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# A constructive proof of the Stone–Weierstrass theorem

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**Abstract**

A constructive version of the Stone–Weierstrass theorem is proved, allowing a globalisation of the Gelfand duality theorem to any Grothendieck topos to be established elsewhere. © 1997 Elsevier Science B.V.

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## 0. Introduction

In this paper, we give a constructive proof of the Stone–Weierstrass theorem, which provides one of the critical steps in establishing an extension of Gelfand duality to the category of commutative C*-algebras in any Grothendieck topos [3, 4]. Indeed the existence of this constructive form of the Stone–Weierstrass theorem also provides one of the stages towards obtaining a completely constructive proof of the Gelfand duality theorem, a task to which we hope to return in a later paper.

For the moment, we note also only that the proof of the Stone–Weierstrass theorem that we obtain carries forward the ideas developed in [12] that the core of Gelfand–Naimark theory is algebraic in nature, with analytic arguments appearing only at certain critical points to provide the context in which the algebraic manipulations can take place. In the present case, this analytic contribution is contained in one simple observation, namely that the Gaussian rationals are dense in the complex numbers. That this provided the content of the Stone–Weierstrass theorem was conjectured in [9], although the context in which this could be established and the details of the proof awaited the present paper.

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## 1. Preliminaries

Classically, the Stone-Weierstrass theorem states that a closed involutive subalgebra of the commutative C*-algebra $\mathbb{C}(X)$ of continuous complex functions on a compact, Hausdorff space $X$ that separates the points of $X$ is necessarily equal to the algebra $\mathbb{C}(X)$. As is well known, within the constructive context, the concept of a compact, Hausdorff topological space needs to be replaced by that of a compact, completely regular locale [1, 2, 8]. Indeed, the Gelfand duality within the constructive context of a Grothendieck topos established by the present authors in [4] takes the form of an adjoint dual equivalence between the category of commutative C*-algebras and the category of compact, completely regular locales.

Remarking that the concept of a compact, completely regular locale is equivalent in the presence of the Axiom of Choice to that of a compact, Hausdorff space, we recall the definitions that are needed in order to provide the setting for a constructive version of the Stone–Weierstrass theorem:

**Definition.** By a *locale* $L$ is meant a lattice having finite meets $\wedge$ and arbitrary joins $\bigvee$ satisfying the condition that

$$u \wedge \bigvee_i v_i = \bigvee_i u \wedge v_i$$

for all $u$, $v_i \in L$.

Of course, the canonical example of a locale is that obtained by taking the lattice $\mathrm{Open}(X)$ of open subsets of a topological space $X$. Indeed, the axiomatisation of a locale in terms of finite meets and arbitrary joins, satisfying the distributivity condition always satisfied within the lattice of subsets of any set, is intended to provide a generalisation of the concept of topological space to situations in which the points of the space no longer play a primary role in its description and properties. This is particularly the case when the absence of the Axiom of Choice within a constructive context means that it may not be possible to demonstrate the existence of the points of a classically considered topological space. That this is likely to be the case in constructing the maximal spectrum of a commutative C*-algebra leads to the approach to the Gelfand–Mazur theorem and to Gelfand duality taken in [3, 4].

**Definition.** A locale $L$ is said to be *compact* provided that for any family $(u_i)_{i \in I}$ of elements of $L$ for which

$$\bigvee_i u_i = 1_L$$

there exist finitely many $i_1, i_2, \ldots, i_n \in I$ such that

$$u_{i_1} \vee u_{i_2} \vee \cdots \vee u_{i_n} = 1_L.$$

Of course, the locale of open subsets of a compact topological space is compact in this sense, the definition evidently being intended to abstract this case. However, it may be remarked that the spectrum Max $A$ of any commutative C*-algebra $A$ [3] is also necessarily compact, even though, constructively, it is not necessarily the locale of open subsets of a compact topological space. The spectrum Max $A$ of a commutative C*-algebra $A$ is also completely regular in the following sense, remarking that of course classically any compact, Hausdorff topological space is necessarily completely regular, hence determines a compact, completely regular locale of open subsets.

**Definition.** A locale $L$ is said to be *regular* provided that each element $u \in L$ may be expressed as the join

$$\bigvee_{v \lhd u} v$$

of those elements $v \in L$ that are *rather below* the element $u \in L$, in the sense that there exists an element $w \in L$ for which

$$v \wedge w = 0_L \quad \text{and} \quad u \vee w = 1_L.$$

The locale $L$ is said to be *completely regular* provided that each element $u \in L$ may be expressed as the join

$$\bigvee_{v \lhd\lhd u} v$$

of those elements $v \in L$ that are *completely below* the element $u \in L$, in the sense that there exists a family of elements $v_q \in L$ indexed by the rationals $0 \leq q \leq 1$ for which

(i) $v = v_0$ and $v_1 = u$; and

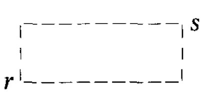
(ii) $v_p \lhd v_q$ whenever $p < q$.

Evidently, the existence of this interpolation between an element $v \in L$ which is completely below an element $u \in L$ is exactly the information that is required in order to specify a continuous real function which separates $v \in L$ and $u \in L$ in the sense of Urysohn's lemma. Of course, complete regularity implies regularity, but even in the presence of compactness of the locale $L$ the converse is not necessarily constructively the case, in the absence of an axiom of countable dependent choice. The Stone–Weierstrass theorem will therefore be established in the context of the commutative C*-algebra $\mathbb{C}(M)$ of continuous complex functions on a compact, completely regular locale $M$, to which we now turn.

We recall firstly that the locale $\mathbb{C}$ of complex numbers may be defined to be that generated by open subsets of the form

$$(r, s)$$

in which $r, s$ denote complex rationals in the ordinary sense, subject to relations which assert that this open subset represents the rational open rectangle



in the complex plane. The details of this description of the locale $\mathbb{C}$ will be omitted, since they may be found elsewhere [3], albeit in a slightly different context. Suffice it to say that the locale $\mathbb{C}$ is again a completely regular locale, of which the sublocale given by taking the closed unit square is necessarily compact.

To describe the commutative C*-algebra $\mathbb{C}(M)$ of continuous complex functions on a compact, completely regular locale $M$, we recall the following:

**Definition.** By a *map of locales*

$$\varphi : L \to M$$

from a locale $L$ to a locale $M$ is meant a mapping

$$\varphi^* : M \to L$$

from the locale $M$ to the locale $L$ which preserves finite meets and arbitrary joins. The mapping will be referred to as the *inverse image mapping* of the map of locales.

By the algebra $\mathbb{C}(M)$ of continuous complex functions on the compact, completely regular locale $M$ is then meant the set of maps of locales

$$a : M \to \mathbb{C}$$

from the locale $M$ into the locale $\mathbb{C}$ of complex numbers, together with the structure of an involutive algebra over the complex rationals inherited from the algebraic structure of the locale $\mathbb{C}$. The details of this construction need not concern us for the present, beyond noting that the presence of a map

$$|\ | : \mathbb{C} \to \mathbb{C}$$

of locales corresponding to the taking of absolute value yields that the algebra $\mathbb{C}(M)$ admits an absolute value induced by composition with this map of locales. In turn, this allows a description of a norm on the algebra $\mathbb{C}(M)$, in the sense required within the constructive context [5, 11], by assigning to each positive rational $q$ the open ball

$$N(q) = \{a \in \mathbb{C}(M) \mid |a| < q\}$$

of continuous complex functions on $M$ of absolute value less than $q$.

That the normed algebra $\mathbb{C}(M)$ is indeed a commutative C*-algebra may be seen by noting firstly that the presence of a map

$$\bar{\ } : \mathbb{C} \to \mathbb{C}$$

of locales corresponding to complex conjugation yields that the algebra $\mathbb{C}(M)$ admits an involution induced by composition with this map of locales, necessarily satisfying the condition, expressed in the constructive context by requiring that

$$a \in N(q) \quad \text{if and only if} \quad aa^* \in N(q^2)$$

for each $a \in \mathbb{C}(M)$ and for each positive rational $q$, which characterises the norm of a C*-algebra with respect to its involution.

Finally, the involutive normed algebra $\mathbb{C}(M)$ is indeed complete, in the sense determined by the following:

**Definition.** By a *Cauchy approximation* $C$ on the algebra $\mathbb{C}(M)$ is meant a mapping that assigns to each natural number $n \in \mathbb{N}$ a subset

$$C_n$$

of $\mathbb{C}(M)$ satisfying the conditions that

(i) for each $n \in \mathbb{N}$ there exists $a \in C_n$; and

(ii) for any $k \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that

$$a - a' \in N(1/k)$$

whenever $a \in C_n$ and $a' \in C_{n'}$ for any $n, n' \geq m$.

The Cauchy approximation $C$ on the algebra $\mathbb{C}(M)$ is said to be *convergent* provided that there exists an element $b \in \mathbb{C}(M)$ for which for any $k \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that

$$a - b \in N(1/k)$$

whenever $a \in C_n$ for any $n \geq m$.

The algebra $\mathbb{C}(M)$ is said to be *complete* provided that any Cauchy approximation on the algebra $\mathbb{C}(M)$ is convergent.

These conditions express constructively that the Cauchy approximation provides the information needed to construct a Cauchy sequence, even though the absence within the constructive context of any Axiom of Countable Choice denies the possibility of such a Cauchy sequence being chosen, and that any choice of Cauchy sequence that might have been made would have converged to the element concerned of the algebra $\mathbb{C}(M)$, necessarily uniquely by the properties of a norm.

Observing that it may indeed be shown [3, 4] that the normed involutive algebra $\mathbb{C}(M)$ is complete, we note the following:

**Theorem 1.** *For any compact, completely regular locale $M$, the normed involutive algebra $\mathbb{C}(M)$ is a commutative $C^*$-algebra.*

Of course, it is the purpose of the Gelfand isomorphism theorem to establish that any commutative C*-algebra is of this form, towards which the Stone–Weierstrass theorem, towards which we now turn, is an important step along the way.

## 2. Partitions of unity

The Stone–Weierstrass theorem is concerned with a closed involutive subalgebra $A$ of the commutative C*-algebra $\mathbb{C}(M)$ of continuous complex functions on a compact, completely regular locale $M$, in the sense of the following:

**Definition.** A subalgebra $A$ of the commutative C*-algebra $\mathbb{C}(M)$ of continuous complex functions on a compact, completely regular locale $M$ is said to be *closed* provided that any element $a \in \mathbb{C}(M)$ such that for each $n \in \mathbb{N}$ there exists an element $b \in A$ for which

$$a - b \in N(1/n)$$

necessarily lies in the subalgebra $A$.

Of course, it may be remarked, on the one hand, that this condition on the element $a \in \mathbb{C}(M)$ exactly describes the requirement that there is a Cauchy approximation on the subalgebra $A$, in the evident sense, which converges to the element $a \in \mathbb{C}(M)$; while, on the other hand, it is also equivalent to the requirement that the closed subalgebra $A$ is again complete, in the evident sense.

An immediate consequence of considering a closed involutive subalgebra $A$ of $\mathbb{C}(M)$ is that it is also closed under the operation of absolute value defined on $\mathbb{C}(M)$, and hence under the lattice operations defined in terms of absolute value on the self-adjoint elements of $\mathbb{C}(M)$. That this is the case is obtained by remarking firstly that the absolute value on $\mathbb{C}(M)$ is characterised by the condition that

$$|a| = (aa^*)^{1/2}$$

for each $a \in \mathbb{C}(M)$, in which the square root is defined to be the unique non-negative self-adjoint element of $\mathbb{C}(M)$ of which the square is the non-negative self-adjoint element $aa^* \in \mathbb{C}(M)$. That this is the case in the commutative C*-algebra $\mathbb{C}(M)$, and that this may be relativised to the closed involutive subalgebra $A$, is a consequence of the following constructivisation of the square root lemma for Banach algebras [13] to this particular context.

It may be remarked firstly that the fact that the absolute value is given by taking the square root of $aa^* \in A$ in the commutative C*-algebra $\mathbb{C}(M)$ is already evident. The question is simply whether this square root, and hence the absolute value of $a \in A$ which it determines, lies in the closed involutive subalgebra $A$. By multiplying by an appropriate positive rational, it is evidently enough to establish this for any $a \in A$ of norm less than 1. Observing that for such an element one also has that $aa^* \in N(1)$, we note that it is then possible to find a natural number $m \in \mathbb{N}$ for which $aa^* + 1/n \in N(1)$ for all $n \geq m$. Then it follows, by the properties of continuous real functions on the locale $M$, that the element $1 - (aa^* + 1/n) \in A$ also has a norm less than 1. Consider now the binomial expansion of the expression

$$(1 - (1 - (aa^* + 1/n)))^{1/2},$$

which yields a series of which the partial sums are polynomials in $aa^* \in A$ over the rationals, convergent under the assumption that $1 - (aa^* + 1/n) \in N(1)$ to the square root of $aa^* + 1/n \in A$. By the closedness of the subalgebra $A$ in the commutative C*-algebra $\mathbb{C}(M)$, it follows that $(aa^* + 1/n)^{1/2} \in A$ for all $n \geq m$.

That this sequence, hence the Cauchy approximation that it determines, in the subalgebra $A$ converges in the commutative C*-algebra $\mathbb{C}(M)$ to the square root of $aa^* \in A$ is an immediate consequence of the fact that

$$b^2 + 1/n^2 \leq (b + 1/n)^2$$

for any self-adjoint element $b \in \mathbb{C}(M)$, considered as a continuous real function on the locale $M$. Hence, again by the closedness of the subalgebra $A$, one has that for any element $a \in A$ of norm less than 1, and hence, by the remarks made above, for *any* element $a \in A$, the square root $(aa^*)^{1/2} \in \mathbb{C}(M)$ also lies in the closed involutive subalgebra $A$. In particular, the self-adjoint elements of $A$ are closed under the lattice operations defined on the self-adjoint elements of the commutative C*-algebra $A$, a fact of which we shall shortly need to take advantage.

Consider now the open subset $P$ of the complex plane obtained by removing the origin: more explicitly, the sublocale $P$ of the locale $\mathbb{C}$ of complex numbers is obtained by taking the join of all those rational open rectangles $(r, s)$ for which $0 \notin (r, s)$. For any element $a \in A$ of the closed involutive subalgebra $A$, denote by

$$D(a)$$

the inverse image of the open subset $P$ of the complex plane along the map of locales

$$a : M \to \mathbb{C}$$

from $M$ to the locale of complex numbers determined by the element $a \in A$.

In the classical context of considering a continuous complex function on a compact Hausdorff space, this subset is exactly that on which the function is non-zero. One way of expressing that the subalgebra $A$ separates the topological space is then to ask that each of its open subsets is a union of these cozero sets. In the present situation, one is therefore led to the following definition:

**Definition.** The subalgebra $A$ of the commutative C*-algebra $\mathbb{C}(M)$ will be said to *separate* the compact, completely regular locale $M$ provided that each open set $U$ of the locale may be expressed in the form:

$$U = \bigvee D(a)$$

taken over those elements $a \in A$ for which $D(a)$ is contained in $U$.

With this definition, we may prove the following result, asserting the existence within any closed involutive subalgebra $A$ that separates a compact, completely regular locale $M$ of a *finite partition of unity* $(p_j)_{j=1,\dots,n} \in A$ subordinate to any open covering $(U_i)_{i \in I}$ of the compact, completely regular locale $M$ and consisting of elements of the closed involutive subalgebra $A$:

**Theorem 2.** *For any closed involutive subalgebra $A$ of the commutative* C*-*algebra* $\mathbb{C}(M)$ *that separates the compact, completely regular locale $M$, and for any open* *covering $(U_i)_{i \in I}$ of the locale $M$, there exist finitely many elements $p_1, \ldots, p_n \in A$, non-negative and self-adjoint, such that each $p_j$ has support within some $U_{i_j}$, in the sense that*

$$D(p_j) \lhd U_{i_j},$$

*and which have the property that*

$$\sum_{j=1}^{n} p_j = 1,$$

*the identity element of $A$.*

**Proof.** The existence of the required partition of unity will be shown in a number of steps, of which the first is to prove that any open subset $U$ of the compact, completely regular locale $M$ may be expressed as a join

$$\bigvee_i [\![ a_i = 1 ]\!]$$

of open subsets determined by elements $a_i \in A$ which have support contained in the open subset $U$, in which for any element $a \in A$ we write

$$[\![ a = 1 ]\!]$$

to denote the open subset which is the pseudocomplement of the element $D(1 - a)$ in the locale $M$.

It may be remarked that this open subset of the locale $M$ is exactly the extent to which the continuous complex function $a \in \mathbb{C}(M)$ is equal to the identity element of the sheaf $\mathbb{C}_M$ of continuous complex functions on the locale $M$, from which this notation is derived. Of course, similarly, we write

$$[\![ a = 0 ]\!]$$

to denote the extent to which $a \in \mathbb{C}(M)$ is equal to the zero element of the sheaf $\mathbb{C}_M$, which is equally the pseudocomplement of the open subset $D(a)$ of the locale $M$.

That any open subset $U$ of the locale $M$ may be expressed in the form

$$U = \bigvee_i [\![ a_i = 1 ]\!]$$

for elements $a_i \in A$ with support in the open subset $U$, moreover that the elements may be chosen to be self-adjoint and satisfying the condition that $0 \le a_i \le 1$ for each $i$, may now be deduced from the condition that the closed involutive subalgebra $A$ separates the compact, completely regular locale $M$ in the following way. For, by the regularity of $M$, this element may be expressed as a join of elements which are rather below it. Because the subalgebra $A$ separates the locale $M$, each of these may in turn be expressed as a join of elements of the form $D(a)$ for $a \in A$. Hence, one may write:

$$U = \bigvee D(a),$$

taken over elements $a \in A$ for which

$$D(a) \lhd U.$$

In particular, this means that each of the elements of $A$ chosen has support contained in $U$. Further, it may be assumed that each of these elements $a \in A$ is chosen to be self-adjoint and non-negative, by the observation that the closed involutive subalgebra $A$ is closed with respect to absolute value, and that $D(a)$ is identically $D(|a|)$.

Now, it is enough to prove that each of these elements

$$D(a)$$

of the locale $M$, in which $a \in A$ is taken to be self-adjoint and non-negative, is indeed a join of elements of the required form. For any rational $r$, we shall denote by

$$[\![r < a]\!]$$

the inverse image along the continuous complex function $a \in A$ of the open right-hand half-plane determined by $r$. Similarly, one defines

$$[\![a < s]\!]$$

for any rational $s$. One sees immediately that, for a non-negative, self-adjoint element $a \in A$, one has

$$D(a) = \bigvee_{q>0} [\![q < a]\!].$$

The idea, then, is to define an element $a_q \in A$, for each positive rational $q$, of which the support is still contained in the element $U$ of the locale, yet for which

$$[\![q < a]\!] \le [\![a_q = 1]\!] \le D(a).$$

To obtain such an element $a_q \in A$ for each positive rational $q$, having the property, intuitively, that it equals 1 whenever $q < a$, one applies the sublattice properties of the self-adjoint elements of the closed involutive subalgebra $A$ of $\mathbb{C}(M)$. For each $q > 0$, define

$$a_q = (a/q) \wedge 1$$

in the closed involutive subalgebra $A$, noting that

$$[\![a_q < s]\!] = \begin{cases} [\![a < qs]\!] & \text{whenever } s \le 1, \\ 1 & \text{otherwise,} \end{cases}$$

and that

$$[\![r < a_q]\!] = \begin{cases} 0 & \text{whenever } 1 \le r, \\ [\![qr < a]\!] & \text{otherwise.} \end{cases}$$

Then, it is asserted that the required conditions hold. Further, we note that $a_q \in A$ also has support contained in $U$: for $a_q \in A$ is also self-adjoint and non-negative, so that

$D(a_q) = [\![0 < a_q]\!]$. But, by the above remarks, this exactly equals $[\![0 < a]\!] = D(a)$, whence the required result. Secondly, $[\![q < a]\!] \le [\![a_q = 1]\!]$: for $[\![a_q = 1]\!]$ is the pseudocomplement in $M$ of $[\![a_q < 1]\!]$, since $a_q \le 1$. But $[\![a_q < 1]\!]$, by the above remarks, is equal to $[\![a < q]\!]$. And, evidently, $[\![q < a]\!]$ is contained in the pseudocomplement, since it is disjoint from $[\![a < q]\!]$. Finally, $[\![a_q = 1]\!] \le D(a)$: for $D(a) = [\![0 < a]\!]$ and $q > 0$ implies that $[\![0 < a]\!]$, $[\![a < q]\!]$ together cover the locale $M$. The pseudocomplement $[\![a_q = 1]\!]$ of $[\![a < q]\!]$ is therefore contained in $[\![0 < a]\!]$. So, it follows that

$$D(a) = \bigvee_{q > 0} [\![a_q = 1]\!],$$

in which we note that each element $a_q \in A$ also satisfies $0 \le a_q \le 1$ and has support contained in the element $U$ of the locale $M$.

Joining these observations together, it has indeed been shown that any element $U$ of the compact, completely regular locale $M$ is the join of elements

$$[\![a = 1]\!]$$

for which $a \in A$ is a self-adjoint element satisfying $0 \le a \le 1$ and having support within $U$.

The existence of a finite partition of unity, consisting of non-negative, self-adjoint elements of the closed involutive subalgebra of $A$, subordinate to any open covering of the locale $M$, may now be shown. Suppose that $M = \bigvee_i U_i$. For each $i$, choose elements $a_{i_j} \in A$, each self-adjoint, satisfying $0 \le a_{i_j} \le 1$, and having support within $U_i$, for which

$$U_i = \bigvee_j [\![a_{i_j} = 1]\!].$$

Now choose, by the compactness of the locale $M$, finitely many $a_1, \ldots, a_n \in A$ say, amongst these $a_{i_j} \in A$, for which

$$M = [\![a_1 = 1]\!] \vee \cdots \vee [\![a_n = 1]\!].$$

Observe that each $a_j \in A$ has support within some $U_{i_j}$ of the covering of $M$. Now, define [6]:

$$p_1 = a_1, \quad p_2 = (1 - a_1)a_2, \quad \ldots, \quad p_n = (1 - a_1) \cdots (1 - a_{n-1})a_n,$$

of which each $p_j$ is a self-adjoint element of $A$, satisfying $0 \le p_i \le 1$, and having support within $U_i$. Observing that

$$\sum_j p_j = 1$$

identically, because $(1 - a_1)(1 - a_2) \cdots (1 - a_n) = 0$, by the choice of the $a_j \in A$, we now have the required finite partition of unity within the closed involutive subalgebra $A$, which completes the proof of the theorem. □

It may be remarked that already in this proof of the existence of finite partititions of unity in the closed involutive subalgebra $A$, we have implicitly been using the fact that the commutative C*-algebra $\mathbb{C}(M)$ is actually the algebra of global sections of the commutative C*-algebra $\mathbb{C}_M$ in the topos of sheaves on the compact, completely regular locale $M$, an observation which lies at the centre of the proof of Stone–Weierstrass theorem to which we now turn.

## 3. The Stone–Weierstrass theorem

With the existence of partitions of unity lying in the closed involutive subalgebra $A$ established, we are now in a position to state the Stone–Weierstrass theorem in a form which will allow it to be proved, in the way conjectured in [9], by applying the techniques introduced in [10, 12] within the context outlined by Burden–Mulvey [5] for the classical situation of continuous complex functions on a topological space, which we now adapt to the case of a compact, completely regular locale $M$:

**Theorem 3.** *Let $M$ be a compact, completely regular locale. Then any closed involutive subalgebra $A$ of the commutative C*-algebra $\mathbb{C}(M)$ which separates the locale $M$ is necessarily equal to $\mathbb{C}(M)$.*

**Proof.** The idea now will be to prove that the closed involutive subalgebra $A$ actually equals the commutative C*-algebra $\mathbb{C}(M)$, by considering the subsheaf which it generates of the sheaf $\mathbb{C}_M$ of continuous complex functions on $M$. It will be proved, firstly, that $A$ is exactly the algebra of sections of this subsheaf, by the existence of finite partitions of unity lying in $A$ that we have already established. Secondly, it will be shown that the subsheaf is actually closed in the sheaf $\mathbb{C}_M$, again by an argument involving these partitions of unity. And, then, finally, it will be remarked that the subsheaf is necessarily also dense in the sheaf $\mathbb{C}_M$, from which the result will follow.

So, consider firstly the sheaf $\mathbb{C}_M$ of continuous complex functions on the locale $M$. The technicalities involved in defining the topos Sh $M$ of sheaves on the locale $M$ need not concern us, except to observe that the constructions involved are entirely constructive, allowing one to talk of sheaves in much the same way as in the classical situation [7]. The sheaf $\mathbb{C}_M$ is, on the one hand, that of Dedekind complex numbers within the constructive context of the topos Sh $M$. On the other hand, it also has its sections over each open subset $U$ of the locale $M$ identifiable with the models of the theory of complex numbers in the corresponding open sublocale of $M$, hence with the continuous complex functions

$$a: U \to \mathbb{C},$$

in which we are also denoting that open sublocale of $M$, of which the inverse image mapping is given by taking meets with $U$ in $M$, by

$$U \to M.$$

The restriction $a \mid V : V \to \mathbb{C}$ of any section $a : U \to \mathbb{C}$ of the sheaf $\mathbb{C}_M$ to an open set $V$ of $M$ contained in $U$ is obtained simply by composing with the inclusion of the open sublocale $V$ in the open sublocale $U$ of $M$. From this observation it is easily verified that the elements $[\![a = 0]\!]$, $[\![a = 1]\!]$, $[\![r < a]\!]$, $[\![a < s]\!]$ of the locale $M$ considered earlier are indeed exactly the interpretations of the corresponding formulae in the sheaf $\mathbb{C}_M$, viewed within the topos $\mathrm{Sh}\,M$.

Now, consider the subsheaf $A_M$ of the sheaf $\mathbb{C}_M$ of continuous complex functions on $M$ determined by the inclusion

$$A \to \mathbb{C}(M)$$

of the closed involutive subalgebra $A$. The sections of $A_M$ over any open set $U$ of the locale $M$ are exactly those continuous complex functions which lie locally in the subalgebra $A$. That is, of those $a \in \mathbb{C}(U)$ for which there is an open covering $(U_i)$ of $U$, together with, for each $i$, an element $a_i \in A$ such that

$$a_i \mid U_i = a \mid U_i.$$

Our first remark is that, in this case, $A$ is exactly the algebra of sections of $A_M$. For if $b \in \mathbb{C}(M)$ is a section of $A_M$ over the locale $M$, then there exists an open covering $(U_i)$ of $M$, together with $a_i \in A$ for each $i$, with

$$a_i \mid U_i = b \mid U_i.$$

Let $(p_j)$ be a partition of unity in the closed involutive subalgebra $A$, subordinate to this open covering of $M$. It may be assumed that $(p_j)$ is finite, and that $p_j \in A$ has support in $U_{i_j}$, say, for each $j$. Letting

$$a = \sum_j p_j a_{i_j},$$

which also lies in $A$, one sees that this actually equals the given element $b$ by the following argument.

Firstly, simplify notation by denoting by $U_i$ the open set in which $p_i \in A$ has its support for each $i$. Hence, for each $i$, one has that $D(p_i) \lhd U_i$, giving the existence of an open set $V_i$ of $M$ for which $D(p_i) \wedge V_i = \emptyset$ and $U_i \vee V_i = M$. Write $a_i \in A$ for the corresponding element of the closed involutive subalgebra $A$. Thus, for each $i$, one has that $a_i \mid V_i = 0$. It may further be supposed that the index set $I$ involved is given by $1 \le i \le n$.

Now, consider the join

$$\bigvee_F \left( \bigvee_{i \in F} U_i \wedge \bigwedge_{i \notin F} V_i \right)$$

taken over all finite subsets $F$ of the index set $I$, which is well-defined since any finite subset of the decidable set $I = \{1, 2, \dots, n\}$ is decidable. Each of the meets considered is finite, since the complement of a finite subset of a finite cardinal is again finite, this join, by distributivity of finite meets over arbitrary joins in the locale $M$, is evidently equal to the identity element, by the observation that

$$U_i \vee V_i = M$$

for each $i \in I$. Hence, the elements

$$W_F = \bigwedge_{i \in F} U_i \wedge \bigwedge_{i \notin F} V_i$$

indexed by the finite subsets $F$ of $I$ form an open covering of the locale $M$.

Next, for each element $W_F$ of this open covering, observe that we have

$$\left(\sum_{i \in I} p_i a_i\right) | W_F = \sum_{i \in I} p_i | W_F a_i | W_F = \sum_{i \in F} p_i | W_F a_i | W_F,$$

since $W_F \subseteq V_i$ for each $i \notin F$, hence $p_i | W_F = 0$. But $i \in F$ implies that $a_i | W_F = b | W_F$ because in that case $W_F \leq U_i$, so

$$\left(\sum_{i \in I} p_i a_i\right) | W_F = \sum_{i \in F} p_i | W_F b | W_F = \left(\sum_{i \in F} p_i | W_F\right) b | W_F.$$

But, again noting that $p_i | W_F = 0$ whenever $i \notin F$, one has

$$\left(\sum_{i \in I} p_i a_i\right) | W_F = \left(\sum_{i \in I} p_i | W_F\right) b | W_F = b | W_F,$$

since $\sum_{i \in I} p_i = 1$. Thus, for each $W_F$ of the open covering of the locale $M$, one has that $a | W_F = b | W_F$; so, $a = \sum_{i \in I} p_i a_i$ is equal to $b$, which therefore lies in the closed involutive subalgebra $A$. Hence, any section over the locale $M$ of this subsheaf actually lies in the subalgebra $A$ of $\mathbb{C}(M)$.

But now, by the fact that $M$ is compact and completely regular, together with the existence of finite, non-negative partitions of unity within $A$, the techniques developed in [10] may be applied constructively to show that the subalgebra $A_M$ of the sheaf $\mathbb{C}_M$ of complex numbers in the topos of sheaves on $M$ is actually complete. Being a subspace of the Banach space $\mathbb{C}_M$, it is therefore closed in $\mathbb{C}_M$.

The argument to be used may be outlined in the following way: given any Cauchy approximation $C$ on the subsheaf $A_M$ of $\mathbb{C}_M$ over an open subset $U$ of the locale $M$, an open covering $(U_\alpha)$ of $U$ may be constructed, over each open subset of which the Cauchy approximation $C$ may be used to define a Cauchy approximation $C^\alpha$ on the closed involutive subalgebra $A$ of $\mathbb{C}(M)$. By the completeness of $A$, an element $b_\alpha \in A$ may then be found to which $C^\alpha$ converges, from which it may be shown that $b_\alpha | U_\alpha$ provides a limit over $U_\alpha$ for the Cauchy approximation $C$. By the uniqueness of limits, these restrictions then patch over the open covering $(U_\alpha)$, providing a limit for the Cauchy approximation $C$ and thereby establishing the completeness of $A_M$. The argument in detail proceeds as follows:

Suppose then that a Cauchy approximation $C$ is given on the subsheaf $A_M$ over an open subset $U$ of $M$. By the compact, complete regularity of the locale $M$ and the fact that the closed involutive subalgebra $A$ of $\mathbb{C}(M)$ separates $M$, choose a family $(q_\alpha)$ of continuous real functions on $M$ lying in $A$, for which the open subsets $(U_\alpha)$ on which each $q_\alpha$ is equal to the identity element of $A$ provide an open covering of $U$, and for which one has

$$0 \leq q_\alpha \leq 1 \quad \text{and} \quad D(q_\alpha) \lhd U$$

for each $\alpha$.

Now, for each $\alpha$ define a Cauchy approximation $C^\alpha$ on the algebra $A$, of which the elements $b \in C^\alpha_n$ for any $n$ are those $b \in A$ for which there exists:

- an open covering $(V_i)$ of $U$ together with for each $i$ an element $a_i \in A$ for which

$$a_i \mid V_i \in C_n(V_i), \quad \text{and}$$

- a finite partition of unity $(p_j)$ in $A$ subordinate to the open covering of $M$ obtained by taking the open covering $(V_i)$ of $U$ together with the pseudocomplement $W_\alpha$ of $D(q_\alpha)$, for which, choosing for each $j$ an index $i_j$ for which $p_j$ has support contained in $V_{i_j}$, it is the case that $a = \sum_j p_j a_{i_j}$ satisfies the condition that

$$b = q_\alpha a.$$

It may be seen that this indeed describes an approximation on $A$, by the observations that $C$ is an approximation on $A_M$ over $U$ and that finite partitions of unity exist in $A$ subordinate to any open covering of $M$. Hence, there exists $b \in C^\alpha_n$ for each $n$, by applying the construction given above to the elements which exist locally in $C_n$ for each $n$. Moreover, the fact that $C$ is Cauchy also allows one to show that $C^\alpha$ satisfies the Cauchy condition in the subalgebra $A$ of $\mathbb{C}(M)$. For any $k$, the cauchyness of $C$, together with the compactness of the locale $M$, implies the existence of an open subset $U'$ of $U$ which together with $W_\alpha$ covers the locale, and of a natural number $m$ for which any elements of $C_n$, $C_{n'}$ are within $1/3k$ over $U'$ whenever $n, n' \geq m$. Now, given $b \in C^\alpha_n$, $b' \in C^\alpha_{n'}$ for $n, n' \geq m$, it will be asserted that

$$b - b' \in N(1/k)$$

in the closed involutive subalgebra $A$. Suppose that $b = q_\alpha a$, $b' = q_\alpha a'$, in which $a = \sum_j p_j a_{i_j}$, $a' = \sum_{j'} p'_{j'} a'_{i'_{j'}}$ are constructed, respectively, from open coverings of $(V_i)$, $(V'_{i'})$ of $U$ and partitions of unity $(p_j)$, $(p'_{j'})$ in $A$, subordinate to these open coverings taken together with $W_\alpha$. Then evidently it is enough to show that

$$a - a' \in N(1/k)$$

over the open subset $U'$, because $0 \leq q_\alpha \leq 1$ and $D(q_\alpha) \lhd U$. For each $j_0, j'_0$, one may write

$$a - a' = \sum_j p_j (a_{i_j} - a_{i_{j_0}}) - \sum_{j'} p'_{j'} (a'_{i'_{j'}} - a'_{i'_{j'_0}}) + (a_{i_{j_0}} - a'_{i'_{j'_0}}),$$

because $(p_j)$, $(p'_{j'})$ are partitions of unity. Then one has that each of these three expressions lie in $N(1/3k)$ over the open subset $V_{i_0} \wedge V'_{i'_0}$ of $M$, by applying, to the observation that $a_i \in C_n$, $a'_{i'} \in C'_{n'}$ over $V_i$, $V'_{i'}$ respectively, an argument which is entirely analogous to that introduced above when considering summations involving partitions of unity. Hence, one has that

$$a - a' \in N(1/k)$$

over each subset $V_{i_0} \wedge V'_{i'_0}$ of the canonical refinement of the open coverings $(V_i)$, $(V'_{i'})$ of $U$, by the triangle inequality, which yields the required result.

The completeness of the closed involutive subalgebra $A$ of $\mathbb{C}(M)$ then implies that the Cauchy approximation $C^\alpha$ converges to an element $b_\alpha \in A$ for each $\alpha$, from which it may be deduced that the Cauchy approximation $C$ on the subsheaf $A_M$ over $U$ converges over $U_\alpha$ to the element $b_\alpha \mid U_\alpha \in A_M(U_\alpha)$, of which the details of the proof will be omitted. The uniqueness of limits then implies that these elements patch over the open covering $(U_\alpha)$ of $U$ to yield a limit for the Cauchy approximation $C$ over $U$. The subspace $A_M$ of the sheaf $\mathbb{C}_M$ is therefore complete, hence closed in $\mathbb{C}_M$.

Finally, the closed involutive subalgebra $A$ of $\mathbb{C}(M)$ contains all rational complex multiples of the identity element of $\mathbb{C}(M)$. The subsheaf of $\mathbb{C}_M$ which it generates therefore contains all continuous complex functions which are locally constant and have rational complex values. But the subsheaf of locally constant complex functions with rational complex values is exactly the rational complex numbers in the topos of sheaves on the locale $M$, hence is dense in the sheaf $\mathbb{C}_M$ of complex numbers in the topos. Hence, in particular, the subsheaf $A_M$ is dense in $\mathbb{C}_M$. But, being also closed in $\mathbb{C}_M$, it then actually equals $\mathbb{C}_M$.

From this, together with the observation that its algebra of sections over the locale $M$ is exactly the commutative C*-algebra $A$, one deduces that the closed involutive subalgebra $A$ is equal to $\mathbb{C}(M)$, which completes the proof of the Stone–Weierstrass theorem. □

For the motivating application of the theorem in this constructive form, the reader is referred to [4], in which the Stone–Weierstrass theorem is applied to show, within the constructive context of any Grothendieck topos $\mathbb{E}$, that the Gelfand representation

$$A \to \mathbb{C}(\operatorname{Max} A)$$

of any commutative C*-algebra $A$ over its maximal spectrum $\operatorname{Max} A$ is indeed an isometric *-isomorphism. In particular, by choosing the topos $\mathbb{E}$ appropriately, this allows the extension of Gelfand duality to contexts such as bundles of commutative C*-algebras over arbitrary topological spaces, and to commutative C*-algebras with actions by semigroups of *-homomorphisms.

Although not all the techniques applied to establishing the existence of Gelfand duality within any Grothendieck topos are constructive, the overall context is one within which it is known metamathematically that indeed a constructive development of these ideas is possible. The present paper, in which the proofs are constructively valid, provides an important step towards establishing Gelfand duality in a constructive manner.

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