *If R is quasistrongly graded, then there exists a natural monomorphism $\theta: \mathrm{Br}^g(R) \to H^2_g(U_0)$. If R is d-quasistrongly graded, then there exists a natural monomorphism $\theta: \mathrm{Br}^g(R) \to H^2_{ggr}(U)$.*

Proof. The first part can be shown in a way which is very similar to the one followed by Knus & Ojanguren in [11], remarking that all mappings are graded, and using Proposition 2.6. We provide the proof for the second part, as some pecularities of a graded nature are involved there.

Let A represent $[A]\in \mathrm{Br}^{\mathrm{g}}(R)$, and (S,P,σ,ϕ,f) be a good graded étale covering splitting datum. We keep the notations of 2.6. ϕ is induced by a graded isomorphism $f:P_1\to P_2\otimes_2 I$, where $I\in \mathbf{Pic}^{\mathrm{g}}(S^{(2)})$. Thanks to the exact sequence 1.4, and the fact that $[I]=1$ in $\mathrm{Pic}^{\mathrm{g}}(S\otimes S)$, we can view $[I]$ as an element of $\mathrm{gr}(S\otimes S)$. Consider $\underline{f}=U(f):\underline{P}_1\to\underline{P}_2$, then $\underline{f}_2^{-1}\underline{f}_3\underline{f}_1$ is multiplication by a unit $\underline{u}\in U(S^{(3)})$ as it induces the identity $\phi_2^{-1}\phi_3\phi_1$, and $\underline{u}$ is a cocycle. Note that $d(\underline{u})=\mathrm{gr}(\underline{f}_2^{-1}\underline{f}_3\underline{f}_1)=I_1\otimes_3 I_3\otimes_3 I_2^{-1}=D_1(I)$, since the isomorphism

$$f_2^{-1}f_3f_1:P_{11}\otimes_3 I_1\otimes_3 I_3\otimes_3 I_2^{-1}\xrightarrow{f_1}P_{13}\otimes_3 I_3\otimes_3 I_2^{-1}\xrightarrow{f_3}P_{23}\otimes_3 I_2^{-1}\xrightarrow{f_2^{-1}}P_1$$

is graded. hence $u=(\underline{u},I)$ is a cocycle in the gr-cohomology sequence $\mathscr{C}_{\mathrm{gr}}(S/R,U)$. Let $[u]$ be the image of u in $H^2_{\mathrm{ggr}}(U)$. From the following lemmas, it follows that $[A]\to[u]$ induces the desired monomorphism.

2.9. Lemma. *If* $[A]=1$ *in* $\mathrm{Br}^{\mathrm{g}}(R)$, *and* (S,P,σ,ϕ) *is an arbitrary* gr-*étale covering splitting datum, then there exists a graded étale covering* S' *of* S, *such that extension of scalars yields a good graded étale splitting* (S',P',σ',ϕ',g), *with* $g_2^{-1}g_3g_1=1$ *and* $\mathrm{gr}(\underline{g})=1$ *in* $H^1(S'/R,\mathrm{gr})$.

Proof. From the Morita equivalence [4, 1.2], $\sigma:(\mathrm{END}_R\,Q)\otimes S\to\mathrm{END}_S\,P$ is induced by a graded $h:Q\otimes S\to P\otimes_1 I$, $I\in\mathbf{Pic}_{\mathrm{g}}\,S$.

Replacing S by a suitable graded étale covering, we can suppose that $[I]=1$ in $\mathrm{Pic}^{\mathrm{g}}\,S$, hence view I as an element of $\mathrm{gr}(S)$. Then define $f:P_1\to P_2$ by commutativity of the following diagram:

$$\begin{array}{ccc}
Q_{23} & \xrightarrow{h\otimes 1} & P_2\otimes_2 I_2\\
\Big\downarrow{\scriptstyle \tau_3} & & \Big\downarrow{\scriptstyle f}\\
Q_{13} & \xrightarrow{1\otimes h} & P_1\otimes_2 I_1
\end{array}$$

From the theorem on faithfully flat descent, it follows that $\underline{f}_2=\underline{f}_3\underline{f}_1$, and $[\mathrm{gr}(\underline{f})]=[I_1\otimes I_2^{-1}]=1$ in $H^1(S/R,\mathrm{gr})$.

2.10. Lemma. *If* (S,P,σ,ϕ,f) *is a good graded étale covering splitting for* A, *and* $f_2^{-1}f_3f_1=1$, *and* $[\mathrm{gr}\,f]=1$ *in* $H^1(S/R,\mathrm{gr})$, *then* $[A]=1$.

Proof. Let $\mathrm{gr}(f)=I_1\otimes_2 I_2^{-1}$, $[I]\in\mathrm{gr}(S)$, then we get a graded isomorphism $f:\mathrm{P}_2\otimes_2 I_2\to P_1\otimes_2 I_1$, which is a graded descent datum, inducing a graded R-progenerator Q:

$$\begin{array}{ccc}
Q_{23} & \longrightarrow & (P\otimes_1 I)\otimes S\\
\Big\downarrow & & \Big\downarrow\\
Q_{13} & \longrightarrow & S\otimes(P\otimes_1 I)
\end{array}$$

inducing

$$\begin{array}{ccc} \mathrm{END}_R(Q_{23}) & \longrightarrow & \mathrm{END}_{S^{(2)}}((P\otimes_1 I)\otimes S) \\ \downarrow & & \downarrow \\ \mathrm{END}_R(Q_{13}) & \longrightarrow & \mathrm{END}_{S^{(2)}}(S\otimes(P\otimes_1 I)) \end{array}$$

From uniqueness in [4, 1.4], it follows that $A\cong_{\mathrm{g}}\mathrm{END}_R(Q)$.

2.11. Lemma. *The cocycle u defined above is independent of the choice of f and I.*

Proof. Let $f:P_1\to P_2\otimes_2 I$, $g:P_1\to P_2\otimes_2 J$ induce ϕ. Put $f_2^{-1}f_3f_1=\underline{u}$, $g_2^{-1}g_3g_1=\underline{v}$. Then $f=tg$ for some $\underline{t}\in U(S^{(2)})$, so $\underline{u}=(\Delta_2\underline{t})\underline{v}$. Also $[I]=\mathrm{gr}(\bar{f})=\bar{d}_2(\underline{t})=\bar{d}_2(\underline{t})\,\mathrm{gr}(\bar{g})=d_2(\underline{t})[\bar{J}]$, hence $[(\underline{u},I)(\underline{v}^{-1},J^{-1})]=[(\Delta_2\underline{t},d_2\underline{t})]=1$ in $H^2_{\mathrm{gr}}(S/R,U)$.

So the cocycle $[u]$ defined above depends only on A and σ. Denote $[u]=\theta(\sigma,A)$.

2.12. Lemma. *If* $[A]=1$, *then* $\theta(\sigma,A)=1$.

Proof. First note that if S' is a graded étale covering of S, then $\theta(\sigma',A)=\theta(\sigma,A)$. Now take S' as in Lemma 2.9. Then it follows that $\theta(\sigma',A')=[(1,I)]=1$, since $[I]=1$ in $H^2(S/R,\mathrm{gr})$.

2.13. Lemma. *Let* (S,P,σ,ϕ,f), (T,Q,τ,ψ,g) *be good graded étale covering splittings for A and B. Then*

$$(S\otimes T,P\otimes Q^*,\sigma\otimes\tau^\circ,\phi\otimes\psi^\circ,f\otimes g^{*-1})$$

is a good graded étale covering splitting for $A\otimes B^\circ$. $\tau^\circ:B^\circ\otimes T\to\mathrm{END}_T Q^*$ *is defined by* $\tau^\circ(x)=(\tau(x))^*$. *Consequently* $\theta(\sigma\otimes\tau^\circ,A\otimes B^\circ)=\theta(\sigma,A)\otimes(\theta(\sigma,B))^{-1}$, *and* $\theta(\sigma,A)$ *is independent of the choice of* σ.

Proof. An easy computation; the last observation then follows from 2.12.

2.14. Lemma. $\theta:\mathrm{Br}^{\mathrm{g}}(R)\to H^2_{\mathrm{ggr}}(U)$ *is a well-defined monomorphism.*

Proof. That θ is a well-defined homomorphism follows from the preceding lemma. Suppose $\theta(A)=1$, and let S define a good graded étale covering splitting datum. Then there exists a graded étale covering S' of S such that the image (u',I') of (u,I) is identical to $(\Delta_2 v,(D_1,J)$. $(d_2V)^{-1})$ for $v\in U(S'^{(2)})$, $[J]\in\mathrm{gr}(S')$. Then consider the splitting $(S',P',\sigma',\gamma',f'v)$ and apply 2.10 to conclude that $[A]=1$.

2.15. Theorem. *If R is quasistrongly graded, then we have an exact sequence*

$$1\to\mathrm{Pic}^{\mathrm{g}}(R)\to H^1_{\mathrm{g}}(U)\to H^1_{\mathrm{g}}(\mathrm{gr})\xrightarrow{\beta}H^2_{\mathrm{g}}(U_0)\xrightarrow{\gamma}H^2_{\mathrm{ggr}}(U)\to 1.$$

Proof. The part $1 \to \mathrm{Pic}^g(R) \to H^1_g(U) \to H^1_g(\mathrm{gr})$ was done previously.

Definition of β. Let $[T] \in H^1(S/R, \mathrm{gr})$ represent $x \in H^1_g(\mathrm{gr})$. Up to replacing S by an étale covering, we can suppose $[T]=1$ in $\mathrm{Pic}_g(S \otimes S)$, hence $[T]=[d(u)]$ for some $u \in U(S^{(2)})$. Since $D_1(T)=1$ in $\mathrm{gr}(S^{(3)})$, $\upsilon = \Delta_1 u = u_2^{-1} u_3 u_1 \in U_0(S^{(3)})$, and is clearly a cocycle in $U_0(S^{(3)})$. Define $\beta(x)=[\upsilon] \in H^2_g(U_0)$. Note that $[\upsilon]$ is not trivial in $H^2_g(U_0)$ in general, although $\upsilon = \Delta_1 u$. Indeed, u could be non-homogeneous.

Exactness at $H^1_g(\mathrm{gr})$. It is clear that $\beta \circ \alpha = 1$. Suppose $[\upsilon]=\beta(x)=1$, then there exists a gr-étale covering S' of S such that $w' = w_2^{-1} w_3 w_1$ for some $w \in U_0(S'^{(2)})$, υ' being obtained by extension of scalars. Since $\deg w = 0$, $d(uw^{-1}) = d(u)$, and $[uw^{-1}] \in H^1(S'/R, U)$, so $x = \alpha[uw^{-1}]$.

Definition of γ. Let $x \in H^2_g(U_0)$ be represented by $u \in U_0(S^{(3)})$, and let $\gamma(x) = [(u, S^{(3)})]$.

Exactness at $H^2_g(U_0)$. If $\gamma(x)=1$, then there exists a gr-étale covering S' such that $[(u', S'^{(3)})]=1$ in $H^2_{\mathrm{gr}}(S'/R, U)$. Hence $u' = \Delta_1 \upsilon$ for some $\upsilon \in U(S^{(2)})$. Then $x = \beta(d(\upsilon))$. Again, $\gamma \circ \beta = 1$ is easy.

γ is surjective. Let $y \in H^2_{\mathrm{ggr}}(U)$ be represented by $[(u, I)]$ in $H^2_{\mathrm{ggr}}(S/R, U)$, and assume $[I]=1$ in $\mathrm{Pic}_g(S \otimes S)$ (if necessary, replace S by a suitable gr-étale covering). Then $[I] = d_1(\upsilon)$ for some $\upsilon \in U(S'^{(2)})$, by 1.4. Now

$$y = [(u, I)] = [(u, d_1 \upsilon) = [(u, d_1 \upsilon)][(\Delta_1 \upsilon, (d_1 \upsilon)^{-1}] = [(u \Delta_1 \upsilon, 1)].$$

As $(u \Delta_1 \upsilon, 1)$ is a cocycle, we have $d_2(u \Delta_1 \upsilon) = 1$ in $\mathrm{gr}(S^{(3)})$, so $u \Delta_1 \upsilon \in U_0(S^{(3)})$, and $y = \gamma(u \Delta_1 \upsilon)$.

2.16. Corollary. *If R is quasistrongly graded, then*

(i) $H^2_g(U_0) \cong H^2_{\mathrm{ggr}}(U)$.

(ii) $H^1_g(U) \to H^1_g(\mathrm{gr})$ *is surjective.*

(iii) $H^2_{\mathrm{ggr}}(U) \to H^2_g(U)$ *is injective.*

Proof. Using [4, §3.1], we have an exact sequence

$$H^1_g(U) \to H^1_g(\mathrm{gr}) \xrightarrow{\delta} H^2_{\mathrm{ggr}}(U) \to H^2_g(U) \to H^2_g(\mathrm{gr}). \qquad (*)$$

It is easily established that $\delta = \gamma \circ \beta$, hence $\delta = 1$, from 2.15. Indeed, for $T = d_1(u)$ representing $x \in H^1_g(\mathrm{gr})$, we have that

$$\gamma \circ \beta([d_1(u)]) = \gamma([\Delta_1 u]) = [(\Delta_1 u, S^{(3)})] = [(1, d_1(u))] = \delta_1(u)).$$

(i), (ii), (iii) are now easily obtained using the exact sequences (*) and 2.15.

2.17. Note. For a quasistrongly graded ring, we now have the following commutative diagram:

$$\begin{array}{ccccc}
\mathrm{Br}^g(R) & \longrightarrow & H^2_g(U_0) \cong H^2_{\mathrm{ggr}}(U) & \longrightarrow & H^2_g(U) \\
\downarrow & & & & \downarrow \\
\mathrm{Br}(R) & & \longrightarrow & & H^2(U)
\end{array}$$

The horizontal morphisms are monomorphisms. Note however that the vertical morphisms $\mathrm{Br}^g(R)\to\mathrm{Br}(R)$ and $H^2_g(U)\to H^2(U)$ are not necessarily monomorphic.

3. On the Grothendieck group of a graded ring.

The aim of this section is to deduce some information on the Grothendieck groups of the categories $\mathbf{FP}^g(R)$ and $\mathbf{P}^g(R)$. It will be used in the proof of Lemma 5.4. The notations are very similar to those in [3]. First, we search for a graded analog to Serre's theorem [3, Ch. IV, §2]. We shall not treat it in its full generality: a restriction to graded projective modules over commutative graded rings will be sufficient for our applications. We also have to make a technical assumption on the gradation of R: assume that every graded localisation at a graded prime ideal of R contains a homogeneous unit of positive degree, or equivalently, the graded residue class fields of the graded localisations are nontrivially graded.

If $Q^g_p(R)$ contains a unit of degree t, then there exists an open neighborhood D of p in $\mathrm{Spec}^g(R)$ such that every $Q^g_p(R)$ contains a unit of degree t $(q\in D)$: if $T\in U(Q^g_p(R))$, with $\deg T=t$, then T or T^{-1} lies in $\mathrm{Im}(R\to Q^g_p(R))$, e.g. T is represented by $f\in R_t$. Then put $D=D^g(f)=\{q\in\mathrm{Spec}^g(R): T\notin q\}$. Using the quasicompactness of $\mathrm{Spec}^g(R)$, we can now find $\delta>0$ such that every graded localisation of R contains a unit of degree δ. Let $\delta(R)$ be the minimal positive number having this property. If $\delta(R)$ does not exist, then we shall say $\delta(R)=+\infty$ (e.g. when $R=k[T]$).

For $d\in\mathbb{Z}^n$, let $R^n(d)$ be the free R-module with basis $\{\alpha_1,\dots,\alpha_n\}$, where $\deg\alpha_i=d_i$. For $k\in\mathbb{Z}$, we write $\mathbf{k}=(k,k,\dots,k)\in\mathbb{Z}^n$. Let $P\in\mathbf{P}^g(R)$, then we call $\alpha\in p_k$ *unimodular of degree* k if the graded ideal $\mathrm{O}_p(\alpha)=\{f(\alpha):f\in P^*\}$ is the whole of R, or, equivalently, if $R(k)\to P: x\to x\alpha$ is a split graded monomorphism. In this case, $R(k)$ embeds as a graded free direct summand in P.

Now suppose R is gr-local (and commutative). For a subset S of $P\in\mathbf{P}^g(R)$, let S be the graded submodule generated by S. For $k\in\mathbb{Z}$, we define $k\text{-rank}_R(S;P)$ as the supremum of all $r\geq 0$ such that (S) contains $R^r(\mathbf{k})$ as a graded free direct summand. if T is a unit of degree t in R, then $(k+t)\text{-rank}(S;P)=k\text{-rank}(S;P)$. If $t=\delta(R)$, then we have $\sum_{k=0}^{\delta-1} k\text{-rank}(P;P)=\mathrm{rank}_R P$ (P is graded free!). Along the lines of [3, Ch. IV, §1], we can prove

3.1. Lemma. *Let (R,m) be a gr-local commutative ring, $k\in\mathbb{Z}$, P, P', $Q\in\mathbf{P}^g(R)$. Then*

(i) $k\text{-rank}(S;P)=k\text{-rank}(S+mP;P)$.

(ii) *If α,β are unimodular of degree k in P, then there exists a graded automorphism ϕ of P leaving invariant all graded submodules containing α and β, and such that $\phi(R\alpha)=R\beta$.*

(iii) *If M is a graded submodule of P, then*

$$k\text{-rank}(R^r(\mathbf{k})\oplus M;R^r(\mathbf{k})\oplus P)=r+k\text{-rank}(M;P).$$

(iv) *If* $\alpha \in P_k$*, then* $k\text{-rank}((S,\alpha); P) \le 1 + k\text{-rank}(S; P)$.

(v) *If* $\alpha_1, \dots, \alpha_r \in P_k$*, and* $k\text{-rank}(\alpha_1, \dots, \alpha_1; P) \ge t$ *for some* $t<r$*, then there exist* $\beta_i = \alpha_i + a_i\alpha_r$ $(a_i \in R_0,\ 1 \le i \le t)$ *such that* $k\text{-rank}(\beta_1, \dots, \beta_t, \alpha_{t+1}, \dots, \alpha_{r-1}; P) \ge t$.

If R is a graded commutative ring, denote $Y = \max^{g}(R)$. For $P \in \mathbf{P}^{g}(R)$, $S \subset P$, $k \in Z$, we define $k\text{-rank}_R(S; P) = \inf_{m \in Y} k\text{-rank}_{Q^g_m(R)}(Q^g_m(S); Q^g_m(P))$, and k-$\text{rank}_R(P) = k\text{-rank}_R(P; P)$.

For each $j \ge 0$, we denote

$$F_j^k(S; P) = \{m \in Y : k\text{-rank}_{Q^g_m(R)}(S; Q^g_m(P)) < j\}.$$

As in [3], we then prove

3.2. Lemma. *With notations as above, we have*

(i) $F_j^k(S; P)$ *is a closed subset of* Y.

(ii) $\alpha_1, \dots, \alpha_j \in P_k$ *form a basis for a graded free direct summand of* P *if and only if* $F_j^k(\alpha_1, \dots, \alpha_j; P) = \emptyset$.

3.3. Theorem. *Let* R *be a graded commutative ring such that* $d = \dim \max^{g}(R) < +\infty$*. Then if* $P \in \mathbf{P}^{g}(R)$*, and* $k\text{-rank}(P) > d$ *for some* $k \in \mathbb{Z}$*, then* P *contains a unimodular element of degree* k.

We omit the (long) proof, as it is a straightforward graded version of [3, IV. 2.5]. Remark that the functions $r_k(P) : \text{Spec}^{g}(R) \to Z : p \to k\text{-rank } Q^g_p(P)$ are not necessary continuous! As an example, consider the scaled Rees ring

$$S = \left(\bigoplus_{i \in \mathbb{Z}} a^{-i} b^{-i} R X^{2i} \right) \oplus \left(\bigoplus_{i \in \mathbb{Z}} a^{-i} b^{-i} R X^{2i+1} \right),$$

where R is a Dedekind-domain containing exactly two maximal ideals aR and bR, and $\deg X = 1$. Then $\text{Spec}^{g}(S) = \{(0), (a, X), (b, X)\}$ is connected. let $P = aS \cap bSX$, which is a graded projective ideal of S. Then we get the following functions:

for k odd: $r_k(P)(0) = r_k(P)(b, X) = 1$; $r_k(P)(a, X) = 0$;

for k even: $r_k(P)(0) = r_k(P)(a, X) = 1$; $r_k(b, X) = 0$.

As a consequence, 3.3 is not sufficient for our purposes. A slight modification is necessary.

3.4. Proposition. *Let* R *be a commutative Noetherian graded ring such that every graded localisation at a graded prime contains a homogeneous unit of positive degree, and* $\delta = \delta(R)$ *defined as above. Then if* P *is a graded projective* R*-module of rank greater than* $d = \dim \max^{g}(R)$*, then* $\tilde{P} = P \oplus P(1) \oplus \cdots \oplus P(\delta - 1)$ *contains* $\tilde{R} = R \oplus R(1) \oplus \cdots \oplus R(\delta - 1)$ *as a graded submodule.*

Proof. Take $m \in \max^{g}(R)$, then $Q^g_m(P) = \bigoplus_{i=1}^{f} Q^g_m(R)(r_i)$, with $f > d$. Hence

$$Q_m^g(\tilde{P}) = \bigoplus_{i=1}^{f} \bigoplus_{k=0}^{\delta-1} Q_m^g(r_i + k) = \bigoplus_{i=1}^{f} \bigoplus_{k=0}^{\delta-1} Q_m^g(k) = \bigoplus_{k=0}^{\delta-1} Q_m^g(k)^f.$$

Hence k-rank$(\tilde{P}) > d$, for $k = 0, \ldots, \delta - 1$. An application of 3.3 then finishes the proof.

We now turn to the study of $K_0\mathbf{P}^g(R)$ and $K_0\mathbf{FP}^g(R)$. Denote $K_0^g(R) = K_0\mathbf{P}^g(R)$.

3.5. Lemma. *The forgetful functors* $U : \mathbf{P}^g(R) \to \mathbf{P}(R)$, $U : \mathbf{FP}^g(R) \to \mathbf{FP}(R)$ *induce monomorphisms*

$$U : K_0^g(R) \to K_0(R) \quad \textit{and} \quad U : K_0\mathbf{FP}^g(R) \to K_0\mathbf{FP}(R).$$

Proof. As these functors are product preserving, we get well-defined homomorphisms. Take $x \in K_0^g(R)$, and let $U(x) = 0$. Since $x = [P] - [R^n]$ for some $P \in \mathbf{F}^g(R)$, $[\underline{P}] = [\underline{R}^n]$ in $K_0(R)$; hence $\underline{P} \oplus \underline{R}^m \cong \underline{R}^n \oplus \underline{R}^m$, so $[P] = [R^n]$, establishing the first half of the lemma. The other part is similar.

Let $X^g = \mathrm{Spec}^g(R)$ be the set of graded primes on R, furnished with the Zarisky topology (thus being a subspace of $X = \mathrm{Spec}(R)$). Using the fact that every idempotent of a graded commutative ring is homogeneous of degree zero, (cf. [5]), we can show that the lattice of clopen subsets of X^g is isomorphic to the lattice of idempotents, the isomorphism being given by $e \to \mathrm{supp}(eR) \subset X^g$. It is therefore very easy to establish the following lemma:

3.6. Lemma. *The space of continuous functions from* $\mathrm{Spec}\, R$ *to* $\mathbb{Z}$, *and from* $\mathrm{Spec}^g R$ *to* $\mathbb{Z}$ *are isomorphic. They are denoted by* $H^0(R)$.

Proof. Suppose $f^g : \mathrm{Spec}^g R \to \mathbb{Z}$ continuous. Then, for $p \in \mathrm{Spec}\, R$, there exists $e \in \mathrm{Idemp}\, R$ such that $p \in \mathrm{supp}(eR)$. Observing that $eR \in \mathrm{Spec}^g R$, define $f(p) = f^g(eA)$. Then f is a well-defined continuous function extending f^g.

3.7. Lemma (concellation). *Let* $P, Q \in \mathbf{FP}^g(R)$; *if* $[P] = [Q]$ *in* $K_0\mathbf{FP}^g(R)$, *then* $P^n \cong Q^n$ *for some* $n > 0$.

Proof. We refer to [3, Ch. IX, Proposition 4.2], and use 3.1 and 3.2.

3.8. Lemma (stability). *Let R be a commutative graded ring, such that every graded localisation of R contains a homogeneous unit outside* R_0, *and* $u \in K_0^g(R)$.

(i) *If R is Noetherian, then there exists a* $\delta > 0$ *such that* $\mathrm{rk}^g\, u > d$ *implies* $u = [P]$ *for some* $P \in \mathbf{P}^g(R)$.

(ii) *If u has nonnegative rank, then* $nu = [P]$ *for some* $n > 0$, *and* $P \in \mathbf{P}^g(R)$.

Proof. Let rk^g be the restriction to $K_0^g(R)$ of the rank function $K_0(R) \to H_0(R)$.

(i) We can write $u=[Q]-[R^n]$ for some $Q\in \mathbf{P}^g(R)$. Let δ be as in 3.4. Then $\mathrm{rk}^g(u)+n>\delta+n$, so from 3.4 it follows that $Q\cong_g P\oplus R^n(d)$, for some $d\in \mathbb{Z}^n$. Hence $u=[P]$.

(ii) Restricting attention to a graded direct factor of R, we may, without loss, assume that u has everywhere positive rank. Write $u=[P]-[R^m]$. Then P is defined over a finitely generated subring $R'\subset R$ which can be chosen to be Noetherian and such that every graded localisation of R' contains a homogenous unit of degree different from zero. Moreover, u is the image of $u'=[P']-[R'^m]$, where $P'\in \mathbf{P}^g(R')$ is such that $P'\otimes_{R'}R\cong_g P$. For n large enough, $nu=[Q']$, by (i), for some Q'. Thus $nu=[Q'\otimes_{R'}R]$.

Recall from [3] that the rank function $\mathrm{rk}:K_0(R)\to H_0(R)$ can be split, yielding an exact sequence

$$0\to \mathrm{rk}_0(R)\to K_0(R)\xrightarrow{\mathrm{rk}} H_0(R)\to 0.$$

Also, $\mathrm{rk}_0(R)$ is a nilideal. Restricting rk to rk^g we still get a split exact sequence

$$0\to \mathrm{rk}_0^g(R)\to K_0^g(R)\xrightarrow{\mathrm{rk}^g} H_0(R)\to 0$$

where $\mathrm{rk}_0^g(R)=\mathrm{rk}_0(R)\cap K_0^g(R)$.

Tensoring up this sequence by $\mathbb{Q}$ still yields an exact sequence; let $U^+(\mathbb{Q}\otimes H_0(R))$ be the set of functions in $\mathbb{Q}\otimes H_0(R)$ reaching only strictly positive values, and $U^+(\mathbb{Q}\otimes K_0^g(R))=(1\otimes \mathrm{rk}^g)^{-1}(U^+(\mathbb{Q}\otimes H_0(R))$. Then $U^+(\mathbb{Q}\otimes K_0^g(R))$ is a subgroup of the groups of units of $\mathbb{Q}\otimes H_0(R)$. The nilradical of $\mathbb{Q}\otimes K_0(R)$ lying in $J(\mathbb{Q}\otimes K_0^g(R))$, an element of $\mathbb{Q}\otimes K_0^g(R)$ is invertible if and only if its rank is. So we have a split exact sequence of groups of units:

$$1\to 1+(\mathbb{Q}\otimes \mathrm{rk}_0^g(R))\to U^+(\mathbb{Q}\otimes K_0^g(R))\to U^+(\mathbb{Q}\otimes H_0^g(R))\to 0.$$

Using the inverse group isomorphisms

$$\mathbb{Q}\otimes \mathrm{rk}_0^g(R)\underset{\log}{\overset{\exp}{\rightleftarrows}} 1+(\mathbb{Q}\otimes \mathrm{rk}_0^g(R))$$

where exp and log are defined by their usual series expansions – nearly all terms being zero here – we get:

3.9. Lemma.

$$U^+(\mathbb{Q}\otimes K_0^g(R))\cong U^+(\mathbb{Q}\otimes H_0(R))\oplus(\mathbb{Q}\otimes \mathrm{rk}_0^g(R)).$$

3.10. Proposition. *The map $[P]\to 1\otimes P$ from $\mathbf{FP}^g(R)\to \mathbb{Q}\otimes K_0^g(R)$ induces a monomorphism $k:K_0\mathbf{FP}^g(R)\to U^+(\mathbb{Q}\otimes K_0^g(R))$, which is an isomorphism if all graded localisations of R contain a homogeneous unit outside R_0.*

Proof. As in [3, IX. 7.1].

3.11. Corollary. *Let R be such that every graded localisation contains a homogeneous unit outside R_0. Then*

(i) $\mathrm{rk}^g: K_0\mathbf{FP}^g(R)\to U^+(\mathbb{Q}\otimes H^0(R))$ *admits section* σ.

(ii) *Each* $x\in K_0\mathbf{FP}^g(R)$ *with* $\mathrm{rk}^g(x)=1$ *has a unique n-th root.*

Proof. (i) is obvious.

For (ii), identify $K_0\mathbf{FP}^g(R)$ with $U^+(\mathbb{Q}\otimes H_0(R))\oplus\mathbb{Q}\otimes\mathrm{rk}_0^g(R)$, using 3.9 and 3.10. An element of rank one is then of the form $(1,x)$. $(1,x/n)$ is therefore the unique n-th root.

4. A Mayer Vietoris sequence for the graded Brauer group

The result of this section is completely analogous to [12, §2]. So we omit the proofs. Consider graded ring homomorphisms

$$\begin{array}{ccc} & & R_1 \\ & & \downarrow \\ R_2 & \longrightarrow & R_3 \end{array}$$

We define a *graded Brauer group* relative to this diagram. Consider data of the form (A_1,A_2,P,Q,ξ), where $A_i\in\mathbf{Az}_g(R_i)$, $P,Q\in\mathbf{FP}_g(R_3)$ such that we have a graded isomorphism

$$\xi: A_1\otimes_{R_1}R_3\otimes_{R_3}\mathrm{END}_{R_3}P\to A_2\otimes_{R_2}R_3\otimes_{R_3}\mathrm{END}_{R_3}Q$$

(i.e. A_1 and A_2 become equivalent on R_3). Call a datum trivial if it is of the form $(\mathrm{END}_{R_1}P_1,\ \mathrm{END}_{R_2}P_2,\ N,\ M,\ \mathrm{END}\phi)$ where ϕ is a graded isomorphism $P_1\otimes_{R_1}R_3\otimes N\to P_2\otimes_{R_2}R_3\otimes M$.

Defining tensor products of data in the obvious way, two data Δ,Δ' are called equivalent if there exist trivial data ε,ε' such that $\Delta\otimes\varepsilon\cong_g\Delta'\otimes\varepsilon'$. The equivalence classes of data then form a group denoted by $\mathrm{Br}^g(R_1,R_2,R_3)$.

4.1. Theorem. *Let R be a quasistrongly graded commutative ring, $f,g\in h(R)$ such that $Rf+Rg=R$. Then there exists a monomorphism $\eta:\mathrm{Br}^g(R_f,R_g,R_{fg})\to H_g^2(R,U_0)$ and a commutative diagram of exact sequences:*

$$\begin{array}{ccccccccc}
\mathrm{Pic}_g R_f\oplus\mathrm{Pic}_g R_g & \xrightarrow{\alpha} & \mathrm{Pic}_g R_{fg} & \xrightarrow{\beta} & \mathrm{Br}^g(R_f,R_g,R_{fg}) & \xrightarrow{\gamma} & \mathrm{Br}^g R_f\oplus\mathrm{Br}^g R_g & \xrightarrow{\delta} & \mathrm{Br}^g R_{fg} \\
\| & & \| & & \downarrow\eta & & \downarrow\theta & & \downarrow\theta \\
H_g^1(R_f,U_0)\oplus H_g^1(R_g,U_0) & \xrightarrow{\alpha'} & H_g^1(R_{fg},U_0) & \xrightarrow{\beta'} & H_g^2(R,U_0) & \xrightarrow{\gamma'} & H_g^2(R_f,U_0)\oplus H_g^2(R_g,U_0) & \xrightarrow{\delta'} & H_g^2(R_{fg},U_0)
\end{array}$$

For details on the construction of the connecting maps, we refer to Knus & Ojanguren [12]. The proof is identical, noting that all units and isomorphisms involved are of degree zero. Let us only mention that (due to 2.4) it suffices to consider graded étale coverings for $R_{fg}=R_f \otimes R_g$ of the form $S_1 \otimes S_2$ in order to calculate $H^i_{\mathrm{g}}(R_{fg}, U_0)$. Also note that α, β are given as formation of quotients.

5. Gabber's theorem for quasistrongly graded rings

Here is the main theorem of this paper:

5.1. Theorem. *If R is a quasistrongly graded ring, then* $\mathrm{Br}^{\mathrm{g}}(R) \cong H^2_{\mathrm{g}}(R, U_0)_{\mathrm{tors}}$.

5.2. Note. The proof of the corresponding ungraded result can be found in Gabber [9], Hoobler [10], and Knus–Ojanguren [12]. The proof of Theorem 5.1 is inspired by the Knus–Ojanguren proof. It is given by the following lemmas. Throughout this section R is assumed to be quasistrongly graded.

5.3. Lemma. *Let $c \in H^2_{\mathrm{g}}(R, U_0)_{\mathrm{tors}}$, $f, g \in h(R)$ such that $Rf + Rg = R$. Denoting c_f, c_g for the images of c in $H^2_{\mathrm{g}}(R_f, U_0)$ and $H^2_{\mathrm{g}}(R_g, U_0)$, and supposing the existence of graded R_f- and R_g-Azumaya algebras A and B such that $\theta[A]=c_f$, $\theta[B]=c_g$, it follows that there exists a graded R-Azumaya algebra C such that $\beta[C]=c$.*

Proof. In view of the results of the two preceding sections the result becomes a mere translation of [12, 3.2].

5.4. Lemma. *Let $c \in H^2_{\mathrm{g}}(R, U_0)_{\mathrm{tors}}$. If for any $m \in \max^{\mathrm{g}} R$, there exists a graded $Q^{\mathrm{g}}_m(R)$-algebra $A(m)$ such that $\theta[A(m)] = Q^{\mathrm{g}}_m(c)$, then $c \in \mathrm{Im}(\beta)$.*

Proof. Take $m \in \max^{\mathrm{g}}(R)$. Then there exists $f \in h(R \setminus m)$, and a graded R_f-Azumaya algebra $A(f)$ such that $\theta[A_f] = c_f$. Consider $\Sigma = \{f \in h(R) : c_f \in \mathrm{Im}\,\beta\}$. It is easily seen that

(1) $a \in h(R)$, $f \in \Sigma \Rightarrow af \in \Sigma$.

(2) $f, g \in \Sigma$, $\deg f = \deg g \Rightarrow f + g \in \Sigma$ (applying 5.3) to R_{f+g}.

Now $\mathrm{Spec}^{\mathrm{g}} R$ can be covered by a set of the form

$$\{U(f(m)) : m \in \max^{\mathrm{g}}(R), f(m) \in \Sigma\}.$$

By quasicompactness, this may be reduced to a finite covering $\{U(f_i) : i = 1, \ldots, n, f_i \in \Sigma\}$. Hence there exist homogeneous $a_1, \ldots, a_n \in h(R)$ such that $\sum a_i f_i = 1$. Hence $1 \in \Sigma$, finishing the proof.

5.5. Lemma. *Theorem* 5.1 *holds for graded rings of the form* $R_0[X, X^{-1}]$ $(\deg X = 1)$.

Proof. In this situation, $\mathrm{Br}^{\mathrm{g}} R = \mathrm{Br}\, R_0$ [21, VI.3.2].

If S is a graded étale covering of R, then $S = S_0[X, X^{-1}]$, where S_0 is an étale covering of R_0. Clearly $S^{(n)} = (S_0 \otimes_{R_0} S_0 \otimes_{R_0} \cdots \otimes_{R_0} S_0)[X, X^{-1}]$, hence $U_0(S^{(n)}) = U(S_0^{(n)})$, and $H^n(S/R, U_0) = H^n(S_0/R_0, U)$, so $H^n_{\mathrm{g}}(R, U_0) = H^n(R_0, U)$.

The result then follows from the classical theorem.

5.6. Lemma. *If R is quasistrongly graded, and gr-local, then* $\mathrm{Br}^{\mathrm{g}}(R) \cong H^2_{\mathrm{g}}(R, U_0)_{\mathrm{tors}}$.

Proof. If R is quasistrongly graded and gr-local, then R has a homogeneous invertible element T, with $\deg T = d > 0$. Then consider $C = R[X]/(X^d - T)$. Then $C = C_0[X, X^{-1}]$, and C is a graded étale covering of R which is finitely generated as an R-module. Hence C is R-projective, so C is graded free as an R-module, $[C:R] = d$.

Let $c \in H^2_{\mathrm{g}}(R, U_0)_{\mathrm{tors}}$ be represented by $u \in U_0(S^{(3)})$ for some graded étale covering S of R. We can choose S such that S/C is a graded étale covering. Now, let c_C be the image of c in $H^2_{\mathrm{g}}(C, U_0)$. By the preceding lemma, $c_C = \beta[A]$ for some graded C-Azumaya algebra A. Now let T be a graded étale covering of S, C, R such that T determines a very good graded splitting

$$\alpha : A \otimes_C T \to M_n(T)(\mathbf{d})$$

where $\mathbf{d} \in \mathbb{Z}^n$ determines a gradation on $M_n(T)$ (cf. [13]).

So $(\alpha \otimes 1) \circ \tau \circ (1 \otimes \alpha)^{-1}$ is induced by a graded $T \otimes_C T$-isomorphism $f : P \to P$, where P is a graded free $T \otimes_C T$-module of rank n. Let

$$f_2^{-1} f_3 f_1 = U \in U_0(T \otimes_C T \otimes_C T),$$

then $[u] = c_C$ in $H^2_{\mathrm{g}}(C, U_0)$, hence there exists a graded étale covering W of C such that $1 \otimes u = (1 \otimes u_C)\Delta_1 w$, for some $w \in U_0(W \otimes_C W)$. f induces $f_w : P' \to P'$, where P' is a graded free $W \otimes_C W$-module. Since $W \otimes_C W = C \otimes W \otimes W$, P' is a graded free $W \otimes W$-module of rank nd, so wf_W induces a graded $W \otimes W$-isomorphism

$$\chi = \mathrm{END}(Wf_w) : M_{nd}(W \otimes W)(d') \to M_{nd}(W \otimes W)(d')$$

which is a graded descent datum (i.e. $\chi_2 = \chi_3 \chi_1$), defining an R-Azumaya algebra $B = \{x \in M_{nd}(W \otimes W) : \chi(1 \otimes x) = x \otimes 1\}$. By construction $\beta[B] = c$.

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