Journal of Pure and Applied Algebra 35 (1985) 287-295
North-Holland

A COMPUTER PROGRAM FOR THE CALCULATION OF A COVERING GROUP OF A FINITE GROUP

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Communicated by K.W. Gruenberg
Received 1 July 1983
Revised 23 January 1984

1. Introduction

In [2], the author described a method for the mechanical computation of the Schur multiplier $M(G)$ of a finite group G given as a permutation group. A disadvantage of this method is that it does not immediately provide any kind of description of the covering group(s) $\hat{G}$ of G. In this paper, we describe the theory and implementation of an algorithm for the computation of a presentation of $\hat{G}$. Of course, a presentation alone does not always provide enough information for computational purposes. With this in mind, we also describe how the elements of $\hat{G}$ can be put into a canonical form which is suitable for computation. In fact, this canonical form is an essential part of the algorithm for computing the presentation of $\hat{G}$.

The theory of the method, which we now summarize, is described in Section 2. The method of computing $M(G)$ (which was explained in full in [2]) was to compute $M(G)_p$ individually, for each prime p dividing the order of G. First we find $P \in \mathrm{Syl}_p(G)$, and then we compute $M(P)$ using the nilpotent quotient algorithm. $M(G)_p$ is then computed as a factor group $M(P)/X$ of $M(P)$. This involves finding a set of double coset representatives of P in G, which is the slowest part of the whole process. If $|G:P|$ is too large, then one or more intermediate subgroups H with $P \subset H \subset G$ may be used to break the calculation down into two or more steps.

The nilpotent quotient algorithm actually produces a presentation for one covering group $\hat{P}$ of P and, after factoring out the subgroup X of $M(P)$, we have a presentation of an extension D of M by P, where $M = M(G)_p$.

D corresponds to an element $[D] \in H^2(P, M)$, and our basic aim is to compute the image of $[D]$ under the corestriction map

$$\mathrm{Cor}_{P,G} : H^2(P,M) \to H^2(G,M).$$

This provides an extension $\hat{G}_p$ of M by G. Again, we may break the problem up into two or more stages, by using intermediate subgroups H.

In order to produce a concrete description of $\hat{G}_p$, we assume that we are given a presentation of G. Then we compute a presentation of $\hat{G}_p$ having relations of form $r=x_p$, where r is a relator of G, and $x_p \in M$. If $p_1 p_2, \ldots, p_n$ are the primes dividing $|G|$, then the relations $r=x_{p_1}x_{p_2}\cdots x_{p_n}$ provide the required presentation of $\hat{G}$. More theoretical details are given in Section 2.

$\hat{G}$ is unique if G is perfect. If not, then different covering groups can be obtained by changing relations of form $g^p=x$ to $g^p=xx_p$, where $x_p \in M(G)_p$, and the image of g in G is a generator of $G/O^p(G)$.

Most of the algorithms required were already available from the Schur multiplier program, and are described in [2]. Others are described in the literature. For example, since G is given as a permutation group, it may be necessary to compute a presentation of G. A technique for doing this is described by Leon in [3].

It was necessary to write two essentially new programs, called CORESTRICT and COVERPERMS. The first of these carries out the computation of the maps $\mathrm{Cor}_{H,G}$, and produces the required presentation of $\hat{G}_p$ as output. The other new problem, which is described in detail in Section 3, was to develop a method for the rapid computation of the elements in the covering groups $\hat{H}$ (where $P \subset H \subset G$). This is vital in the computation of $\mathrm{Cor}_{H,G}$. To this end, we introduce a canonical form for the elements of $\hat{H}$. The information needed for putting elements into canonical form can be obtained by using the techniques of the modified Todd–Coxeter Algorithm, which is described (with references) in [4]. COVERPERMS is a version of this algorithm which is adapted to our particular requirements.

In Section 4, we describe some results of these computations. We list presentations for $\hat{G}$, for the three groups $G=\mathrm{PSL}(3,4)$, M_{22} and $\mathrm{PSU}(4,3)$, which are permutation groups of degrees 21, 22 and 112 respectively.

2. The theory of the algorithm

First, we must justify the assertion that the procedure described in the Introduction for producing $\hat{G}$ really does result in a covering group for G. Unfortunately, this is not quite so clear as it may appear to be at first sight, since, owing to the non-uniqueness of $\hat{P}$, $[D]$ need not itself be stable.

Since $\hat{G}$ is certainly a central extension of $M(G)$ by G, it is enough to show that $M(G) \subseteq [\hat{G}, \hat{G}]$. But, since $M(G)$ is a direct product of its p-primary parts, it suffices to show that $M(G)_p \subseteq [\hat{G}_p, \hat{G}_p]$, for each prime p, dividing $|G|$.

We recall from [2], that D was defined to be $\hat{P}/X$, where $\hat{P}$ is a covering group of P, and $X \subseteq M(P)$. Let us repeat the details of the computation of X, as described in [2].

(i) From Section 10, Chap. XII of [1], the corestriction map $\mathrm{Cor}: H^2(P,\mathbb{C}) \to H^2(G,\mathbb{C})$ maps the stable elements of $H^2(P,\mathbb{C})$ isomorphically onto $H^2(G,\mathbb{C})_p$, where $\phi \in H^2(P, \mathbb{C})$ is called stable, if and only if

$$\mathrm{Res}_{P,Q}(\phi) = \mathrm{Con}^g(\mathrm{Res}_{P,R}(\phi)),$$

for all $g \in G$, where $Q = P \cap P^g$ and $R = P \cap P^{g^{-1}}$. It is also shown that it is sufficient to let g range over a set of double coset representatives of P in G. Furthermore, when ϕ is stable, we have $\mathrm{Res}_{G,P}\mathrm{Cor}_{P,G}(\phi) = |G:P|\,\phi$.

(ii) By the Universal Coefficient Theorem, there is a natural isomorphism $\tau : H^2(P, \mathbb{C}) \to \mathrm{Hom}(M(P), \mathbb{C})$. By dualizing, we see that ϕ is stable if and only if $X \subseteq \mathrm{Ker}(\phi)$, where X is defined as follows. For each double coset representative g of P in G, let $M(\bar{Q})$ and $M(\bar{R})$ denote the images of $M(Q)$ and $M(R)$ in $M(P)$ induced by the insertion maps $Q \to P$ and $R \to P$, where Q and R are as in (i). Then conjugation by g^{-1} induces a map $\chi : M(\bar{Q}) \to M(\bar{R})$, and X is generated by the subgroups $\langle x^{-1}\chi(x) \mid x \in M(\bar{Q})\rangle$ of $M(P)$.

To proceed further, it is necessary to describe the map τ more explicitly. We may replace $\mathbb{C}$ by any trivial G-module M, and τ will still be an epimorphism, although it will not in general be an isomorphism. (N.B. When working in extensions E of modules M by groups G, we usually use additive notation for the elements of M inside E, but multiplicative notation for general elements of E.)

Let $1 \to R \to F \to G \to 1$ be a free presentation of G, and let F_P be the complete inverse image of P in G. Then

$$M(P) = (R \cap [F_P, F_P])/[R, F_P] \quad \text{and} \quad M(G) = (R \cap [F, F])/[R, F].$$

Regard $R/[R, R]$ as a right G-module and let T be a left transversal for P in G. Then, to each $\phi \in H^2(P, M)$, there corresponds a P-homomorphism $\psi \in \mathrm{Hom}_P(R/[R, R], M)$. Since M is a trivial P-module, $\mathrm{Ker}(\psi)$ contains $[R, F_P]/[R, R]$. Then $\tau(\phi)$ is the map induced by ψ on $M(P)$. $\mathrm{Cor}_{P,G}(\phi)$ is represented by the G-homomorphism $\sum_{g \in T} \psi^g$, where $\psi^g(x) = \psi(xg)$. Now, on restricting to $M(P)$, we have

$$\mathrm{Res}_{M(P)}\left(\sum (\psi_{M(P)})^g\right) = |G:P|\,\psi_{M(P)}$$

if and only if $\psi_{M(P)}$ is a G-homomorphism, which is the case if and only if $[R, F]/[R, R] \subseteq \mathrm{Ker}(\psi)$. Since τ is an isomorphism in the case $M = \mathbb{C}$, we conclude that $[R, F]/[R, F_P] = X$.

Now we put $M = M(G)_p$ and consider the diagram

$$\begin{array}{ccc} H^2(P, M) & \xrightarrow{\mathrm{Cor}} & H^2(G, M) \\ \downarrow{\scriptstyle \tau} & & \downarrow{\scriptstyle \sigma} \\ \mathrm{Hom}(M(P), M) & \longrightarrow & \mathrm{Hom}(M(G), M) \end{array}$$

Let $[D]$ be the element of $H^2(P, M)$ corresponding to the extension $D = \hat{P}/X$ of $M = M(P)/X$ by P. Then

$$X = [R, F]/[R, F_P] \subseteq \mathrm{Ker}(\tau[D]),$$

and so

$$\overline{\text{Res}}(\overline{\text{Cor}}(\tau[D])) = |G:P|\,\tau[D],$$

where $\overline{\text{Res}}$ and $\overline{\text{Cor}}$ are the maps induced by $\text{Res}_{G,P}$ and $\text{Cor}_{P,G}$ under τ and σ. Now, since $M(P) \subseteq [\hat{P}, \hat{P}]$, we have $M \subseteq [D, D]$, which means that $\tau[D]$ is surjective. Thus, since $|G:P|$ is prime to p, $|G:P|\,\tau[D]$ is surjective, and hence $\overline{\text{Cor}}(\tau[D])$ must be surjective. We conclude that $M \subseteq [\hat{G}_p, \hat{G}_p]$ where $[\hat{G}_p] = \text{Cor}[D]$, which is what we were trying to prove. Note that, since τ is not an isomorphism in this case, $[D]$ itself need not necessarily be stable, but this is not important. Of course, $\text{Res}(\text{Cor}[D])$ is stable.

The second theoretical problem involved is the method of computing the corestriction map $\text{Cor}_{H,G}: H^2(H,M) \to H^2(G,M)$, where $H \subseteq G$ and M is a trivial G-module. let

$$\mathcal{R}(G) = \quad \cdots \longrightarrow P_2 \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\partial_0} \mathbb{Z} \longrightarrow 0,$$

$$\mathcal{R}(H) = \quad \cdots \longrightarrow Q_2 \longrightarrow Q_1 \longrightarrow Q_0 \longrightarrow \mathbb{Z} \longrightarrow 0$$

be the standard G-free and H-free resolutions of $\mathbb{Z}$, using right modules. Then $P_0 = \mathbb{Z}G$, and P_1 and P_2 have free $\mathbb{Z}G$-bases $\{[g] \mid g \in G\}$ and $\{[g_1, g_2] \mid g_1, g_2 \in G\}$ respectively, where

$$\partial_1[g] = 1 - g \quad \text{and} \quad \partial_2[g_1, g_2] = [g_2] - [g_1 g_2] + [g_1]g_2,$$

and similarly for $\mathcal{R}(H)$. Of course, $\mathcal{R}(G)$ is also an H-free resolution, and so we can define H-homomorphisms $\alpha_i: P_i \to Q_i$ to give a commutative diagram. In particular, we can define α_0, α_1 and α_2 as follows. Let T be a left transversal of H in G and, for $g \in G$, let $\bar{g}$ denote the coset representative of g in T. Then

$$\alpha_0(k) = 1, \qquad \alpha_1([g])k) = [\overline{gk}^{-1} gk], \quad \text{and}$$

$$\alpha_2([g_1, g_2]k) = [\overline{g_1 g_2 k}^{-1} g_1 \overline{g_2 k}, \overline{g_2 k}^{-1} g_2 k],$$

for all $g, g_1 g_2 \in G$ and $k \in T$.

So, if an element of $H^2(H,M)$ is represented by $\phi \in \text{Hom}_H(Q_2, M)$, then its image in $H^2(G,M)$ under the map $\text{Cor}_{H,G}$ is represented by $\psi \in \text{Hom}_G(P_2, M)$, where

$$\psi[g_1, g_2] = \sum_{k \in T} \phi[\overline{g_1 g_2 k}^{-1} g_1 \overline{g_2 k}, \overline{g_2 k}^{-1} g_2 k]. \tag{1}$$

This is the formula that we use to compute the corestriction map.

In practice, we are given relators for G of form $g_{i_n} \cdots g_{i_2} g_{i_1}$, and we wish to evaluate these relators in the extension E of M by G defined by ψ. Let D be the extension of M by H defined by ϕ. Then, according to the standard interpretation of $H^2(-, M)$, we can find transversals $T(H)$ and $T(G)$ of M in H and G respectively, with respect to which ϕ and ψ represent factor sets. We denote the elements of $T(G)$ and $T(H)$ that map onto g_i and h_i respectively, by $y(g_i)$ and $z(h_i)$. Then we have

$$y(g_i) y(g_j) = \psi[g_i, g_j] y(g_i g_j) \quad \text{and} \quad z(h_i) z(h_j) = \phi[h_i, h_j] z(h_i h_j).$$

For $k \in T$ (the left transversal of H in G), let $g_j k = l h(j,k)$, where $l = \overline{g_j k}$ and $h(j,k) \in H$. Then, from the formula (1), we get

$$\psi[g_i, g_j] = \sum_{k \in T} \phi[h(i,l), h(j,k)]. \tag{2}$$

More generally, we extend the definitions of ψ and ϕ, by putting

$$y(g_{i_n}) \cdots y(g_{i_2}) y(g_{i_1}) = \psi[g_{i_n}, \ldots, g_{i_1}] y(g_{i_n} \cdots g_{i_1}),$$

and

$$z(h_{i_n}) \cdots z(h_{i_2}) z(h_{i_1}) = \phi[h_{i_n}, \ldots, h_{i_1}] z(h_{i_n} \cdots h_{i_1}).$$

for $g_{i_j} \in G$, $h_{i_j} \in H$. For $k \in T$, define $k_1 = k$, and $k_x = \overline{g_{i_{x-1}} k_{x-1}}$, for $2 \le x \le n$. Then we have the following formula.

$$\psi[g_{i_n}, \ldots, g_{i_1}] = \sum_{k \in T} \phi[h(i_n, k_n), \ldots, h(i_1, k_1)]. \tag{3}$$

Proof. We proceed by induction on n. The smallest case $n=2$ is precisely the equation (2), so let us assume that (3) holds for some $n \ge 2$. Then

$$\begin{aligned} &\psi[g_{i_{n+1}}, \ldots, g_{i_1}] y(g_{i_{n+1}} \cdots g_{i_1}) \\ &= y(g_{i_{n+1}}) \cdots y(g_{i_1}) = y(g_{i_{n+1}}) \psi[g_{i_n}, \ldots, g_{i_1}] y(g_{i_n} \cdots g_{i_1}) \\ &= (\psi[g_{i_{n+1}}, g_{i_n} \cdots g_{i_1}] + \psi[g_{i_n}, \ldots, g_{i_1}]) y(g_{i_{n+1}} \cdots g_{i_1}) \\ &= \sum_{k \in T} (\phi[h(i_{n+1}, k_{n+1}), h(j, k_1)] + \phi[h(i_n, k_n), \ldots, h(i_1, k_1)]) y(g_{i_{n+1}} \cdots g_{i_1}), \end{aligned}$$

(where $g_j = g_{i_n} \cdots g_{i_1}$).

But, since $h(j, k_1) = h(i_n, k_n) \cdots h(i_1, k_1)$, we have

$$\begin{aligned} &\phi[h(i_{n+1}, k_{n+1}), \ldots, h(i_1, k_1)] z(h(i_{n+1} k_{n+1}) \cdots h(i_1, k_1)) \\ &\quad = z(h(i_{n+1}, k_{n+1})) \cdots z(h(i_1, k_1)) \\ &\quad = z(h(i_{n+1}, k_{n+1})) \phi[h(i_n, k_n), \ldots, h(i_1, k_1)] z(h(i_n, k_n) \cdots h(i_1, k_1)) \\ &\quad = (\phi[h(i_{n+1}, k_{n+1}), h(j, k_1)] + \phi[h(i_n, k_n), \ldots, h(i_1, k_1)] z(h(i_{n+1}, k_{n+1}) \cdots h(i_1, k_1)) \end{aligned}$$

Putting these two equations together yields the formula (3) for $n+1$, which completes the induction.

We use (3), in particular, when $g_{i_n} \cdots g_{i_1}$ is a relator of G, to obtain relations in E of the form $y(g_{i_n}) \cdots y(g_{i_1}) = m$, where $m \in M$. To compute the right hand side of (3), we need to be able to compute the functions $h(i,k)$ and ϕ. The latter problem is discussed in Section 3. The $h(i,k)$ can be computed and stored at the same time as T, as described in Section 4 of [2]. We need only do this for those g_i which lie in the relators of the given presentation of G. If the method of Leon and Sims is used to compute this presentation, then these g_i form a strong generating set of G with respect to the given permutation representation. One complication is that $y(g_i^{-1})$ is not in general the same as $y(g_i)^{-1}$. We can get round this by computing

$y(g_i)y(g_i^{-1})$ for each i, which enables us to replace $y(g_i^{-1})$ by $y(g_i)^{-1}$ in the other relations of E. This is clearly more convenient.

3. The calculation of ϕ

Since T may be fairly large, and the elements $h(i,k)$ may be quite long as words in the generators of H, it is vital to have a method of computing ϕ rapidly. Indeed, a high percentage of the total computing time for the whole algorithm is taken up with the computation of values of ϕ.

Let us recapitulate the details of the problem at this stage. H is defined as a permutation group on Ω with respect to a given base $\alpha_1,\alpha_2,\ldots,\alpha_r$ and strong generating set $S=\{h_i\}$. (We assume that the reader is fully conversant with these concepts.) So, we have $\langle H^s \cap S\rangle = H^s$ $(=H_{\alpha_1\alpha_2\cdots\alpha_{s-1}})$ for $1\le s\le r$, and $H^{s+1}=1$. Using the algorithm of Leon and Sims, we can find a strong presentation $\langle S\,|\,R\rangle$ of H. This means that, for each $1\le s\le r$, $\langle H^s\cap S\,|\,R^s\rangle$ is a presentation of G^s, where R^s consists of those relators in R that lie entirely in G^s. In this presentation we may replace S by S_1, which is chosen to be a minimal subset of S satisfying $S=S_1\cup S_1^{-1}$. If $H=P$, then we have a power-commutator presentation of the extension D of M by P, and so we can compute the relators in R as elements of M. On the other hand, if $P\subset H\subset G$, then the relations in D have been computed already, in a previous application of the formula (3). In any case, we may assume that we have a presentation of D of the following form

$$D=\langle z(S_1)\cup X\,|\,\hat{R}\cup Y\rangle,$$

where $z(S_1)=\{z(h_i)\,|\,h_i\in S_1\}$, X is the set of generators of M, Y consists of relations of M together with relations which make $M\subseteq Z(D)$, and $\hat{R}$ consists of relations of the form $\bar{z}(r)=m$, where $m\in M$, and $\bar{z}(r)$ is the word in the $z(h_i)$ corresponding to r (with h_i^{-1} replaced by $z(h_i)^{-1}$, *for* $h_i\in S_1$).

The problem is to find a suitable canonical form for the elements of D and, in particular, a rapid method of computing $\bar{z}(w)\in M$, where w is a word in S equal to the identity in H. We have adopted the following approach to this problem. By using the base and strong generating set in the usual way, the elements of H can be put into canonical form

$$h_{i_r,r}\cdots h_{i_2,2}h_{i_1,1},$$

where each $h_{i_s,s}$ is an element of a right transversal of H^{s+1} in H^s, for $1\le s\le r$. In practice, the $h_{i_s,s}$ are words in S. Let D^s be the complete inverse image of H^s in D, for $1\le s\le r$, and $D^{r+1}=M$. Then each element of D has a canonical expression as

$$m\bar{z}(h_{i_r,r})\cdots\bar{z}(h_{i_1,1}).$$

We now carry out modified Todd–Coxeter coset enumeration algorithms on the presentations $\langle X\cup(D^s\cap z(S_1))\,|\,\hat{R}^s\cup Y\rangle$, using the subgroup D^{s+1}, for $s=r,r-1,\ldots,1$, successively. This algorithm, together with various implementations, is discussed in Section 4 of [4]. The usual aim is to compute a presentation of the given subgroup, which is not necessary in our case, because we already have one.

However, in order to achieve this aim, it is necessary to compute the coefficients $u(i,j,s) \in D^{s+1}$ in the equations

$$\bar{z}(h_{i_s,s})z(h_j)^{\pm 1} = u(i,j,s)\bar{z}(h_{k_s,s}),$$

for all i,j and s with $h_j \in H^s \cap S_1$. In most versions of the algorithm one avoids computing and storing these $u(i,j,s)$ explicitly as far as possible, because there are many of them, and they can grow very long as words in the generators.

However, it is essential for our purposes that we store each $u(i,j,s)$. For these reasons, we had to write a new programme for these calculations which we called COVERPERMS. This programme has the additional facility of checking the consistency of the given strong presentation of D, if required.

Once we have computed the $u(i,j,s)$, we can put an element $d \in D$ into canonical form, as follows. Let

$$d = mz(h_{i_1})^{\pm 1} \cdots z(h_{i_n})^{\pm 1}.$$

Then we perform the following operation, for $s = 1, 2, \ldots, r$ in turn. At the end of the $(s-1)$th step, d has the form

$$m'z(h_{j_1})^{\pm 1} \cdots z(h_{j_{n'}})^{\pm 1}\bar{z}(h_{i_{s-1},s-1}) \cdots \bar{z}(h_{i_1,1})$$

where $m' \in M$ and each $h_{j_k} \in H^s \cap S_1$. For the sth step, we compute the image of 1 (the coset representative of the identity coset of D^{s+1} in D^s) under $z(h_{j_1})^{\pm 1} \cdots z(h_{j_{n'}})^{\pm 1}$, introducing coefficients $u(i,j,s+1)$ as we go along. This brings d into canonical form after the rth step.

In the programme COVERPERMS, we prevent the words $u(i,j,s)$ from growing too long by putting them into their canonical form, using the values of $u(i,j,s)$ for higher values of s, which have already been calculated. This helps to prevent the storage problem for the $u(i,j,s)$ from becoming too acute. We also save space by packing the words $u(i,j,s)$ as tightly as possible into machine registers. In the examples run so far, H has never had more than 16 generators (including inverses), and it has always been possible to fit each $u(i,j,s)$ into two 48-bit machine words.

4. Some examples

With the moderate computing power available from the Burroughs B 6700 machine at Warwick, it is feasible to compute the corestriction map for indices $|G:H|$ up to a few hundred. Beyond that, the process time and storage space required start to grow very rapidly.

We list the results of three strong presentations of covering groups that were computed using the procedures described above.

Example 1. $G = \mathrm{PSL}(3,4)$; $|M(G)| = 48$; $|\Omega| = 21$; $|G| = 21 \cdot 20 \cdot 16 \cdot 3$.

For $p = 2$, we used the sequence of subgroups $P_2 \subset H \subset G$, with $P_2 \in \mathrm{Syl}_2(G)$, and

H is the stabilizer of a point. Then $|H:P_2|=15$ and $|G:H|=21$. For $p=3$, we used $P_3 \subset N \subset G$, with $N=N(P_3)$. $|N:P_3|=8$ and $|G:N|=280$.

$$\hat{G}=\langle a,b,c,d,e,x,y,t\rangle$$

with $t^3=x^4=y^4=1$ and $t,x,y \in Z(\hat{G})$. The other relations are:

$$a^3=b^3=1, \quad c^2=x^2, \quad (ab^{-1})^2=y^2,$$

$$(acb^{-1})^2=(ab^{-1}c)^2=(a^{-1}c)^3=1, \quad (cd)^2=x^2y^2,$$

$$a^{-1}ca^{-1}dad^{-1}=x^2, \quad acb^{-1}dbda^{-1}cb^{-1}d^{-1}=x^3y,$$

$$ad^2ca^{-1}cd^{-1}=x^2y, \quad abadbcdb^{-1}d^{-1}=x^3y^2, \quad e^2=(ce)^2=1,$$

$$beae=t, \quad abadbacedeabd^{-1}e=xyt.$$

Example 2. $G=M_{22}$; $|M(G)|=12$; $|\Omega|=22$; $|G|=22\cdot 21\cdot 20\cdot 16\cdot 3$.

For $p=2$, we used $P_2 \subset H \subset G$, where H is the stabilizer of a block in the Steiner System $S(3,6,22)$, and, for $p=3$, we used $P_3 \subset N \subset H \subset G$, with $N=N(P_3)$. Then $|H:P_2|=45$, $|G:H|=77$, $|N:P_3|=8$ and $|H:N|=80$.

$$\hat{G}=\langle a,b,c,d,e,f,x,t\rangle$$

with $t^3=x^4=1$ and $t,x \in (\hat{G})$. The other relations are:

$$a^3=b^3=1, \quad c^2=x^2,$$

$$(ab^{-1})^2=(acb^{-1})^2=(ab^{-1}c)^2=(a^{-1}c)^3=1,$$

$$(cd)^2=a^{-1}ca^{-1}dad^{-1}=x^2, \quad acb^{-1}dbda^{-1}cb^{-1}d^{-1}=x,$$

$$ad^2ca^{-1}cd^{-1}=1, \quad abadbcdb^{-1}d^{-1}=x^3, \quad e^2=(ce)^2=1,$$

$$beae=t, \quad abadbacedeabd^{-1}e=x^3t,$$

$$cf^2=b^{-1}a^{-1}fbf^{-1}=ca^{-1}dcafdf^{-1}=b^{-1}ca^{-1}efedfa^{-1}cd^{-1}e^{-1}f^{-1}=1.$$

Example 3. $G=\mathrm{PSU}(4,3)$; $|M(G)_3|=9$; $|\Omega|=112$; $|G|=112\cdot 81\cdot 30\cdot 4\cdot 3$.

In this case, we only computed the presentation for $\hat{G}_3$, using $P_3 \subset H \subset G$, with H the point stabilizer, $|H:P_3|=40$ and $|G:H|=112$.

$$\hat{G}=\langle a,b,c,d,e,f,x,y\rangle$$

with $x^3=y^3=1$ and $x,y \in Z(\hat{G}_3)$. The other relations are:

$$a^3=b^2=(ba)^3=c^2=d^2=(bac)^2=(bad)^2=aba^{-1}caca^{-1}c$$

$$=aba^{-1}dada^{-1}d=(cd)^3=(abcad)^2=e^2=babeceba^{-1}be=1,$$

$$cacbebcbaece=y, \quad ba^{-1}cdabedabade=y,$$

$$baea^{-1}bea^{-1}cba^{-1}e=1, \quad abdacbeabeda^{-1}bea^{-1}e=x,$$

$$f^3=1, \quad acacbdedfba^{-1}bf^{-1}=y^2,$$

$$ba^{-1}cabdedafcf^{-1} = dadbfea^{-1}baef^{-1} = cacbfedef^{-1} = 1,$$

$$cacdbefa^{-1}fda^{-1}ef^{-1} = xy^2, \qquad a^{-1}ba^{-1}cdbeaedbfdfa^{-1}ea^{-1}f^{-1} = x^2.$$

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