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A convenient basis for a graded ring

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Abstract

The graded ring $\mathbf{Z}[e, t_1, t_2, t_3, \ldots]/(et_n = \sum_{i+j=n+1} t_i t_j$ for $n \geq 1)$ is considered, where e has degree 2, t_n has degree $2n$, and $0 \leq i, j \leq n+1$ with $t_0 = 1$. This ring is encountered in calculating the E_2-term of an Adams–Novikov spectral sequence converging to the homotopy ring of a spectrum related to MU. New ring generators $\{a_n\}_{n \geq 1}$ are defined which possess simple computational properties and an additive basis is presented in each degree.

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1. Introduction

Let T be the graded polynomial ring $\mathbf{Z}[e, t_1, t_2, t_3, \ldots]$, where e has degree 2 and t_n has degree $2n$. Let I be the graded ideal in T generated by the homogeneous relations

$$et_n = \sum_{i+j=n+1} t_i t_j \quad \text{for } n \geq 1, \tag{1}$$

where i and j run from 0 to $n+1$ and t_0 is interpreted as 1. The subject of this paper is the structure of the graded quotient ring $A = T/I$.

The unusual definition of this ring has its origin in algebraic topology. For each positive integer k, Alexander [4] defined a ring spectrum MkU in which the knth space is $M(k\xi_n)$, the Thom space of the k-fold Whitney sum of the canonical unitary bundle over $BU(n)$. There is a canonical map $S \to MkU$ of the sphere spectrum to each of these spectra as the multiplicative unit. Obviously, $M1U$ is the unitary bordism spectrum MU. This collection of spectra is interesting since it forms an inverse system

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in which the natural map $S_*(X) \to \operatorname{inv\,lim} MkU_*(X)$ is an isomorphism whenever X is a complex with finitely generated homology groups. Since each ring $\pi_*(MkU)$ contains a significant torsion-free part, it may be easier to attack than $\pi_*(S)$. The ring A is encountered in calculating the E_2-term of an Adams–Novikov spectral sequence converging to $\pi_*(M2U)$. This connection is sketched in Section 2.

For an element u in T, let $[u]$ stand for the equivalence class of u in A. It is easy to write $[t_n]$ in terms of $[e]$ and $[t_1]$. In fact, when A is tensored with $\boldsymbol{Z}[1/2]$, it becomes the polynomial ring $\boldsymbol{Z}[1/2][[e],[t_1]]$. It is convenient to imbed A as a subring of the graded polynomial ring $\boldsymbol{Z}[X, Y]$, with X and Y of degree 2, by sending $[e]$ to $4Y$ and $[t_1]$ to $2(Y - X)$. In view of this, we will also use symbols A, e, and t_n to denote the images of A, $[e]$, and $[t_n]$, respectively, in $\boldsymbol{Z}[X, Y]$.

Nevertheless, the given ring generators t_n do not promote calculations in A. It is the purpose of this paper to construct convenient new generators a_n of A, with $\deg a_n = 2n$, and then to exhibit an explicit additive basis in each degree.

Theorem 1.1. *The ring A is multiplicatively generated by the set $\{e, a_1, a_2, a_3, \ldots\}$.*

A salient feature of the a_n is that each square a_n^2 can be expressed in terms of other generators. The a_n are defined in Section 3 together with a related set of monomials M_n and a collection of integer coefficients $m_{n,q}$. There we also prove the following theorem.

Theorem 1.2. *For each $d \geq 0$ and with $h = 2X$, the elements $\{m_{n,q} h^q M_n \colon n + q = d\}$ form a basis for the $\boldsymbol{Z}$-module A_d of homogeneous elements of degree 2d. Furthermore, for $d \geq 1$, a_d is a member of this basis. In particular, for d even, $d = (2i + 1)n$, $n = 2^k$, $k \geq 1$, we have $a_d = m_{n,d-n} h^{d-n} a_n$ and $a_{d+1} = m_{n,d+1-n} h^{d+1-n} a_n$.*

Section 4 looks a some ideals in A, and Section 5 is devoted to a proof of Theorem 1.1.

2. Background

Returning to algebraic topology, we know from [1] that $MU_*(MkU)$ is a graded comodule over the Hopf algebroid $MU_*(MU)$. Recall that $MU_*(MU)$ can be identified with $\pi_*(MU) \otimes B$, where $B = \boldsymbol{Z}[b_1, b_2, b_3, \ldots]$ with $\deg b_n = 2n$. The coproduct $\Delta \colon B \to B \otimes B$ is given by the well-known [2, 5] equation $\Delta(b_n) = \sum_{i \geq 0} (b^{i+1})_{n-i} \otimes b_i$, where $b = \sum_{i \geq 0} b_i$ and $b_0 = 1$. This equation comes with the understanding that for an indeterminate x of degree -2, b is really the formal power series $b = 1 + b_1 x + b_2 x^2 + \cdots$ of degree 0 in $B[[x]]$ and $(b^{i+1})_{n-i}$ denotes the coefficient of x^{n-i}. Similarly, $MU_*(MkU)$ can be identified with $\pi_*(MU) \otimes G$, where $G = \boldsymbol{Z}[g_1, g_2, g_3, \ldots]$ with $\deg g_n = 2n$ and where the coaction map $\psi \colon G \to B \otimes G$ is given by $\psi(g_n) = \sum_{i \geq 0} (b^{i+k})_{n-i} \otimes g_i$ [8].

In order to calculate the E_2-term of an Adams–Novikov spectral sequence converging to $\pi_*(M2U)$, one might start by constructing a resolution of G by extended B-comodules

$$0 \to G \xrightarrow{\varepsilon} B \otimes R^0 \xrightarrow{d_0} B \otimes R^1 \xrightarrow{d_1} \cdots,$$

where each R^m is a graded $\boldsymbol{Z}$-module. General references for these notions are [3, 7]. In the present case, let P be the graded polynomial algebra $\boldsymbol{Z}[X]$ with $\deg X = 2$, and let U be the bigraded $\boldsymbol{Z}$-module with resolution grading given by $U^m = P \otimes P^{\otimes m}$, $m \geq 0$, and topological grading imposed by P. Here, $P^{\otimes m}$ denotes the m-fold tensor product of P. For each m, we choose $R^m \subseteq U^m$ and informally identify e with $4X$ in P, t_1 with $2(1 \otimes X - X \otimes 1)$ in R^1, and $\boldsymbol{Z}[X, Y]$ of Section 1 with U^1.

Let $C^m = B \otimes R^m$, $m \geq 0$. On each B-comodule C^m, the coaction map $\psi : B \otimes R^m \to B \otimes B \otimes R^m$ is $\Delta \otimes 1$. As a tensor product of $\boldsymbol{Z}$-algebras, each C^m is actually a commutative $\boldsymbol{Z}$-algebra with the usual *internal* product $C^m \otimes C^m \to C^m$. A non-commutative *resolution* product $\bullet$ can also be defined on $C = \sum_{m \geq 0} C^m$ with respect to which (C, d) is a differential graded $\boldsymbol{Z}$-algebra. On the R^m, this resolution product is different from the usual product on the tensor algebra $T(P)$ as defined in [6]. Restricted to C^0, however, the resolution product is identical to the internal product.

The maps ε and d_m must be B-comodule maps obeying $(\Delta \otimes 1)\varepsilon = (1 \otimes \varepsilon)\psi$ and $(\Delta \otimes 1)d_m = (1 \otimes d_m)(\Delta \otimes 1)$, respectively. A suitable definition for ε as a $\boldsymbol{Z}$-algebra map is $\varepsilon(g_n) = \sum_{i \geq 0}(b^{i+2})_{n-i} \otimes e^i$. For d_0, the key structure formulas are $d_0(1 \otimes e) = 2t_1$ and $d_0((b^{-1})_n \otimes 1) = \sum_{i \geq 0}(b^i)_{n-i-1} \otimes t_{i+1}$. Note that as a power series, b has constant coefficient 1 and is invertible and that B is multiplicatively generated by the homogeneous coefficients of $b^{-1} = 1 - b_1 x + (-b_2 + b_1^2)x^2 + \cdots$.

Similar considerations in $C^0[[x]]$ show that the image of ε is multiplicatively generated by the homogeneous coefficients of $b^{-2} \otimes 1 - b^{-1} \otimes e$. The interaction of the internal and resolution products together with the need for $\mathrm{im}(\varepsilon) \subseteq \ker(d_0)$ leads to the equation $(1 \otimes t_n) \bullet (1 \otimes e) = 1 \otimes (\sum_{i+j=n+1} t_i t_j)$ in C^1, where $t_i t_j$ involves the internal product and where we may view the resolution product as giving a right action $R^1 \otimes R^0 \to R^1$ of R^0 on R^1. The resolution product also gives a left action $R^0 \otimes R^1 \to R^1$, related to the right action by $(1 \otimes e) \bullet (1 \otimes t_n) = (1 \otimes t_n) \bullet (1 \otimes e) - 2(1 \otimes t_n t_1)$. It is the right action that leads to the relations (1).

3. Ring structure

3.1 New ring generators

The proposed multiplicative generators $\{a_n\}_{n \geq 1}$ of A are defined in $\boldsymbol{Z}[X, Y]$ as follows. Setting $h = 2X$, let $a_1 = 2(Y - X) = t_1$ and $a_2 = 2(Y - X)(Y + X) =$

$ha_1 + a_1^2/2$. Then, for $n > 2$ inductively define the remaining elements by

$$a_n = \begin{cases} h^{n-2}a_2 - \frac{1}{2}a_{n/2}^2 & \text{for } n = 2^k,\ k \geq 2, \\ h^{n-2^k}a_{2^k} & \text{for } n = (2i+1)2^k,\ i \geq 1,\ k \geq 1, \\ ha_{n-1} & \text{for } n = 2i+1,\ i \geq 1. \end{cases} \tag{2}$$

The elements for which n is a power of 2 form the foundation of this collection. For later, we extract

$$a_{(2i+1)2^k} = h^{(2i)2^k}a_{2^k} \quad \text{for } i \geq 0. \tag{3}$$

Although h is central to the definition and $2h = e - 2t_1 \in A$, it is important to note that h itself is not in A. Consequently, the defining equations in $\boldsymbol{Z}[X, Y]$ do not provide a decomposition of the a_n in A; nor is it immediately evident that the collection of elements is contained in A.

An alternate description of the a_n, $n \geq 2$, is provided by

$$a_n = \begin{cases} ha_{n-1} - \frac{1}{2}a_{n/2}^2 & \text{for } n \equiv 0 \pmod 4, \\ ha_{n-1} & \text{for } n \equiv 1, 3 \pmod 4, \\ ha_{n-1} + \frac{1}{2}a_{n/2}^2 & \text{for } n \equiv 2 \pmod 4. \end{cases} \tag{4}$$

The case where n is odd is immediate. For the other cases, first note that (2) gives

$$a_{4i} = h^{4i-2}a_2 - \tfrac{1}{2}a_{2i}^2 \quad \text{for } i \geq 1 \tag{5}$$

when $4i$ is a power of 2. Otherwise, (5) follows from the calculation

$$a_{4i} = h^{4i-2^k}a_{2^k} = h^{4i-2^k}(h^{2^k-2}a_2 - \tfrac{1}{2}a_{2^{k-1}}^2).$$

Combining (5) with (3) gives

$$a_{4i} = h^2(h^{4(i-1)}a_2) - \tfrac{1}{2}a_{2i}^2 = h^2 a_{4i-2} - \tfrac{1}{2}a_{2i}^2,$$

which establishes (4) when $n \equiv 0 \pmod 4$. For the final case with $n \equiv 2 \pmod 4$, note that the definition of a_2 conforms to (4); for $n > 2$, multiplication of (5) by h^2 and an application of (3) establishes the formula.

Let R be the subring of $\boldsymbol{Z}[X, Y]$ multiplicatively generated by $\{2h, a_1, a_2, a_3, \ldots\}$. Theorem 1.1 is just the statement that $A = R$.

3.2 Additive bases

It is helpful to have notation that reflects the binary representation of n. Given $n \geq 0$, let J_n be the set of distinct integers $j \geq 0$ such that $n = \sum_{j \in J_n} 2^j$, let r_n denote the cardinality of J_n, and when $n > 0$ let $k_n = \min(J_n)$. Thus, $r_n = 0$ implies $n = 0$; $r_n = 1$ and $k_n \geq 1$ imply n is a positive power of 2; $r_n \geq 2$ and $k_n \geq 1$ imply n is even but not a power of 2; and $k_n = 0$ implies n is odd. When the context n is clear, we will simply use the notation J, r, and k, respectively.

Table 1
Some values of $m_{n,q}$ for small n and q

n \ q	0				4				8				12				16				20				24				28				32	
36	1	1	1	2	2	2	2	2	1	1	1	2	2	2	2	2	1	1	1	2	2	2	2	2	1	1	1	2	2	2	2	2	1	1
34	1	1	1	2	1	1	1	2	1	1	1	2	1	1	1	2	1	1	1	2	1	1	1	2	1	1	1	2	1	1	1	2	1	1
32	1	1	2	4	8	2^4	2^4	2^4	8	8	8	8	8	2^4	2^4	2^4	4	4	4	4	8	2^4	2^4	2^4	8	8	8	8	8	2^4	2^4	2^4	2	2
30	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
28	1	1	1	1	2	2	2	2	1	1	1	1	2	2	2	2	1	1	1	1	2	2	2	2	1	1	1	1	2	2	2	2	1	1
26	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
24	1	1	1	2	4	4	4	4	2	2	2	2	4	4	4	4	1	1	1	2	4	4	4	4	2	2	2	2	4	4	4	4	1	1
22	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
20	1	1	1	2	2	2	2	2	1	1	1	2	2	2	2	2	1	1	1	2	2	2	2	2	1	1	1	2	2	2	2	2	1	1
18	1	1	1	2	1	1	1	2	1	1	1	2	1	1	1	2	1	1	1	2	1	1	1	2	1	1	1	2	1	1	1	2	1	1
16	1	1	2	4	8	8	8	8	4	4	4	4	8	8	8	8	2	2	2	4	8	8	8	8	4	4	4	4	8	8	8	8	1	1
14	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
12	1	1	1	2	2	2	2	2	1	1	1	2	2	2	2	2	1	1	1	2	2	2	2	2	1	1	1	2	2	2	2	2	1	1
10	1	1	1	2	1	1	1	2	1	1	1	2	1	1	1	2	1	1	1	2	1	1	1	2	1	1	1	2	1	1	1	2	1	1
8	1	1	2	4	4	4	4	4	2	2	2	4	4	4	4	4	1	1	2	4	4	4	4	4	2	2	2	4	4	4	4	4	1	1
6	1	1	1	2	1	1	1	2	1	1	1	2	1	1	1	2	1	1	1	2	1	1	1	2	1	1	1	2	1	1	1	2	1	1
4	1	1	2	4	2	2	2	4	1	1	2	4	2	2	2	4	1	1	2	4	2	2	2	4	1	1	2	4	2	2	2	4	1	1
2	1	1	2	2	1	1	2	2	1	1	2	2	1	1	2	2	1	1	2	2	1	1	2	2	1	1	2	2	1	1	2	2	1	1

Within R, let M_n be the monomial $\prod_{j \in J_n} a_{2^j}$ of degree $2n$. When n is a power of 2, M_n is just a_n. For a given $q \geq 0$, $h^q M_n$ need not be in R, although $2^q h^q M_n$ certainly is. We now define a collection of minimal coefficients $m_{n,q}$ for $n, q \geq 0$, each a non-negative power of 2 in the range $1 \leq m_{n,q} \leq 2^q$ such that $m_{n,q} h^q M_n \in R$:

$$m_{n,q} = \begin{cases} 2^q & \text{for } r = 0, \\ 2 & \text{for } r = 1,\ k = 1,\ q \equiv 2, 3 \pmod 4, \\ 1 & \text{for } r = 1,\ k \geq 1,\ q \equiv 0, 1 \pmod{2n}, \\ 2\gcd\{m_{n/2,q-1}, m_{n/2,q}\} & \text{for } r = 1,\ k \geq 2,\ q \not\equiv 0, 1 \pmod{2n}, \\ \gcd\{m_{2^k,q-i} : 0 \leq i \leq r-1\} & \text{for } r \geq 2,\ k \geq 1, \\ m_{n-1,q} & \text{for } r \geq 1,\ k = 0. \end{cases} \tag{6}$$

Here, we interpret $m_{n,q}$ as 0 whenever $q < 0$. For small $n > 0$ and $q \geq 0$, n even, these coefficients are displayed in Table 1. The claimed properties are addressed in Section 3.3.

Note that for n even, the coefficients are either all 1 (this first occurs when $n = 14$) or are periodic in q with period 2^{k+1}. Also, $m_{i,q} = m_{j,q}$ whenever i and j are even, $r_i = r_j$, and $k_i = k_j$.

Although the definition appears daunting, the coefficients are easily computed and reflect the following relationship: whenever a multiple of $m_{n,q} h^q M_n$ is decomposable in R in terms of elements of lower degree, $m_{n,q}$ is the minimum coefficient to result over all such decompositions. An example to consider is $m_{6,3} h^3 M_6$, where $m_{6,3} = 2$.

3.3 Proof of Theorem 1.2

We first establish some properties held by our coefficients.

Lemma 3.3.1. *For coefficients $m_{n,q}$ given by* (6),
(a) $m_{n,q}$ *divides* $2m_{n,q-1}$ *whenever* $q > 0$ *and* $n \geq 0$, *and*
(b) $m_{n,q}$ *divides* $m_{2n,q}$ *whenever* $q \geq 0$ *and* $n = 2^k$ *with* $k > 0$.

Proof. Obviously, (a) holds for $n = 0, 1, 2$, so suppose $n = 2^k$, $k \geq 2$. By way of induction on k, assume that $m_{n/2,i}$ divides $2m_{n/2,i-1}$ for $i > 0$. From (6), we have $m_{n,q} = 1$ when $q \equiv 0, 1 \pmod{2n}$ and $m_{n,q} = 2$ when $q \equiv 2 \pmod{2n}$. Thus, (a) holds for $q \equiv 0, 1, 2 \pmod{2n}$. But when $q \not\equiv 0, 1, 2 \pmod{2n}$, the induction hypothesis and (6) imply that $m_{n,q}$ divides $2\gcd\{2m_{n/2,q-2}, 2m_{n/2,q-1}\} = 2m_{n,q-1}$. Thus, (a) holds whenever n is a power of 2. But then, for any $k \geq 0$, it follows that

$$\gcd\{m_{2^k,q-i}\colon 0 \leq i \leq r-1\} \text{ divides } 2\gcd\{m_{2^k,q-i-1}\colon 0 \leq i \leq r-1\}.$$

Thus, when n is even but not a power of 2, (a) holds from (6). Finally, the case where n is odd follows from the even case.

Part (b) is obviously true when $q \equiv 0, 1 \pmod{2n}$ and otherwise follows from $m_{2n,q} = \gcd\{2m_{n,q-1}, 2m_{n,q}\}$ and part (a). □

Lemma 3.3.2 *Let* $1 \leq k_1 \leq k_2 \leq \cdots \leq k_r$, *with* $r \geq 1$. *Then*

$$\gcd\{m_{2^{k_1},q-i}\colon 0 \leq i \leq r-1\}$$
$$= \gcd\{m_{2^{k_1},i_1} m_{2^{k_2},i_2} \cdots m_{2^{k_r},i_r}\colon i_1 + i_2 + \cdots + i_r = q\}$$

for each $q \geq 0$.

Proof. When $r = 1$, the result is trivial. Consider the special case where $r = 2$ and $k_1 = k_2 = k \geq 1$. When $k = 1$, the lemma clearly holds by examining the bottom row of Table 1. So suppose that $k > 1$ and set $n = 2^k$. Recalling that $m_{n,0} = m_{n,1} = 1$, we have

$$\gcd\{m_{n,i}m_{n,j}\colon i+j=q\} = \gcd(\{m_{n,q-1}, m_{n,q}\} \cup \{m_{n,i}m_{n,j}\colon i+j=q \text{ and } i,j \geq 2\}),$$

and it suffices to show that $g = \gcd\{m_{n,q-1}, m_{n,q}\}$ divides each $m_{n,i}m_{n,j}$ on the right-hand side. This is clear using periodicity whenever i or $j \equiv 0, 1 \pmod{2n}$ since then either $m_{n,i}m_{n,j} = m_{n,q}$ or $m_{n,i}m_{n,j} = m_{n,q-1}$ holds. So suppose that both $i, j \not\equiv 0, 1 \pmod{2n}$. Then, by (6),

$$m_{n,i}m_{n,j} = \gcd\{4m_{n/2,i-1}m_{n/2,j-1}, 4m_{n/2,i-1}m_{n/2,j}, 4m_{n/2,i}m_{n/2,j-1}, 4m_{n/2,i}m_{n/2,j}\} \tag{7}$$

and by Lemma 3.3.1(a), g divides $g' = 2\gcd\{m_{n,q-2}, m_{n,q-1}, m_{n,q}\}$. If it happens that $q \equiv 0, 1, 2, 3 \pmod{2n}$, then $g' = 2$ so g divides $m_{n,i}m_{n,j}$. Otherwise, by (6),

$$g' = 4\gcd\{m_{n/2,q-3}, m_{n/2,q-2}, m_{n/2,q-1}, m_{n/2,q}\}.$$

By induction on k, we may assume that $\gcd\{m_{n/2,i+j-1}, m_{n/2,i+j}\}$ divides $m_{n/2,i}m_{n/2,j}$ for all $i, j \geq 0$. It follows that g' divides each product on the right-hand side of (7) and so, g divides $m_{n,i}m_{n,j}$ as needed. Thus, the lemma is established when $r = 2$ and $k_1 = k_2 = k \geq 1$.

Before turning to the general case, for $n = 2^k$, $k = k_1$, define

$$c_{r,q} = \gcd\{m_{n,i_1}m_{n,i_2}\cdots m_{n,i_r}\colon i_1 + i_2 + \cdots + i_r = q\} \quad \text{for } r \geq 1.$$

It is clear that $c_{r,q} = m_{n,q}$ when $r = 1$ and, from the special case just established, that

$$c_{r,q} = \gcd\{c_{r-1,q-1}, c_{r-1,q}\} \tag{8}$$

when $r = 2$. We now show that this equation also holds for all $r \geq 2$. Using induction on r, assume that (8) holds for all $c_{s,q}$ with $2 \leq s < r$. Also note that $c_{r,q} = \gcd\{c_{r-1,i}m_{n,j}\colon i + j = q\}$ for all q. Thus,

$$c_{r,q} = \gcd(\{c_{r-1,q-1}, c_{r-1,q}\} \cup \{c_{r-2,i-1}m_{n,j}, c_{r-2,i}m_{n,j}\colon i + j = q \text{ and } j \geq 2\}).$$

But, since $c_{r-1,q} = \gcd\{c_{r-2,i}m_{n,j}\colon i + j = q\}$ for all q, $c_{r-1,q}$ divides each $c_{r-2,i}m_{n,j}$ and $c_{r-1,q-1}$ divides each $c_{r-2,i-1}m_{n,j}$. Thus, Eq. (8) is established for $r \geq 2$. But then, successive application of (8) shows that $c_{r,q}$ equals the left-hand side of the equation of Lemma 3.3.2.

For the general case, let c denote the right-hand side of the equation of the lemma. Clearly, $c_{r,q}$ divides c by Lemma 3.3.1(b). But c divides $c_{r,q}$ because the coefficients defining the left-hand side of the lemma are just those defining c with the restriction that $i_j \in \{0, 1\}$ for $2 \leq j \leq r$. Thus, $c_{r,q} = c$. □

Lemma 3.3.3. *For each n, $q \geq 0$, $m_{n,q}h^q M_n \in R$.*

Proof. We systematically work through (6). When $r = 0$, we have $m_{n,q} = 2^q$, and the elements $(2h)^q M_0$ are clearly in R.

Now suppose that $r = 1$ and $n = 2^k$. When $k = 1$ and $q \equiv 2, 3 \pmod 4$, we see from (5) that $2h^{4i+2}M_2 = 2a_{4i+4} + a_{2i+2}^2 \in R$ and, hence, $2h^{4i+3}M_2 = 2a_{4i+5} + a_{2i+3}a_{2i+2} \in R$. When $k \geq 1$ and $q \equiv 0 \pmod{2n}$, write $q = 2ni$, $i \geq 0$. Here $1h^q M_n = a_{q+n} \in R$, by (3), and $1h^{q+1}M_n = a_{q+n+1} \in R$.

When $k \geq 2$ and $q \not\equiv 0, 1 \pmod{2n}$, we proceed by induction on k. Denoting 2^{k-1} by $n/2$, assume that $m_{n/2,q-1}h^{q-1}a_{n/2}$ and $m_{n/2,q}h^q a_{n/2}$ are in R. Since $n/2$ is even, $ha_{n/2} = a_{n/2+1} \in R$ and, hence, $m_{n/2,q-1}h^q a_{n/2}^2 \in R$. So clearly, if $m = \gcd\{m_{n/2,q-1}, m_{n/2,q}\}$, then $mh^q a_{n/2}^2 \in R$. But according to (2), $2mh^q M_n = 2mh^{q+n-2}a_2 - mh^q a_{n/2}^2$ for any m. And from the previous paragraph, we know that $2mh^i a_2 \in R$ for any i-$1 \geq 0$. Setting $m_{n,q} = 2m$, it follows that $m_{n,q}h^q M_n \in R$.

When $r \geq 2$ and $k \geq 1$, consider the obvious formula

$$m_{2^k,q-i}h^q M_n = (m_{2^k,q-i}h^{q-i}a_{2^k})(m_{2^{k_2},i_2}h^{i_2}a_{2^{k_2}})(m_{2^{k_3},i_3}h^{i_3}a_{2^{k_3}})\cdots(m_{2^{k_r},i_r}h^{i_r}a_{2^{k_r}}), \tag{9}$$

where $k < k_2 < k_3 < \cdots < k_r$ enumerate the integers in J_n, $0 \leq i \leq r-1$, $i_2 + i_3 + \cdots + i_r = i$, and $i_2, i_3, \ldots, i_r \in \{0, 1\}$. From the previous two paragraphs, each factor is in R. Thus, if we set $m_{n,q} = \gcd\{m_{2^k,q-i}: 0 \le i \le r-1\}$, then $m_{n,q}h^q M_n$ is in R.

Finally, when $r \ge 1$ and $k = 0$, then $n-1$ is even. We have already seen that $m_{n-1,q}h^q M_{n-1} \in R$. But $m_{n,q} = m_{n-1,q}$, and multiplication by a_1 shows that $m_{n,q}h^q M_n \in R$. □

Lemma 3.3.4. *Let $n = 2^k$, $k \ge 1$. Then $(m_{n,i}h^i a_n)(m_{n,j}h^j a_n)$ is a linear combination of $m_{2,q}h^q a_2$ and $m_{2n,i+j}h^{i+j}a_{2n}$, where $q = 2n + i + j - 2$.*

Proof. Since $a_n^2 = 2h^{2n-2}a_2 - 2a_{2n}$ by (5), the product equals $mh^q a_2 - mh^{i+j}a_{2n}$ where $m = 2m_{n,i}m_{n,j}$. But $m_{2,q}$ is 1 or 2 and obviously divides m. And if $i + j \equiv 0, 1 \pmod{4n}$, then $m_{2n,i+j} = 1$; otherwise, by Lemma 3.3.2, $m_{2n,i+j} = \gcd\{2m_{n,u}m_{n,v}: u + v = i + j\}$. In either case, the coefficient divides m. □

Lemma 3.3.5. *For $u, v, i, j \ge 0$, suppose that $u + v + i + j = d$. Then $(m_{u,i}h^i M_u)(m_{v,j}h^j M_v)$ is a linear combination of the elements $\{m_{n,q}h^q M_n: n + q = d\}$.*

Proof. If u is even, then $m_{u,i}h^i M_u$ may be written as the expression on the right-hand side of (9) for some choice of $i_2, i_3, \ldots, i_r$. If u is odd, it may also be written in this form, but multiplied by a_1. We interpret the expression to be $(2h)^q$ when $u = 0$, $(2h)^q a_1$ when $u = 1$, and $m_{n,q}h^q a_n$ when $n = 2^k$, $k \ge 1$. Expand both factors of $(m_{u,i}h^i M_u)(m_{v,j}h^j M_v)$ accordingly. If a_1^2 is found in the result, replace it by $2a_2 - (2h)a_1 = 2(m_{2,0}h^0 a_2) - m_{1,1}h^1 a_1$. If any other squares $a_{2^k}^2$, $k \ge 1$, exist in the result, iteratively replace them according to Lemma 3.3.4. This process is finite since each replacement reduces the number of a's in each of the resulting terms by 1. Eventually, we obtain a linear combination of terms, each consisting of $(2h)^w$ or $(2h)^w a_1$, $w \ge 0$, multiplied by a product of factors of the form $m_{2^{k_s},i_s}h^{i_s}a_{2^{k_s}}$, $1 \le s \le r$, with $1 \le k_1 < k_2 < \cdots < k_r$. For any specific term, let n be $\sum 2^{k_s}$ if a_1 is absent; otherwise, let n be one larger. Also let $q = w + i_1 + \cdots + i_r$ and note that $n + q = d$. If it happens that $r_n \ge 1$, then $m_{n,q-w} = \gcd\{m_{2^{k_1},q-w-i}: 0 \le i \le r-1\}$ divides $m_{2^{k_1},i_1}m_{2^{k_2},i_2}\cdots m_{2^{k_r},i_r}$ by Lemma 3.3.2. And by Lemma 3.3.1(a), $m_{n,q}$ divides $2^w m_{n,q-w}$. Thus, the term is a multiple of $m_{n,q}h^q M_n$. The conclusion follows. □

This leads to a proof of Theorem 1.2 (with A_d replaced by R_d until we prove Theorem 1.1), because each multiplicative generator $2h, a_1, a_2, a_3, \ldots$ of R has the form $m_{n,q}h^q M_n$ for some n and q. In particular, $2h = m_{0,1}h^1 M_0$ and $a_1 = m_{1,0}h^0 M_1$. For $n \ge 2$, let k be the smallest positive integer in J_n. Then from (2), $a_n = m_{2^k,n-2^k}h^{n-2^k}M_{2^k}$, where the coefficient is 1 since $n - 2^k \equiv 0, 1 \pmod{2^{k+1}}$. Thus, by Lemma 3.3.5, any homogeneous element in R_d is a linear combination of the stated elements. This is a good place to note that our equation for a_n places a_d in the claimed basis for R_d as stated.

Now the degree of $a_{2^k} \in \boldsymbol{Z}[X, Y]$ in the variable Y is 2^k for $k \ge 0$. This is clear for a_1 and a_2. Assume, by induction, that $k \ge 2$ and that the Y-degree of $a_{2^{k-1}}$ is 2^{k-1}. Then by (2), the Y-degree of a_{2^k} is that of $a_{2^{k-1}}^2$, and the assertion holds for all k. But then, the Y-degree of $m_{n,q}h^q M_n$ is n for all n, $q \geq 0$. Therefore, the stated elements are linearly independent in each degree and Theorem 1.2 follows. □

4. Ideals in R

Let S be the ideal $\{s \in R\colon h^n s \in R$ for all $n \geq 0\}$. Of course, $s \in S$ implies that $h^n s \in S$ for any $n \geq 0$. It is also clear from an examination of the bottom row of Table 1 that $2a_2 \in S$.

Lemma 4.1. *Suppose that $n > 0$ is even, and let $r > 0$ and $k > 0$ be related as in* Section 3.2. *Then $M_n \in S$ if and only if $r \geq 2^{k+1} - 1$.*

Proof. For any $q \geq 0$, choose $i, j \geq 0$ such that $q = j2^{k+1} + i$ with $i \leq 2^{k+1} - 1$. Thus

$$h^q M_n = (h^{j2^{k+1}+i_1} a_{2^k})(h^{i_2} a_{2^{k_2}}) \cdots (h^{i_r} a_{2^{k_r}}),$$

where $i_1 + i_2 + \cdots + i_r = i$. If $r \geq 2^{k+1} - 1$, then we may choose $i_s \in \{0, 1\}$ for each s so that each factor on the right is in R and $M_n \in S$. Conversely, if we know that $M_n \in S$, then $h^q M_n \in R$ for the choice $q = 2^{k+1} - 1$. Hence, $m_{n,q} = 1$, and from (6), we see that $m_{2^k, q-i} = 1$ for some $0 \leq i \leq r - 1$ with $i \leq q$. Consequently, again by (6), we must have $q - i \equiv 0, 1 \pmod{2^{k+1}}$. But since $q < 2^{k+1}$, this implies that either $i = q$ or $i + 1 = q$. It follows that $i + 1 \geq q$, and thus $r \geq i + 1 \geq 2^{k+1} - 1$. □

The definition of S insures that equivalence modulo S is preserved under multiplication by h. That this equivalence is also preserved under $s \to s^2/2$ is established next.

Lemma 4.2. *Whenever $s \in S$, $s^2 \in 2S$.*

Proof. Let $s \in S \cap R_d$ and represent $s = \sum_{n+q=d} c_n(m_{n,q}h^q M_n)$. Then, for $i \geq 0$, each term of $h^i s = \sum_{n+q=d}(c_n m_{n,q})h^{q+i}M_n$ is in R. Denoting $s_n = (c_n m_{n,q})h^q M_n$, we see that $s_n \in S$ for each n. From (6), it is evident that $c_0 = c_1 = 0$. For the conclusion to hold, it is only necessary to show that $s_n^2 \in 2S$ for $n \geq 2$.

For such an n, if 2 divides $c_n m_{n,q}$, then $s_n^2/2 = (c_n m_{n,q}/2)M_n(h^q s_n)$ is in the ideal S. On the other hand, if 2 does not divide $c_n m_{n,q}$, then $s_n \in S$ implies that $m_{n,i} = 1$ for each $i \geq q$. But by periodicity, $m_{n,i} = 1$ for all $i \geq 0$, so $M_n \in S$.

If n is even, Lemma 4.1 applies and the number r of positive integers in J_n obeys $r \geq 2^{k+1} - 1 \geq 3$. Denoting these integers by $k < k_2 < \cdots < k_r$, let $i = 2^{k_{r-1}}$ and $j = 2^{k_r}$ and write $M_n = M_{n-i-j}\, a_i a_j$. From (5) and using $2a_2 \in S$, we have $(a_i a_j)^2/2 = 2(h^{2i-2}a_2 - a_{2i})(h^{2j-2}a_2 - a_{2j}) \equiv 2a_{2i}a_{2j} \pmod{S}$. Thus, $M_n^2/2 \equiv 2M_{n-i-j}^2 a_{2i}a_{2j} \pmod{S} = 2M_{n-i-j}M_{n+i+j}$. Since k and r for J_{n+i+j} are the same as for J_n, M_{n+i+j} is in the ideal S by Lemma 4.1; consequently, $s_n^2/2 \in S$.

Similarly, if n is odd, $m_{n-1,q} = m_{n,q} = 1$ for all q, so $M_{n-1} \in S$. The lemma then applies to M_{n-1} and the conclusion follows. □

Let L be the augmentation ideal $\sum_{d \geq 1} R_d$ of R. Then L^2 is the ideal of decomposable elements. But R is generated by $\{2h, a_1, a_2, a_3, \ldots\}$, and from (4) it follows that $2a_n \in L^2$ whenever $n \geq 2$. Hence, $2R_d \subseteq L^2$ for $d \geq 2$.

Lemma 4.3. *If* $d \geq 2$ *and* $r \in R_d$, *then* $r \equiv ca_d (\operatorname{mod} L^2)$ *for some integer* c.

Proof. Let $d \geq 2$. We first determine which of the additive basis elements of R_d are in L^2. From the proof of Lemma 3.3.3, it is clear that $m_{n,q} h^q M_n$ is decomposable whenever $n + q \geq 2$, except possibly when $n = 2^k$, $k \geq 1$. But since $2a_i \in L^2$ for $i \geq 2$, it also follows from the same proof in the case $n = 2^k$, $k = 1$, that

$$2h^q a_2 \in L^2 \quad \text{for } q \geq 0. \tag{10}$$

Thus, when $k = 1$ and $q \not\equiv 0, 1 (\operatorname{mod} 4)$, we have $m_{n,q} h^q M_n = 2h^q a_2 \in L^2$. Now if $n = 2^k$, $k \geq 2$, and $q \not\equiv 0, 1 (\operatorname{mod} 2n)$, then by (2) we have $m_{n,q} h^q M_n = m_{n,q} h^{n+q-2} a_2 - m_{n,q} h^q a_{n/2}^2/2$. By (6), $m_{n,q}$ is either $2m_{n/2,q-1}$ or $2m_{n/2,q}$. In the former case, $m_{n,q} h^q a_{n/2}^2/2 = (m_{n/2,q-1} h^{q-1} a_{n/2})(ha_{n/2}) \in L^2$ and in the latter the term is $(m_{n/2,q} h^q a_{n/2}) a_{n/2} \in L^2$. By (10), we know that $m_{n,q} h^{n+q-2} a_2$ is in L^2 since $m_{n,q}$ is divisible by 2. Thus, $m_{n,q} h^q M_n \in L^2$.

Thus, all basis elements of R_d are in L^2 except possibly when $n = 2^k$, $k \geq 1$, and $q \equiv 0, 1 (\operatorname{mod} 2n)$, where $m_{n,q} = 1$. The exceptional case is unique for each d; from the restriction $q = d - n$, it follows that k must be the smallest positive integer in J_d and here $m_{n,q} h^q M_n = a_d$ by the proof of Theorem 1.2. The lemma follows immediately. □

Corollary 4.4. $S \subseteq L^2$.

Proof. Since $S \cap R_d = \{0\}$ when $d = 0, 1$, consider $r \in S \cap R_d$ with $d \geq 2$. By Lemma 4.3, $r \equiv ca_d (\operatorname{mod} L^2)$ for some integer c, where we know that a_d is the basis element $m_{n,d-n} h^{d-n} a_n$ with $n = 2^k$, $k \geq 1$, and $d - n \equiv 0, 1 (\operatorname{mod} 2n)$. Thus $m_{n,d-n} = 1$, and 2 divides $m_{n,d-n+2}$ since $d - n + 2 \not\equiv 0, 1 (\operatorname{mod} 2n)$. Since $h^2 r \in R$, it follows that $m_{n,d-n+2}$ divides $cm_{n,d-n}$. Thus, 2 divides c; consequently, ca_d, and therefore r, are in L^2. □

5. Conclusion

Our attention now turns to the relationship of the generators t_n of A to the generators a_n of R. We have already seen that $t_1 = a_1$ and $e = 2h + 2a_1$. From (1) it is easy to derive $t_2 = ht_1 + 1/2t_1^2 = a_2$. Similarly, for $n \geq 3$, after substituting for e and canceling terms containing t_1, we obtain

$$t_n = \begin{cases} ht_{n-1} - \frac{1}{2}t_{n/2}^2 - \sum_{2 \le i \le n/2-1} t_{n-i}t_i & \text{for } n \text{ even,} \\ ht_{n-1} - \sum_{2 \le i \le (n-1)/2} t_{n-i}t_i & \text{for } n \text{ odd.} \end{cases} \tag{11}$$

Referring to (4), it then follows directly that $t_3 = a_3$, $t_4 = a_4$, and $t_5 = a_5 - t_3t_2$.

The ideal S is valuable in simplifying the remaining calculations. In particular, whenever $u \equiv v \pmod S$, we have $hu \equiv hv \pmod S$ by the definition of S and $u^2/2 \equiv v^2/2 \pmod S$ by Lemma 4.2. The next result makes use of this preservation of equivalence.

Proposition 5.1. *For each $n \ge 1$, the following modulo S equivalences hold:*

$$a_{4n-2} - t_{4n-2} \equiv \sum_{1 \le i \le n-1} t_{4n-2i-2}t_{2i}, \tag{12a}$$

$$a_{4n-1} - t_{4n-1} \equiv \sum_{1 \le i \le n-1} t_{4n-2i-2}t_{2i+1}, \tag{12b}$$

$$a_{4n} - t_{4n} \equiv \sum_{1 \le i \le n-1} t_{4n-2i}t_{2i} + \tfrac{1}{2}(t_{2n}^2 - a_{2n}^2), \tag{12c}$$

$$a_{4n+1} - t_{4n+1} \equiv \sum_{1 \le i \le n} t_{4n-2i}t_{2i+1} + \tfrac{1}{2}h(t_{2n}^2 - a_{2n}^2). \tag{12d}$$

Furthermore, $a_{4n-2} - t_{4n-2} \equiv l_{4n-2}$, $a_{4n-1} - t_{4n-1} \equiv l_{4n-1}$, $a_{4n} - t_{4n} \equiv l_{4n}$, and $a_{4n+1} - t_{4n+1} \equiv l_{4n+1}$, where $l_{4n-2}, l_{4n-1}, l_{4n}$, and l_{4n+1} are in L^2.

Proof. We will actually prove a somewhat stronger result: that also, for each $n \ge 1$, we have $ht_{4n-2} + t_{4n-1}$, $ht_{4n} + t_{4n+1} \in S$ and we may choose l_{4n-2}, $l_{4n} \in E$, where $E = L^2 \cap E'$ with E' the subring of $\mathbf{Z}[X, Y]$ generated by evenly subscripted elements $\{a_{2i}\colon i \ge 1\}$. The calculations $ht_2 + t_3 = h(2a_2) \in S$ and $ht_4 + t_5 = h^3(2a_2) - (2a_2)a_3 \in S$ show that this stronger result holds for the case $n = 1$. We now proceed by induction on $n > 1$, assuming the proposition holds for integers smaller than n.

Beginning with the decomposition of t_{4n-2} given by (11), we replace t_{4n-3} using (12d) and then apply $(ht_{4n-2i-4} + t_{4n-2i-3})t_{2i+1} \equiv 0$ for $1 \le i \le n-2$:

$$\begin{aligned} t_{4n-2} &= ht_{4n-3} - \tfrac{1}{2}t_{2n-1}^2 - \sum_{2 \le i \le 2n-2} t_{4n-i-2}t_i \\ &\equiv ha_{4n-3} - \sum_{1 \le i \le n-1} ht_{4n-2i-4}t_{2i+1} - \tfrac{1}{2}h^2(t_{2n-2}^2 - a_{2n-2}^2) \\ &\quad - \tfrac{1}{2}t_{2n-1}^2 - \sum_{2 \le i \le 2n-2} t_{4n-i-2}t_i \end{aligned}$$

$$\equiv ha_{4n-3} - ht_{2n-2}t_{2n-1} - \tfrac{1}{2}h^2(t_{2n-2}^2 - a_{2n-2}^2) - \tfrac{1}{2}t_{2n-1}^2$$
$$- \sum_{1 \le i \le n-1} t_{4n-2i-2}t_{2i}.$$

By (4), $ha_{4n-3} + h^2a_{2n-2}^2/2 = a_{4n-2}$, and by Lemma 4.2, $ht_{2n-2}t_{2n-1} + h^2t_{2n-2}^2/2 + t_{2n-1}^2/2 = (ht_{2n-2} + t_{2n-1})^2/2 \equiv 0$; thus, (12a) follows. But using our additional inductive assumptions, $t_{4n-2i-2}t_{2i} \equiv (a_{4n-2i-2} - l_{4n-2i-2})(a_{2i} - l_{2i}) \in E$ for $1 \le i \le n-1$; thus $a_{4n-2} - t_{4n-2} \equiv l_{4n-2} \in E$.

In a similar manner, (12b) follows from (12a) with $a_{4n-1} - t_{4n-1} \equiv l_{4n-1} \in L^2$. Furthermore,

$$ht_{4n-2} + t_{4n-1} \equiv ha_{4n-2} + a_{4n-1} - \sum_{1 \le i \le n-1} t_{4n-2i-2}(ht_{2i} + t_{2i+1})$$
$$\equiv h^{4n-3}(2a_2) \equiv 0$$

using the inductive assumption and (3).

Proceeding as with (12a), we have

$$t_{4n} = ht_{4n-1} - \tfrac{1}{2}t_{2n}^2 - \sum_{2 \le i \le 2n-1} t_{4n-i}t_i$$
$$\equiv ha_{4n-1} - \sum_{1 \le i \le n-1} ht_{4n-2i-2}t_{2i+1} - \tfrac{1}{2}t_{2n}^2 - \sum_{2 \le i \le 2n-1} t_{4n-i}t_i$$
$$\equiv ha_{4n-1} - \tfrac{1}{2}t_{2n}^2 - \sum_{1 \le i \le n-1} t_{4n-2i}t_{2i},$$

where by (4), $ha_{4n-1} - t_{2n}^2/2 = a_{4n} - (t_{2n}^2 - a_{2n}^2)/2$. This yields (12c).

Next note that for any $i, j \ge 1$, Eq. (5) leads to

$$\tfrac{1}{2}(a_{2i}a_{2j})^2 = 2(h^{4i-2}a_2 - a_{4i})(h^{4j-2}a_2 - a_{4j}) \equiv 2a_{4i}a_{4j}.$$

Thus for any element $a \in E$, there exists an $l \in E$ such that $a^2/2 \equiv l$. It follows by induction that $(t_{2n}^2 - a_{2n}^2)/2 = (t_{2n} - a_{2n})^2/2 + (t_{2n} - a_{2n})a_{2n}$ is equivalent to some $l \in E$. As with (12a), the summation in (12c) is in E, so $a_{4n} - t_{4n} \equiv l_{4n} \in E$.

Now (12d) follows from (12c), with the summation in L^2 as before. But $hl \in L^2$ whenever $l \in E$, so the remaining term is in L^2. Hence, $a_{4n+1} - t_{4n+1} \equiv l_{4n+1} \in L^2$. Finally,

$$ht_{4n} + t_{4n+1} \equiv ha_{4n} + a_{4n+1} - \sum_{1 \le i \le n-1} t_{4n-2i}(ht_{2i} + t_{2i+1})$$
$$- t_{2n}t_{2n+1} - h(t_{2n}^2 - a_{2n}^2)$$
$$\equiv 2ha_{4n} - t_{2n}(ht_{2n} + t_{2n+1}) + ha_{2n}^2$$
$$\equiv 2h^{4n-1}a_2$$
$$\equiv 0$$

follows from the inductive assumption and (5). □

An immediate consequence of this is that each generator t_n of A is in R and, therefore, $A \subseteq R$. By way of induction, assume that $a_i \in A$ for $i < n$. Since $a_n \equiv t_n (\bmod L^2)$, it follows that a_n is in A for all n. This establishes Theorem 1.1. □

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