A Cramer Rule for Least-Norm Solutions of Consistent Linear Equations

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ABSTRACT

The least (Euclidean-) norm solution of a consistent linear equation $Ax = b$ is given a determinantal form, which reduces to Cramer's rule if A is nonsingular.

1. INTRODUCTION

We denote by

C^n	the n-dimensional complex vector space,
$C^{m\times n}$	the space of $m \times n$ complex matrices,
$C_r^{m\times n}$	$\{X \in C^{m\times n} : \text{rank } X = r\}$,
N	$\{1,2,\dots,n\}$.

For any $A \in C^{m\times n}$, $x \in C^m$, $j \in \mathrm{N}$ we denote by

$$A(j \to x)$$

the matrix obtained from A by replacing column j with x.

Steve Robinson [2] gave an elegant proof of Cramer's rule, rewriting

$$Ax = b \tag{1}$$

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as

$$AI(j \to x) = A(j \to b), \qquad j \in \mathbf{N} \tag{2}$$

and taking determinants (assuming ($A \in C_n^{n \times n}$):

$$\det A \det I(j \to x) = \det A(j \to b). \tag{3}$$

Since

$$\det I(j \to x) = x_j, \qquad j \in \mathbf{N}, \tag{4}$$

it follows from (3) that

$$x_j = \frac{\det A(j \to b)}{\det A}, \qquad j \in \mathbf{N}, \tag{5}$$

which is Cramer's rule. This method[1] of Robinson's is used here to derive a Cramer rule for the least-norm solution of a consistent linear equation.

2. RESULTS

For $A \in C^{m \times n}$ let

A^*	be the *conjugate transpose*,
A^+	be the *Moore-Penrose inverse*,
$R(A)$	be the *range*,
$N(A)$	be the *null space*

of A.

The linear equation (1) is called *consistent* if $b \in R(A)$, in which case the *least* (Euclidean-) *norm solution* [i.e. the unique solution of (1) which lies in $R(A^*)$] is given by $A^+ b$ (e.g. [1, Section 3.2]).

A Cramer rule for the least-norm solution is the following:

THEOREM. *Let $A \in C_r^{m \times n}$, $b \in R(A)$. Then the least-norm solution of*

$$Ax = b \tag{1}$$

[1]A method is a trick which succeeded twice.

is given, componentwise, by

$$(6)\qquad x_j = \frac{\det\begin{bmatrix} A(j\to b) & U \\ V^*(j\to 0) & O \end{bmatrix}}{\det\begin{bmatrix} A & U \\ V^* & O \end{bmatrix}}, \qquad j\in \mathbf{N},$$

where

(7) $U\in C_{m-r}^{m\times(m-r)}$ *is any matrix whose columns are a basis for* $N(A^*)$,

(8) $V\in C_{n-r}^{n\times(n-r)}$ *is any matrix whose columns are a basis for* $N(A)$,

(9) $V^*(j\to 0)$ *is* V^* *with column* j *replaced by* 0, *and*

(10) O *is a zero matrix of appropriate size* [*here* $(n-r)\times(m-r)$].

NOTE. In (6) we interpret U and O as absent if $r=m$, and similarly for V^* and O if $r=n$. Thus (6) reduces to the Cramer rule (5) if A is nonsingular.

Proof. The least-norm solution x of (1) satisfies

$$(11)\qquad \begin{bmatrix} A & U \\ V^* & O \end{bmatrix}\begin{bmatrix} I(j\to x) & O \\ O & I \end{bmatrix} = \begin{bmatrix} A(j\to b) & U \\ V^*(j\to 0) & O \end{bmatrix},$$

which follows from (2) and $V^*x=0$ [since $x\in R(A^*)=N(A)^\perp$]. The matrix

$$\begin{bmatrix} A & U \\ V^* & O \end{bmatrix}$$

[where U, V^*, O are chosen by (7), (8), and (10) respectively] is nonsingular (e.g. [1, Section 5.6]), and (6) follows from (11) by taking determinants. ■

An easy consequence of (11) is the following computation of the absolute values of the components of the least-norm solution:

$$(12)\qquad |x_j|^2 = \frac{\det[A(j\to b)^*A(j\to b)+V(j\to 0)V(j\to 0)^*]}{\det(A^*A+VV^*)}, \qquad j\in\mathbf{N},$$

where

$$A(j \to b)^* \quad \text{and} \quad V(j \to 0)^*$$

are the conjugate transposes of $A(j \to b)$ and $V(j \to 0)$ respectively.

REFERENCES

1 A. Ben-Israel and T. N. E. Greville, *Generalized Inverses: Theory and Applications*, Wiley-Interscience, 1974; Krieger, 1980.

2 S. M. Robinson, A short proof of Cramer's rule, *Math. Mag.* 43:94–95 (1970); reprinted in *Selected Papers on Algebra* (S. Montgomery et al., Eds.), Mathematical Association of America, 1977, pp. 313–314.

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