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A DECOMPOSITION FOR THE CYCLIC COHOMOLOGY OF A COMMUTATIVE ALGEBRA*

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0. Introduction

Let $\mathscr{X}$ be a compact C^∞-manifold and $C^\infty(\mathscr{X})$ be the algebra of C^∞-functions on $\mathscr{X}$. Connes has shown that the (continuous) cyclic cohomology of $C^\infty(\mathscr{X})$ decomposes as

$$H_\lambda^n(C^\infty(\mathscr{X}))_{\text{cont}} = Z_n(\mathscr{X}) \oplus H_{n-2}(\mathscr{X}, \mathbb{C}) \oplus H_{n-4}(\mathscr{X}, \mathbb{C}) \oplus \cdots,$$

where $Z_n(\mathscr{X})$ is the space of closed n-currents on $\mathscr{X}$ and $H_\bullet(\mathscr{X}, \mathbb{C})$ is the de Rham homology [1]. Similar decompositions hold for the cyclic homology and cohomology of a smooth commutative k-algebra whenever k is a commutative ring containing $\mathbb{Q}$ [5]. It seems natural to seek a uniform description of these decompositions in terms of the cyclic cochain complex and certain algebraic operations.

Now, if k is a commutative unital ring, A is an associative k-algebra, and $A^* = \operatorname{Hom}_k(A, k)$ is viewed as an A-bimodule in which afa' is defined by $(afa')(x) = f(a'xa)$, then there is a canonical monomorphism $C_\lambda^\bullet(A) \rightarrowtail C^\bullet(A, A^*)$ from the cyclic cochain complex, $C_\lambda^\bullet(A)$, to the Hochschild cochain complex, $C^\bullet(A, A^*)$. If, further, k is a $\mathbb{Q}$-algebra, A is commutative, and M is a *symmetric* A-bimodule – i.e., $am = ma$ for all $a \in A$ and $m \in M$ – then the Hochschild cohomology $H^\bullet(A, M)$ has a natural 'Hodge' decomposition, introduced in [2] by the second author and Gerstenhaber. (In an April, 1986 preprint of [2], they asserted that this decomposition agrees with the usual Hodge decomposition of the complex cohomology of a smooth complex projective variety. This claim is substantiated in [3].)

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Since the dual of a commutative algebra is a symmetric bimodule, it is natural to expect a decomposition of $H_\lambda^\bullet(A)$ similar to that of $H^\bullet(A,A^*)$ when A is commutative and k contains $\mathbb{Q}$. In this paper we produce such a decomposition[1] (together with explicit projection operators) and show that it agrees with those noted earlier for smooth algebras and $C^\infty(\mathscr{X})$. There is likewise a decomposition of the standard periodicity sequence. An examination of the component sequences then shows that the Harrison cohomology groups $\mathrm{Har}^n(A,A^*)$ are summands of the cyclic cohomology $H_\lambda^n(A)$ for $n>2$. (Since $k\supset\mathbb{Q}$, the Harrison cohomology agrees with the cohomologies introduced by André and Quillen for commutative algebras.) These assertions apply equally well to homology and the reader will have no trouble translating the arguments below to that setting. We shall therefore confine our attention to cohomology, making no further mention of cyclic homology.

The following notational conventions will be in force throughout this paper. First, k will be a fixed commutative unital $\mathbb{Q}$-algebra. If A and V are k-modules, then $C^n(A,V)$ will be the k-module consisting of the k-multilinear maps of n variables $A\times\cdots\times A\to V$. Of course, when A is an associative k-algebra and V is an A-bimodule $C^n(A,V)$ is the familiar module of Hochschild n-cochains. Coproduct (or, in older terminology, direct sum) will be represented by $\coprod$. Finally, the group of permutations of a set $\mathscr{I}$ will be denoted $\mathscr{I}!$ and the sign of a permutation π will be denoted $(-1)^\pi$.

1. Hodge-type decomposition

A permutation $\pi\in\{1,\dots,n\}!$ is a *p-shuffle*, where $0<p<n$, if

$$\pi(1)<\cdots<\pi(p) \quad\text{and}\quad \pi(p+1)<\cdots<\pi(n);$$

that is, π preserves the order of each of the sets $\{1,\dots,p\}$ and $\{p+1,\dots,n\}$. The set of such p-shuffles will be denoted $\mathrm{Sh}_p\{1,\dots,n\}$ or Sh_p. Note that the identity permutation, Id, is a p-shuffle for all p. Also, $\mathrm{Sh}_p\cap\mathrm{Sh}_q=\{\mathrm{Id}\}$ if $p\neq q$. The *disjoint* union of the sets of p-shuffles will be denoted $\mathrm{Sh}\{1,\dots,n\}$; it consists of one copy of each nonidentity shuffle and $n-1$ copies of the identity permutation.

Let $k[\{1,\dots,n\}!]$ be the group algebra of $\{1,\dots,n\}!$ and define the (total) shuffle operator $s_n\in k[\{1,\dots,n\}!]$ by

$$s_n=\sum_{\pi\in\mathrm{Sh}\{1,\dots,n\}}(-1)^\pi\pi.$$

In [2] the minimal polynomial of s_n is shown to have n distinct roots, namely $\alpha_r=2^r-2$ for $1\le r\le n$. These roots determine Lagrange interpolation polynomials

[1] A similar decomposition has been obtained, using entirely different techniques, by Burghelea and Vigué-Poirrier (Springer Lecture Notes in Mathematics 1318).

whose values at s_n will be denoted by $e_n(1),\ldots,e_n(n)$. Thus, $e_n(r)\in k[\{1,\ldots,n\}!]$ is defined by

$$e_n(r) = \prod_{i\neq r} (s_n-\alpha_i)/(\alpha_r-\alpha_i).$$

The basic properties of the Lagrange polynomials immediately imply that $e_n(1),\ldots,e_n(n)$ are pairwise orthogonal idempotents such that

$$\sum e_n(r)=1 \quad \text{and} \quad \sum (2^r-2)e_n(r)=s_n.$$

(In different terminology, $\{e_n(r)\}$ is the set of spectral projections of the operator s_n.) It is convenient to extend our definitions by setting $e_0(0)=1$ and $e_n(0)=0$ for $n\neq 0$.

Now let A be an associative k-algebra. The symmetric group $\{1,\ldots,n\}!$ has a natural left action on $A^{\otimes_k n}$, namely: $\pi(a_1\otimes\cdots\otimes a_n)=a_{\pi^{-1}1}\otimes\cdots\otimes a_{\pi^{-1}n}$. Hence, if V is any k-module, then $C^n(A,V)$ is a *right* $k[\{1,\ldots,n\}!]$-module in which $(f\pi)(a_1,\ldots,a_n)=f(a_{\pi^{-1}1},\ldots,a_{\pi^{-1}n})$. If, further, A is commutative and V is a symmetric A-bimodule, then, as is shown in [2], $\delta(fe_n(r))=(\delta f)e_{n+1}(r)$, where δ is the Hochschild coboundary. This yields the Hodge decomposition of the Hochschild cohomology: set $C^{r,n-r}(A,V)=C^n(A,V)e_n(r)$, so that an n-cochain f is in $C^{r,n-r}(A,V)$ for $1\le r\le n$ if and only if $fe_n(r)=f$ or, equivalently, $fs_n=(2^r-2)f$. Thus, if we set $C^{r,t}(A,V)=H^{r,t}(A,V)=0$ when $t<0$ then $C^{r,\bullet-r}(A,V)$ is a complex, $C^\bullet(A,V)=\coprod_{r\ge 0} C^{r,\bullet-r}(A,V)$ and, in an obvious notation, $H^\bullet(A,V)=\coprod_{r\ge 0} H^{r,\bullet-r}(A,V)$.

For the cyclic cohomology, first recall that there is a natural isomorphism $I:C^{n+1}(A,k)\to C^n(A,A^*)$ given by $((If)(a_1,\ldots,a_n))(a_0)=f(a_0,\ldots,a_n)$. Using this isomorphism, the Hochschild coboundary can be transferred to $C^{\bullet+1}(A,k)$, where it is commonly denoted by $\boldsymbol{b}$. The coboundary of a multilinear functional $f\in C^{n+1}(A,k)$ is then the multilinear functional $\boldsymbol{b}f\in C^{n+2}(A,k)$ described by

$$(\boldsymbol{b}f)(a_0,\ldots,a_{n+1}) = \sum_{i=0}^{n} (-1)^i f(\ldots,a_ia_{i+1},\ldots)+(-1)^{n+1}f(a_{n+1}a_0,\ldots,a_n).$$

Now, if $\mathcal{I}$ is an arbitrary finite linearly ordered set with n elements, then shuffles are defined in the evident manner using the unique order-preserving bijection $\mathcal{I}\to\{1,\ldots,n\}$. Thus, $\pi\in\mathcal{I}!$ is a p-shuffle for some $p\in\mathcal{I}$ if π preserves the order of each of the subsets $\{i\le p\}$ and $\{i>p+1\}$. The $(n+1)$-cycle $\lambda=\lambda_n=(n,n-1,\ldots,0)\in\{0,\ldots,n\}!$ is a particularly important 0-shuffle: it is used to define the cyclic cochain complex, $C_\lambda^\bullet(A)$, as a subcomplex of $C^{\bullet+1}(A,k)$. To be precise, $C^{n+1}(A,k)$ is a right $k[\{0,\ldots,n\}!]$-module (as above), a multilinear functional $f\in C^{n+1}(A,k)$ is called a *cyclic n-cochain* if $f\lambda=(-1)^\lambda f=(-1)^n f$, and $C_\lambda^n(A)$ is the sub-k-module consisting of such cochains. The restriction of I to $C_\lambda^\bullet(A)$ is then a cochain map $I:C_\lambda^\bullet(A)\to C^\bullet(A,A^*)$.

The evident map $\{1,\ldots,n\}!\to\{0,\ldots,n\}!$ – which regards each permutation of $\{1,\ldots,n\}$ as a permutation of $\{0,\ldots,n\}$ fixing 0 – is a monomorphism. Using this map, $C^{n+1}(A,k)$ becomes a right $k[\{1,\dots,n\}!]$-module in which $(f\pi)(a_0,a_1,\dots,a_n)=f(a_0,a_{\pi^{-1}1},\dots,a_{\pi^{-1}n})$. In particular, fs_n and $fe_n(r)$ are defined for any multilinear functional $f\in C^{n+1}(A,k)$ and the isomorphism $C^{n+1}(A,k)\to C^n(A,A^*)$ is a $k[\{1,\dots,n\}!]$-module map. Also, $b(fe_n(r))=(bf)e_{n+1}(r)$ since, as noted earlier, this identity holds in $C^\bullet(A,A^*)$. Unfortunately, $C^n_\lambda(A)$ is not a $k[\{1,\dots,n\}!]$-submodule of $C^{n+1}(A,k)$. Nonetheless, we have the following proposition, whose proof (being computational) will be deferred until Section 4:

Proposition 1. *If A is commutative and $f\in C^n_\lambda(A)$ is a cyclic cochain, then the multilinear functional $fe_n(r)$ is also a cyclic cochain.*

As before, if we set $C^{r,n-r}_\lambda(A)=C^n_\lambda(A)e_n(r)$, then a cyclic cochain f is in $C^{r,n-r}_\lambda(A)$ for $1\le r\le n$ if and only if $fe_n(r)=f$ or, equivalently, $fs_n=(2^r-2)f$. Moreover,

$$\begin{aligned} C^n_\lambda(A) &= \coprod_{r=0}^{n} C^{r,n-r}_\lambda(A) \\ &= C^{0,n}_\lambda(A)\oplus C^{1,n-1}_\lambda(A)\oplus\cdots\oplus C^{n,0}_\lambda(A), \end{aligned}$$

and $C^{r,\bullet-r}_\lambda(A)$ is a subcomplex of $C^\bullet_\lambda(A)$ whose homology is denoted $H^{r,\bullet-r}_\lambda(A)$. Of course, $C^{0,n}_\lambda(A)=H^{0,n}_\lambda(A)=0$ for $n\ne 0$ and $C^{r,t}_\lambda(A)=H^{r,t}_\lambda(A)=0$ if $t<0$. Summarizing, we have

Theorem 2. *If A is a commutative algebra, then its cyclic cohomology $H^\bullet_\lambda(A)$ decomposes as $H^\bullet_\lambda(A)=\coprod_{r\ge 0} H^{r,\bullet-r}_\lambda(A)$ where $H^{r,n-r}_\lambda(A)$ is the eigenspace for the eigenvalue 2^r-2 of the shuffle operator s_n whenever $1\le r\le n$. Moreover, the restriction of the induced map $I:H^\bullet_\lambda(A)\to H^\bullet(A,A^*)$ to $H^{r,\bullet-r}_\lambda(A)$ is a map $I_r:H^{r,\bullet-r}_\lambda(A)\to H^{r,\bullet-r}(A,A^*)$.* □

2. The periodicity sequence

The cohomology map I is part of a long exact 'periodicity' sequence

$$\cdots\longrightarrow H^n_\lambda(A)\xrightarrow{I} H^n(A,A^*)\xrightarrow{B} H^{n-1}_\lambda(A)\xrightarrow{S} H^{n+1}_\lambda(A)\longrightarrow\cdots$$

which will be described below. In this section we examine the restrictions of B and S to the individual summands of their domains and show that the periodicity sequence has a natural decomposition. To make our statements concise we again extend our definitions, this time by setting $H^{-1,t}_\lambda(A)=0$ for all t. The basic result is then

Theorem 3. *When A is commutative, the exact sequence above is a coproduct of long exact periodicity sequences*

$$\cdots \longrightarrow H_\lambda^{r,n-r}(A) \xrightarrow{I_r} H^{r,n-r}(A,A^*) \xrightarrow{B_r} H_\lambda^{r-1,n-r}(A)$$

$$\xrightarrow{S_{r-1}} H_\lambda^{r,n+1-r}(A) \longrightarrow \cdots$$

for $r \geq 0$ where I_r, B_r, and S_{r-1} are restrictions of I, B, and S, respectively.

The essential observation needed to prove the theorem is the following proposition whose proof (being computational) is deferred until Section 4, where we also recall the definition of the cochain map $B: C^\bullet(A,A^*) \to C_\lambda^{\bullet-1}(A)$.

Proposition 4. *If A is commutative and $f \in C^n(A,A^*)$ is a Hochschild cochain, then $B(fs_n) = 2(Bf)s_{n-1} + 2(Bf)$. In particular, if $f \in C^{r,n-r}(A,A^*)$, then $(Bf)s_{n-1} = (2^{r-1}-2)(Bf)$.*

Proof of Theorem 3. The proposition certainly implies that $B(C^{r,n-r}(A,A^*)) \subset C_\lambda^{r-1,n-r}(A)$ for all $r>1$. To see that the same is true when $r=1$, pick $f \in C^{1,n-1}(A,A^*)$. Then, since $(Bf)s_{n-1} = (1-2)Bf$ and -1 is not an eigenvalue of s_{n-1}, we see that $Bf=0$, as needed. At this point the maps I and B have been shown to have the appropriate properties: $I = \coprod I_r$ and $B = \coprod B_r$. To finish the proof we recall a description of S using B: consider the short exact sequence of complexes $\mathscr{E}: 0 \to C_\lambda^\bullet(A) \xrightarrow{I} C^\bullet(A,A^*) \to \mathrm{Cok}^\bullet I \to 0$ in which $\mathrm{Cok}^\bullet I$ is the cokernel of I. Then $BI=0$ and, so, B induces a cochain map $\bar{B}: \mathrm{Cok}^\bullet I \to C_\lambda^{\bullet-1}(A)$ which, in turn, induces *isomorphisms* $H^n(\bar{B}): H^n(\mathrm{Cok}^\bullet I) \to H_\lambda^{n-1}(A)$ [1]. The map $S: H_\lambda^{n-1}(A) \to H_\lambda^{n+1}(A)$ is just $\Delta^n \circ H^n(\bar{B})^{-1}$ where $\Delta^n: H^n(\mathrm{Cok}^\bullet I) \to H_\lambda^{n+1}(A)$ is the connecting homomorphism induced by $\mathscr{E}$. Now $\mathscr{E}$ is the coproduct of exact sequences $0 \to C_\lambda^{r,\bullet-r}(A) \to C^{r,\bullet-r}(A,A^*) \to \mathrm{Cok}^{\bullet-r} I_r \to 0$ (since $I = \coprod I_r$) and, so, Δ^n is the coproduct of the connecting homomorphisms $H^n(\mathrm{Cok}^{\bullet-r} I_r) \to H_\lambda^{r,n+1-r}(A)$. Moreover, since $B = \coprod B_r$, the induced map $H^n(\bar{B})$ is the coproduct of its restrictions $H^n(\mathrm{Cok}^{\bullet-r} I_r) \to H_\lambda^{r-1,n-r}(A)$. It follows immediately that the restriction of S to $H_\lambda^{r-1,n-r}(A)$ is, for each $r \geq 1$, a map $S_{r-1}: H_\lambda^{r-1,n-r}(A) \to H_\lambda^{r,n+1-r}(A)$. Thus, $S = \coprod S_{r-1}$ and the theorem follows – the exactness being a trivial consequence of the decomposition. □

These sequences have the same computational utility as the one first mentioned. Suppose, for example, that A is an algebra for which $H^\bullet(A,A^*) = H^{\bullet,0}(A,A^*)$. (As shown in [3], this occurs when A is smooth, in which case it follows from the Hochschild–Kostant–Rosenberg computation of $H^\bullet(A,-)$ in [4]. An examination of the periodicity sequences then shows $S_{r-1}: H_\lambda^{r-1,t-1}(A) \to H_\lambda^{r,t}(A)$ to be an isomorphism for $t>1$. It follows that $H_\lambda^{r,t}(A) \cong H_\lambda^{r-t+1,1}(A)$ if $r \geq t \geq 1$ while $H_\lambda^{r,t}(A)=0$ if $t>r \geq 1$, and, so,

$$H_\lambda^n(A) \cong H_\lambda^{n,0}(A) \oplus \coprod_{n>r\geq(n/2)} H_\lambda^{2r+1-n,1}(A)$$

whenever $H^\bullet(A,A^*)=H^{\bullet,0}(A,A^*)$. This decomposition of $H_\lambda^n(A)$ is, as we shall see in the next section, the one mentioned in the introduction. The summands of $H_\lambda^n(A)$ have a nice interpretation in terms of the differential operator $I\circ B$ defined on $H^\bullet(A,A^*)$. To describe it, note first that for each p the sequence

$$0 \longrightarrow H_\lambda^{p,0}(A) \xrightarrow{I_p} H^p(A,A^*) \xrightarrow{B} H_\lambda^{p-1,0}(A)$$
$$\xrightarrow{S_{p-1}} H_\lambda^{p,1}(A) \longrightarrow 0$$

is a segment of a periodicity sequence and, so, is exact. Also, $I\circ B: H^p(A,A^*)\to H^{p-1}(A,A^*)$ coincides with $I_{p-1}\circ B$. Then, since $I_\bullet: H_\lambda^{\bullet,0}(A)\to H^\bullet(A,A^*)$ is a monomorphism, we have

Corollary 5. *Let A be a commutative algebra for which $H^\bullet(A,A^*)=H^{\bullet,0}(A,A^*)$. Then $H_\lambda^n(A)$ has a decomposition as $H_\lambda^n(A)\cong H_\lambda^{n,0}(A)\oplus \coprod_{n>r\geq(n/2)} H_\lambda^{2r+1-n,1}(A)$. Also, $H_\lambda^{p,0}(A)$ is the kernel of $I\circ B=I_{p-1}\circ B: H^p(A,A^*)\to H^{p-1}(A,A^*)$ for each $p\geq 0$, while $H_\lambda^{p,1}(A)$ is isomorphic to the quotient group* Kernel$(I_{p-2}\circ B)$/Image$(I_{p-1}\circ B)$, *which is the $(p-1)$th homology group of the differential operator $I\circ B$.* □

Theorem 3 also reveals an interesting relationship between cyclic cohomology and commutative algebra cohomology. There are various formulations of the latter, all of which are known to coincide when k contains $\mathbb{Q}$ (as we always assume). The simplest is due to Harrison, who – in different language – defined the cohomology of a commutative algebra A with coefficients in a symmetric bimodule V for $n>0$ to be $\mathrm{Har}^n(A,V)=H^{1,n-1}(A,V)$ (cf., [2]). Since, $H_\lambda^{0,n}(A)=0$ when $n>0$, the periodicity sequence for $r=1$ yields

Corollary 6. *Let A be a commutative algebra. Then the map $I_1: H_\lambda^{1,n-1}(A)\to H^{1,n-1}(A,A^*)=\mathrm{Har}^n(A,A^*)$ is an isomorphism for $n\geq 3$, an epimorphism for $n=2$, and a monomorphism for $n=1$. In particular, the Harrison cohomology* $\mathrm{Har}^n(A,A^*)$ *is a direct summand of the cyclic cohomology $H_\lambda^n(A)$ for $n\geq 3$.* □

As a second easy corollary of Theorem 3 we obtain a decomposition of the stabilized cyclic cohomology groups, $H^{\mathrm{even}}(A)$ and $H^{\mathrm{odd}}(A)$, which are defined to be the colimits of $H_\lambda^2(A)\xrightarrow{S} H_\lambda^4(A)\xrightarrow{S}\cdots$ and $H_\lambda^1(A)\xrightarrow{S} H_\lambda^3(A)\xrightarrow{S}\cdots$. Specifically, for each $r\in\mathbb{Z}$ let $H^r(A)$ be the colimit of the system $H_\lambda^{r,0}(A)\xrightarrow{S_r} H_\lambda^{r+1,1}(A)\xrightarrow{S_{r+1}}\cdots$. (We have tacitly set $H_\lambda^{r,\bullet}(A)=0$ for $r<0$, so that $H^r(A)$ is the colimit of $H_\lambda^{1,1-r}(A)\xrightarrow{S_1} H_\lambda^{2,2-r}(A)\xrightarrow{S_2}\cdots$ when $r<0$.) Theorem 3 immediately yields

Corollary 7. *When A is commutative, the stabilized cyclic cohomology decomposes as $H^{\mathrm{even}}(A)=\coprod H^{2r}(A)$ and $H^{\mathrm{odd}}(A)=\coprod H^{2r+1}(A)$. If, further, $H^\bullet(A,A^*)=$*

$H^{\bullet,0}(A,A^*)$, then $H^r(A)\cong H_\lambda^{r+1,1}(A)$ for $r\geq 0$ and $H^r(A)=0$ for $r<0$. This occurs, in particular, when A is smooth. □

3. Connes' decomposition

The arguments in the preceding sections apply equally well to the continuous cyclic cohomology of an arbitrary commutative locally convex algebra. We shall consider the particular case $A=C^\infty(\mathscr{X})$, the Fréchet algebra of C^∞-functions on a smooth compact manifold $\mathscr{X}$. Connes has calculated the continuous Hochschild cohomology, $H^\bullet(A,A^*)_{\text{cont}}$, of this algebra. [1, Lemma 45]. Specifically, let $D_p(\mathscr{X})$ be the space of de Rham p-currents on $\mathscr{X}$. There is a map $D_p(\mathscr{X})\to H^p(A,A^*)_{\text{cont}}$ which associates to each current C the cohomology class of the p-cocycle φ_C defined by $\varphi_C(f_0,f_1,\ldots,f_p)=\langle C, f_0\,\mathrm{d}f_1\wedge\cdots\wedge\mathrm{d}f_p\rangle$. Connes shows that this map is an isomorphism and that it transforms the de Rham boundary for currents into the differential operator $I\circ B: H^p(A,A^*)_{\text{cont}}\to H^{p-1}(A,A^*)_{\text{cont}}$. Now, it is immediate that $\varphi_C\pi=(-1)^\pi\varphi_C$ for every permutation $\pi\in\{1,\ldots,p\}!$, so that - there being 2^p-2 shuffles - $\varphi_C s_p=(2^p-2)\varphi_C$. Thus, $\varphi_C\in C^{p,0}(A,A^*)_{\text{cont}}$ and $H^\bullet(A,A^*)_{\text{cont}}=H^{\bullet,0}(A,A^*)_{\text{cont}}$. These comments, combined with Corollaries 5 and 7, yield

Theorem 8 (Connes' decomposition). *Let $\mathscr{X}$ be a smooth compact manifold. Then $H_\lambda^{p,0}(C^\infty(\mathscr{X}))_{\text{cont}}=Z_p(\mathscr{X})$, the space of closed p-currents on $\mathscr{X}$, and $H_\lambda^{p,1}(C^\infty(\mathscr{X}))_{\text{cont}}\cong H_{p-1}(\mathscr{X},\mathbb{C})$, a de Rham homology group; $H_\lambda^n(C^\infty(\mathscr{X}))_{\text{cont}}$ decomposes as $H_\lambda^n(C^\infty(\mathscr{X}))_{\text{cont}}=Z_n(\mathscr{X})\oplus H_{n-2}(\mathscr{X},\mathbb{C})\oplus H_{n-4}(\mathscr{X},\mathbb{C})\oplus\cdots$. Also, the stabilized continuous cyclic cohomology is the de Rham homology: $H^r(C^\infty(\mathscr{X}))_{\text{cont}}\cong H_r(\mathscr{X},\mathbb{C})$ for $r\geq 0$ and otherwise vanishes.* □

Similar interpretations can be given for the decomposition of the (ordinary) cyclic cohomology and homology of a smooth algebra. As above, they require only the identification of $I\circ B$ with a suitable de Rham boundary or coboundary.

4. Deferred proofs

We conclude this paper by proving Propositions 1 and 4. For the first of these, let f be a cyclic n-cochain. We must show that $fe_n(r)$ is again a cyclic cochain. Now, $e_n(r)$ is defined as a polynomial in the shuffle operator, s_n. It therefore suffices to show that fs_n is a cyclic cochain, i.e. $fs_n\lambda=(-1)^n fs_n$ where $\lambda=\lambda_n=(n,\ldots,0)$. This will be a consequence of the following lemma:

Lemma 9. *For each $\pi\in\{0,\ldots,n\}!$ let $\hat\pi=\lambda^{\pi(n)}\pi\lambda$ where $\lambda=\lambda_n=(n,\ldots,0)$. Then $\hat\pi$ fixes 0 and is in the image of the inclusion map $\{1,\ldots,n\}!\to\{0,\ldots,n\}!$. Further, if π is a shuffle of $\{0,\ldots,n\}$, then $\hat\pi$ is a shuffle which fixes 0. The restriction of* *the function* $\pi \mapsto \hat{\pi}$ *to* $\{1,\dots,n\}!$ *is a bijection* $\{1,\dots,n\}! \to \{1,\dots,n\}!$ *which carries shuffles to shuffles.*

Proof. The final assertion will follow from the others if we show that $\{1,\dots,n\}! \to \{0,\dots,n\}! : \pi \mapsto \hat{\pi}$ is an injection. This is trivial: if $\hat{\pi}=\hat{\sigma}$ then $\lambda^{\pi(n)-\sigma(n)} = \sigma\pi^{-1} \in \{1,\dots,n\}!$ and, so, $\sigma\pi^{-1}=\mathrm{Id}$. For the other assertions, note that $\lambda(j) = j-1$ for $j\neq 0$ and $\lambda(0)=n$. It follows immediately that $\hat{\pi}(0)=0$ and, equivalently, $\hat{\pi}\in\{1,\dots,n\}!$. Moreover, if $j\neq 0$, then $\hat{\pi}(j)=\pi(j-1)-\pi(n)$ when $\pi(j-1)\geq\pi(n)$ and $\hat{\pi}(j)=\pi(j-1)-\pi(n)+n+1$ otherwise.

Now let π be a p-shuffle of $\{0,\dots,n\}$ for some $p\geq 0$. Then either $\pi(n)=n$ or $\pi(p)=n$. In the first case $\pi(j-1)<\pi(n)$ for all $j\neq 0$ and, so, $\hat{\pi}(j)=\pi(j-1)+1$ for $j\neq 0$. In particular, $\hat{\pi}(0)=0<\hat{\pi}(1)<\cdots<\hat{\pi}(p+1)$ and $\hat{\pi}(p+2)<\cdots<\hat{\pi}(n)$; that is, $\hat{\pi}$ is a $(p+1)$-shuffle. In the second case – namely, $\pi(p)=n$ – there is some $r<p$ for which $\pi(r-1)<\pi(n)$ and $\pi(r)>\pi(n)$. Now, for $j\leq r$ and $j\geq p+2$ we have $\hat{\pi}(j)=\pi(j-1)-\pi(n)+n+1$ while if $r+1\leq j\leq p+1$, then $\hat{\pi}(j)=\pi(j-1)-\pi(n)$. This implies that $\hat{\pi}$ preserves the order of each of the three sets $\{0,\dots,r\}$, $\{r+1,\dots,p+1\}$, and $\{p+2,\dots,n\}$. Finally, it is trivial that $\hat{\pi}(p+1)=n-\pi(n)< \pi(p+1)-\pi(n)+n+1=\hat{\pi}(p+2)$. Hence, $\hat{\pi}$ is an r-shuffle. □

Proof of Proposition 1. Let $\mathrm{Sh}\{1,\dots,n\}\to\mathrm{Sh}\{1,\dots,n\}$ be the function defined by sending each copy of the identity permutation to itself and each nonidentity shuffle π to $\hat{\pi}$. Since $(\mathrm{Id})\hat{}=\mathrm{Id}$, this function is a bijection and we may safely denote it by $\pi\mapsto\hat{\pi}$. Now observe that if f is a cyclic cochain, then $f\pi\lambda=f\lambda^{-\pi(n)}\hat{\pi}=(-1)^{n\pi(n)}f\hat{\pi}$. Since the sign of $\hat{\pi}$ is $(-1)^{\hat{\pi}}=(-1)^{n\pi(n)}(-1)^{\pi}(-1)^{n}$, it follows that $fs_n\lambda= (-1)^n\sum_{\pi\in\mathrm{Sh}\{1,\dots,n\}}(-1)^{\hat{\pi}}f\hat{\pi}$. But the latter is just $(-1)^n fs_n$, as $\pi\mapsto\hat{\pi}$ is a bijection. □

We turn now to the proof of Proposition 4, beginning with the definition of $C^\bullet(A,A^*)\xrightarrow{B}C_\lambda^{\bullet-1}(A)$: for $f\in C^n(A,A^*)$, the cyclic $(n-1)$-cochain Bf is given by

$$(Bf)(a_1,\dots,a_n)=f(1-(-1)^n\lambda_n)\left(\sum_{j=0}^{n-1}(-1)^{(n-1)j}\lambda_{n-1}^j\right)(1,a_1,\dots,a_n)$$

where $\lambda_{n-1}=(n,\dots,1)$ and $\lambda_n=(n,\dots,0)$. Note that we are indexing the arguments of Bf by $\{1,\dots,n\}$ rather than $\{0,\dots,n-1\}$. Consequently, the shuffle operator s_{n-1} used to decompose $C_\lambda^{n-1}(A)$ shuffles the set $\{2,\dots,n\}$ rather than $\{1,\dots,n-1\}$ and lies in $k[\{2,\dots,n\}!]$. Now the identity we wish to prove is $B(fs_n)=(Bf)(2s_{n-1}+2)$. It clearly suffices, therefore, to show that

$$s_n\Lambda=\Lambda(2s_{n-1}+2) \tag{1}$$

and

$$s_n\lambda_n\Lambda=\lambda_n\Lambda(2s_{n-1}+2). \tag{2}$$

where $\Lambda=\sum_{j=0}^{n-1}(-1)^{(n-1)j}\lambda_{n-1}^j$. As the proofs of these two identities are virtually the same, we consider only the first and henceforth write λ for λ_{n-1}. Expanding the left-hand side of (1) gives

$$s_n\Lambda = \sum_{\pi\in \mathrm{Sh}\{1,\dots,n\}} \sum_{j=0}^{n-1} (-1)^{\pi}(-1)^{(n-1)j}\pi\lambda^j$$
$$= \sum_{r=0}^{n-1} \sum_{(\pi,j)\in E_r} (-1)^{\pi}(-1)^{(n-1)j}\pi\lambda^j,$$

where E_r consists of pairs $(\pi,j)\in(\mathrm{Sh}\{1,\dots,n\})\times\{0,\dots,n-1\}$ satisfying $\pi\lambda^j(1)=r+1$ or, equivalently, $\pi(n-j+1)=r+1$. We shall examine the second of these summations.

First, Lemma 9 may be rephrased as follows: if π is an arbitrary shuffle of $\{0,\dots,n\}$, then there is an r such that $\lambda_n^r\pi\lambda_n$ is a shuffle of $\{0,\dots,n\}$ which fixes 0, namely $r=\pi(n)$. Shifting the indexing and iterating we find: if π is a shuffle of $\{1,\dots,n\}$ and $0\le j\le n-1$, then there is an r such that $\lambda^r\pi\lambda^j$ is a shuffle of $\{1,\dots,n\}$ which fixes 1 and $0\le r\le n-1$. Note that r is then uniquely determined by the pair (π,j): since $\pi\lambda^j(1)=\pi(n-j+1)$ we must have $r=\pi(n-j+1)-1$, which means that $(\pi,j)\in E_r$. Thus, for each r with $0\le r\le n-1$, the image of the function

$$E_r\to\{\sigma\in\{1,\dots,n\}!\mid \sigma(1)=1\}:(\pi,j)\mapsto\overline{(\pi,j)}=\lambda^r\pi\lambda^j$$

is contained in the shuffles. Also, the formula above implies that

$$s_n\Lambda = \sum_{r=0}^{n-1}(-1)^{-r(n-1)}\lambda^{-r}\sum_{(\pi,j)\in E_r}(-1)^{\overline{(\pi,j)}}\,\overline{(\pi,j)}.$$

Thus, (1) will follow if we show that

$$\sum_{(\pi,j)\in E_r}(-1)^{\overline{(\pi,j)}}\,\overline{(\pi,j)} = 2s_{n-1}+2 \tag{3}$$

for each r. Before proving this we gather a few observations in a lemma.

Lemma 10. *Fix an r with $0\le r\le n-1$. If $\overline{(\pi,j)}=\mathrm{Id}$, then either $j\ne n-r$ and $\pi=\lambda^{-(r+j)}$ or $j=n-r$ and π is one of the $n-1$ copies of the identity permutation in $\mathrm{Sh}\{1,\dots,n\}$. If π is a p-shuffle and $\overline{(\pi',j')}=\overline{(\pi,j)}\ne\mathrm{Id}$, then either $(\pi',j')=(\pi,j)$ or $\pi'=\pi\lambda^{n-p}$; in the second case $j'=p+j$ if $p+j<n$ and $j'=p+j-n$ otherwise. If σ is a shuffle of $\{1,\dots,n\}$ which fixes 1, then $\sigma=\overline{(\pi,j)}$ for some $(\pi,j)\in E_r$.*

Proof. The first statement is trivial. The second then easily reduces to the assertion: If π is a p-shuffle, $0<m<n$, and $\pi\ne\lambda^t$ for any t, then $\pi\lambda^m$ is a shuffle if and only if $m=n-p$. For this, suppose first that $0<m<n-p$, set $\tilde\pi=\pi\lambda^m$, and note that $\tilde\pi(i)=\pi(i-m+n)$ for $i\le m$ while $\tilde\pi(i)=\pi(i-m)$ for $i>m$. It is then immediate that $\tilde\pi$ preserves the order of each of the three sets $\{1,\dots,m\}$, $\{m+1,\dots,m+p\}$, and $\{m+p+1,\dots,n\}$. Thus, it is not a shuffle unless either $\tilde\pi(m)<\tilde\pi(m+1)$ or $\tilde\pi(m+p)<\tilde\pi(m+p+1)$. The first of these conditions means that $\pi(n)<\pi(1)$ which, of course, implies that $\pi=\lambda^p$; the second says $\pi(p)<\pi(p+1)$ and, so, forces π to be the identity. As these possibilities for π have been excluded, $\tilde{\pi}=\pi\lambda^m$ is not a shuffle. A similar argument applies when $m>n-p$. On the other hand, $\pi\lambda^{n-p}$ is readily seen to be an $(n-p)$-shuffle.

For the final assertion, let σ be a shuffle which fixes 1 and choose m to be minimal with the property that $\sigma(m)\leq n-r$ but $\sigma(m+1)>n-r$. Then it is easy to check that $\sigma'=\lambda^{-r}\sigma\lambda^m$ is a shuffle, $(\sigma', n-m)\in E_r$, and $\overline{(\sigma', n-m)}=\sigma$. □

Finally, let $\mathscr{S}\subset \mathrm{Sh}\{1,\ldots,n\}$ be the subset consisting of those shuffles which fix 1. If $\sigma\in\mathscr{S}$ and $\sigma\neq\mathrm{Id}$, then, according to the lemma, $\sigma=\overline{(\pi,j)}$ for *exactly* two choices of $(\pi,j)\in E_r$. Likewise, there are $n-1$ copies of the identity permutation in $\mathscr{S}$ and $2(n-1)$ elements $(\pi,j)\in E_r$ for which $\overline{(\pi,j)}=\mathrm{Id}$. The left-hand side of (3) thus reduces to $2\sum_{\sigma\in\mathscr{S}}(-1)^\sigma\sigma$. On the other hand, $\mathrm{Sh}\{2,\ldots,n\}\subset\mathscr{S}$ and if $p\geq 2$, then every p-shuffle in $\mathscr{S}$ is, in fact, a shuffle of $\{2,\ldots,n\}$. Since the only 1-shuffle in $\mathscr{S}$ is a copy of the identity, we see that $\sum_{\sigma\in\mathscr{S}}(-1)^\sigma\sigma=s_{n-1}+1$. Equation (3) now follows, thereby finishing our proof of Proposition 4.

References

[1] A. Connes, Non-commutative differential geometry, Parts I and II, Publ. I.H.E.S. 62 (1985) 257–360.

[2] M. Gerstenhaber and S.D. Schack, A Hodge-type decomposition for commutative algebra cohomology, J. Pure Appl. Algebra 48 (1987) 229–247.

[3] M. Gerstenhaber and S.D. Schack, Algebraic cohomology and deformation theory, Proc. NATO-ASI Conference on "Deformation Theory and Applications", Castelvecchio-Pascoli, Italy, 1986 (Reidel, Dordrecht).

[4] G. Hochschild, B. Kostant, and A. Rosenberg, Differential forms on regular affine algebras, Trans. Amer. Math. Soc. 102 (1962) 383–408.

[5] J.-L. Loday and D. Quillen, Cyclic homology and the Lie algebra homology of matrices, Comment. Math. Helvetica 59 (1984) 565–591.