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A GEOMETRIC INTERPRETATION OF ONE-DIMENSIONAL QUASINORMAL RINGS

Luciana RAMELLA
Istituto Matematico Università, via L.B. Alberti 4, Genova, Italy

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Introduction

A commutative noetherian reduced ring A is said to be *seminormal* (SN) [resp. *quasinormal* (QN)] if its integral closure is a finite A-module and the canonical homomorphism of the Picard groups $\mathrm{Pic}\, A \to \mathrm{Pic}\, A[T]$ [resp. $\mathrm{Pic}\, A \to \mathrm{Pic}\, A[T, T^{-1}]$] is an isomorphism.

In what follows, A is a reduced, 1-dimensional noetherian ring with integral closure $\bar{A}$, a finite A-module.

We define $\nu(A) = h_0(\bar{A}/\mathfrak{c}) - h_0(A/\mathfrak{c}) - h_0(\bar{A}) + h_0(A)$, where $\mathfrak{c}$ is the conductor of A in $\bar{A}$ and $h_0(A)$ denotes the number of the connected components of Spec(A).

Bass and Murty proved in [2] the following

Theorem 1. (a) $\mathrm{Pic}\, A[T] \cong \mathrm{Pic}\, A \oplus X$; $X = 0$ *if and only if* $\sqrt[\bar{A}]{\mathfrak{c}} = \mathfrak{c}$.

(b) $\mathrm{Pic}\, A[T, T^{-1}] \cong \mathrm{Pic}\, A \oplus X \oplus Y$ *and* Y *is a free abelian group whose rank is* $\nu(A)$.

As a consequence of Theorem 1 we have:

Theorem 2. A *is* QN *if and only if* A *is* SN *and* $\nu(A) = 0$.

In Section 1, by using combinatorial methods, we give a geometric interpretation of $\nu(A)$. This allows a geometric description of a quasinormal ring of dimension 1. Thus we get an affirmative answer to a problem posed by Greco in [5].

In Section 2, we study relations between the numbers $\nu(A)$ and $\nu(B)$, where B is a finite étale A-algebra. We show that quasinormality is not so well behaved as seminormality.

Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ be the minimal primes of A. Let $A_i = A/\mathfrak{p}_i$ and assume $\dim A_i = 1$ for every $i = 1, \ldots, n$. The affine curves $X_i = \mathrm{Spec}(A_i)$, for $i = 1, \ldots, n$, are the irreducible components of the curve $X = \mathrm{Spec}(A)$.

$\bar{X} = \mathrm{Spec}(\bar{A})$ is the normalization of X; we have $\bar{X} = \coprod_{i=1}^{n} \bar{X}_i$, since $\bar{A} = \bigoplus_{i=1}^{n} \bar{A}_i$.

Let $P_1, \dots, P_m$ be the singular points of X and let $x_1, \dots, x_M$ be the branches of X ($x \in \bar{X}$ is a branch of X over P if $x \in \phi^{-1}(P)$, where $\phi : \bar{X} \to X$ is the canonical morphism and P is a singular point of X).

We have $n = h_0(\bar{A})$, $m = h_0(A/\mathfrak{c})$ and $M = h_0(\bar{A}/\mathfrak{c})$.

1. Incidence cycles of a curve

In this section, we give a geometric interpretation of the number $\nu(A)$, by using combinatorial methods concerning graph theory. Similar methods are also applied by Roberts in [11] and [12] to calculate the Picard group of a class of curves.

We associated to the curve $X = \mathrm{Spec}(A)$ the matrices $\beta = (\beta_{ij}) \in Z_2^{M \times m}$ and $\gamma = (\gamma_{ij}) \in Z_2^{M \times n}$, where β_{ij} is 1 if $\phi(x_i) = P_j$ and 0 otherwise, γ_{ij} is 1 if $x_i \in \bar{X}_j$ and 0 otherwise.

We say that $\alpha = (\beta, \gamma)$ is the *matrix associated* to X (or to A).

Proposition 1.1. *Let α be the matrix associated to the curve X, let $\varrho(\alpha)$ be the rank of the matrix α, we have $\varrho(\alpha) < m + n$. If the curve X is connected, then $\varrho(\alpha) = m + n - 1$.*

Proof. Every row of the matrix $\alpha = (\beta, \gamma)$ contains two 1's, one in β and one in γ. Then the sum of columns of β equals the sum of the columns of γ and we have $\varrho(\alpha) < m + n$. If X is connected it is easily shown that the above column relation is the only one, up to scalar factors.

We associated to the curve $X = \mathrm{Spec}(A)$ the graph Γ whose vertices are $P_1, \dots, P_m, X_1, \dots, X_n$ and whose edges represent the M branches of X in this way: if x_r is a branch over P_i and $x_r \in \bar{X}_j$ we construct an edge by joining P_i and X_j.

Observe that several edges may join two vertices.

The graph Γ just constructed is said to be the *graph associated* to X (or to A).

Let α be the matrix and let Γ be the graph associated to X, then α is the incidence matrix of the graph Γ.

It is known that (see e.g. [3]):

(a) The number $\nu(\Gamma)$ of independent cycles of the graph Γ is equal to the number of the edges of Γ minus the rank of the matrix α.

(b) The rank of α is equal to the number of the edges of Γ minus the number of the connected components of Γ.

One can prove the following

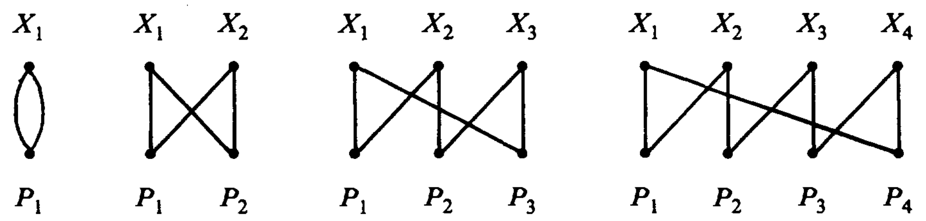
Proposition 1.2. *Let Γ be the graph associated to the curve $X = \mathrm{Spec}(A)$. If X is connected, then also the graph Γ is connected and we have $\nu(\Gamma) = M - m - n + 1$.*

Then we can conclude that $\nu(A)$ coincides with the number of independent cycles of the graph Γ associated to the curve $X = \mathrm{Spec}(A)$.

Theorem 1.3. *Let Γ be the graph associated to the curve X. Γ contains cycles if and only if X satisfies one of the following conditions:*

(a) *An irreducible component of X is not locally unibranche.*
(b) *Two irreducible components of X meet in more then one point.*
(c) *X contains polygons.*

Proof. An 'elementary' cycle of the graph Γ is a cycle of one of the following types



(1) (2) (3) (4)

The cycle $X_1P_1X_1$ of the case (1) represents two distinct branches belonging to $\bar{X}_1$ and lying over the same singular point P_1, therefore X_1 is not locally unibranche.

The cycle $X_1P_1X_2P_2X_1$ of the case (2) represents two distinct points belonging to the intersection of two irreducible components of X.

If $r \geq 3$, the cycle $X_1P_1 \cdots X_rP_rX_1$ represents a polygon of X with r vertices.

The following corollary gives and affirmative answer to a problem posed by Greco in [5, 4.12].

Corollary 1.4. *A is* QN *if and only if A is* SN *and the following conditions are satisfied:*

(i) *All irreducible components of $X = \mathrm{Spec}(A)$ are locally unibranche.*
(ii) *Two distinct irreducible components of X meet in at most one point.*
(iii) *X does not contain polygons.*

Proof. A is QN if and only if A is SN and $\nu(A) = 0$. Moreover $\nu(A) = \nu(\Gamma)$, where Γ is the graph associated to A, then the conclusion follows from Theorem 1.3.

Definition 1.5. Any cycle of the graph Γ associated to $X = \mathrm{Spec}(A)$ is said to be an *incidence cycle* of X (or of A)

Then an incidence cycle of X is a composition of cycles of the types (1), (2), (3), ... described above. Thus $\nu(A)$ represents the number of the independent incidence cycles of A.

We can restate Theorem 2 in the following way: "A is QN if and only if A is SN and A has no incidence cycle".

If A is an integral domain, A has no incidence cycle if and only if A is locally unibranche. Therefore we also get the following: "If A is an integral domain, A is QN if and only if A is SN and locally unibranche" (see [5]).

Now we recall the following results about seminormality (see e.g. [6]):

(i) If A is SN, then A_S is SN for every multiplicative subset S of A.

(ii) A is SN if and only if $A_{\mathfrak{m}}$ is SN for every maximal ideal $\mathfrak{m}$ of A.

(iii) If A is SN, any intermediate ring $A \subset B \subset \bar{A}$ is SN.

We will see that (i) and (iii) also hold for quasinormality, but (ii) may not.

Proposition 1.6. (a) *Let S be a multiplicative subset of A, then we have $\nu(A_S) \leq \nu(A)$.*

(b) *A_i is locally unibranche for every $i = 1, \dots, n$ if and only if $\nu(A_{\mathfrak{m}}) = 0$ for every maximal ideal $\mathfrak{m}$ of A.*

Proof. (a) Since $(\overline{A_S}) = \bar{A}_S$ and $\mathfrak{c}A_S$ is the conductor of A_S in $\bar{A}_S$, where $\mathfrak{c}$ is the conductor of A in $\bar{A}$, the graph Γ_{A_S} associated to A_S is a subgraph of the graph Γ_A associated to A.

(b) There is maximal ideal $\mathfrak{m}$ of A such that $\nu(A_{\mathfrak{m}}) \neq 0$ if and only if there is an irreducible component $X_i = \mathrm{Spec}(A_i)$ of $X = \mathrm{Spec}(A)$ with two distinct branches lying over a singular point P of X_i.

Corollary 1.7. (a) *Let S be a multiplicative subset of A. If A is* QN, *then A_S is* QN.

(b) *A is* SN *and A_i is locally unibranche for every $i = 1, \dots, n$ if and only if $A_{\mathfrak{m}}$ is* QN *for every maximal ideal $\mathfrak{m}$ of A.*

Notice that the ring $A = \mathbb{C}[X, Y]/(Y-1)(Y-X^2)$ is not QN, but $A_{\mathfrak{m}}$ is QN for every maximal ideal $\mathfrak{m}$ of A.

If A is an integral domain, then A is QN if and only if $A_{\mathfrak{m}}$ is QN for every maximal ideal $\mathfrak{m}$ of A (see [5]).

Proposition 1.8. *Let B be a subring of $\bar{A}$ containing A, then we have $\nu(B) \leq \nu(A)$.*

Proof. If the maximal ideals $\mathfrak{M}_1, \dots, \mathfrak{M}_r$ of B, the minimal prime ideals $\mathfrak{P}_1, \dots, \mathfrak{P}_r$ of B and the maximal ideals $M_1, \dots, M_{2r}$ of $\bar{B} = \bar{A}$ form an elementary cycle of the graph Γ_B associated to B, then $\mathfrak{M}_1 \cap A, \dots, \mathfrak{M}_r \cap A, \mathfrak{P}_1 \cap A, \dots, \mathfrak{P}_r \cap A, M_1, \dots, M_{2r}$ form a cycle of the graph Γ_A associated to A.

Corollary 1.9. (a) *If A is* Qn, *then any intermediate ring $A \subset B \subset \bar{A}$ is* QN.

(b) *If $\nu(A) = 0$, then the seminormalization of A is* QN.

2. Etale coverings and incidence cycles

We recall the following results (see [6]).

(a) Let $f: A \to B$ be a faithfully flat homomorphism. If B is SN, then A is SN.

(b) Let $f: A \to B$ be an étale homomorphism. If A is SN, then B is SN.

Let A and B be (1-dimensional) integral domains and let $f: A \to B$ be a faithfully flat homomorphism. If B is QN, then A is QN (see [5, 4.10]).

If B is not an integral domain, the above assertion is false. Indeed let $A = \mathbb{C}[X, Y]/(Y^2 - X^2(X-1))$ be the coordinate ring of the cubic node in $\mathbb{A}^2_{\mathbb{C}}$ and let $\mathfrak{m}$ be the maximal ideal of A corresponding to the singular point 0. Let $\hat{A}_{\mathfrak{m}}$ be the completion of the local ring $A_{\mathfrak{m}}$. Then the canonical homomorphism $f: A_{\mathfrak{m}} \to \hat{A}_{\mathfrak{m}}$ is a faithfully flat homomorphism, $\hat{A}_{\mathfrak{m}}$ is QN, $A_{\mathfrak{m}}$ is SN but it is not locally unibranche, hence $A_{\mathfrak{m}}$ is not QN.

Quasinormality is not preserved by étale extensions. The ring $A = \mathbb{R}[X, Y]_{(X,Y)}/(X^2 + Y^2)$ is QN, but there exists a local finite étale A-algebra B which is SN but not QN (see [5, 4.13]).

From now on the ring A will be assumed to be a finitely generated k-algebra, where k is an algebraically closed field.

In this section we study the incidence cycles of a finite étale A-algebra B, and therefoe the incidence cycles of an étale covering of $X = \operatorname{Spec}(A)$.

We simply say that a homomorphism $f: A \to B$ is an étale covering (of A) if the induced morphism $F: \operatorname{Spec}(B) \to \operatorname{Spec}(A)$ is an étale covering (of $\operatorname{Spec}(A)$).

Proposition 2.1. *Let $f: A \to B$ be an étale covering. Then we have $\nu(B) \geq \nu(A)$.*

Proof. We may assume that A and B are connected. Let q be the degree of f. Since the ground field k is algebraically closed, any fiber of the morphism $F: \operatorname{Spec}(B) \to \operatorname{Spec}(A)$ at a closed point consists of q points. Since f is an étale homomorphism, we have $\mathfrak{c}_A B = \mathfrak{c}_B$ and $\bar{B} = B \otimes_A \bar{A}$, where $\mathfrak{c}_A$ and $\mathfrak{c}_B$ denote the conductor of A in $\bar{A}$ and of B in $\bar{B}$ respectively. The induced homomorphisms $A/\mathfrak{c}_A \to B/\mathfrak{c}_B$, $\bar{A} \to \bar{B}$, $\bar{A}/\mathfrak{c}_A \to \bar{B}/\mathfrak{c}_B$ are étale coverings of degree q. Then $h_0(B/\mathfrak{c}_B) = q h_0(A/\mathfrak{c}_A)$, $h_0(\bar{B}/\mathfrak{c}_B) = q h_0(\bar{A}/\mathfrak{c}_A)$ and $h_0(\bar{B}) \leq q h_0(\bar{A})$. It follows that $\nu(B) - 1 \geq q(\nu(A) - 1)$ and we are done.

The following example shows that the case $\nu(B) > \nu(A)$ can occur.

Example 2.2. Let $C = \mathbb{C}[X, Y]/(Y^2 - X^3 + X)$ be the coordinate ring of the non singular cubic curve of $\mathbb{A}^2_{\mathbb{C}}$, let $\mathfrak{m}$ be a maximal ideal of C. Put $A' = C \oplus C$; $\mathfrak{m}_1 = \mathfrak{m} \oplus C$ and $\mathfrak{m}_2 = C \oplus \mathfrak{m}$ are maximal ideals of A'. Let A be the ring obtained from A' by glueing $\mathfrak{m}_1$ with $\mathfrak{m}_2$ (see [9]). A is a connected ring, A' is the integral closure of A, A has no incidence cycles. There exists an étale covering $h: C \to E$ of degree 2, with E a connected ring (see [10]). Let $\mathfrak{M}'$ and $\mathfrak{M}''$ be the two maximal ideals of E lying over $\mathfrak{m}$. Put $B' = E \oplus E$, let B the ring obtained from B' by glueing $\mathfrak{M}' \oplus E$ with $E \oplus \mathfrak{M}'$ and $\mathfrak{M}'' \oplus E$ with $E \oplus \mathfrak{M}''$. The étale covering h induces an étale covering $f: A \to B$ of degree 2 (see [14]). B is a connected ring and it has an incidence cycle. We have $\nu(A) = 0$ and $\nu(B) = 1$. Notice that A is QN, whereas B is SN but not QN.

The above example shows that quasinormality is not preserved by étale extensions, not even for finitely generated k-algebras, with k algebraically closed.

If the irreducible components of the curve $X=\mathrm{Spec}(A)$ are rational curves and if X has no incidence cycles, then any étale covering of X has no incidence cycles, but in this case it is a trivial covering.

Proposition 2.3. *Suppose that all irreducible components of* Spec(A) *are rational curves. Let $f: A \to B$ be an étale covering, with B a connected ring. Let q be the degree of f. Then $v(B)-1=q(v(A)-1)$.*

Proof. We have $h_0(\bar{B})=qh_0(\bar{A})$. In fact let $\mathfrak{P}$ be a minimal prime ideal of $\bar{A}$, we have $q=\dim_{\bar{A}_{\mathfrak{P}}} \bar{B}\otimes_{\bar{A}} \bar{A}_{\mathfrak{P}}$. $\bar{A}_{\mathfrak{P}}$ is the residue field of $\bar{A}$ at $\mathfrak{P}$, it is isomorphic to the quotient field of $\bar{A}/\mathfrak{P}$, which is $k(T)$ by assumption. We have $\bar{B}\otimes_{\bar{A}} \bar{A}_{\mathfrak{P}} \cong \bigoplus_{j=1}^{r} \bar{B}_{\mathfrak{P}_j}$, where $\{\mathfrak{P}_1, \ldots, \mathfrak{P}_r\}$ is the set of the minimal prime ideals of $\bar{B}$ lying over $\mathfrak{P}$. Since f is an étale homomorphism, $\bar{B}_{\mathfrak{P}_j}$ is a separable finite extension of $\bar{A} \cong k(T)$. Hence we have $\bar{B}_{\mathfrak{P}_j} \cong k(T)$ for all $i=1,\ldots,r$, because $k(T)$ has not a proper separable finite extension (see [7]). It follows that $r=q$ and then $h_0(\bar{B})=qh_0(\bar{A})$. By using the same methods of the proof of Proposition 2.1, the conclusion follows.

Corollary 2.4. *Suppose that all irreducible components of* Spec(A) *are rational curves. Let $f: A \to B$ be an étale covering. If $v(A)=0$, then $v(B)=0$ and B is a trivial covering.*

Proof. We may suppose that A and B are connected. Let q be the degree of f. We have $v(B)-1=q(v(A)-1)$; $v(A)=0$ implies $v(B)=1-q$, which is non-negative, then $q=1$.

In Proposition 2.3 and in Corollary 2.4, the assumption that the ground field k is algebraically closed cannot be omitted, as the following example shows.

Put $A=\mathbb{R}[T^2+1, T(T^2+1)]$. $B=\mathbb{C}[T^2+1, T(T^2+1)]$ is a finite étale a-algebra. We have $\bar{A}=\mathbb{R}[T]$, the conductor $\mathfrak{c}_A$ is a maximal ideal in A and in $\bar{A}$, hence $v(A)=0$. We have $\bar{B}=\mathbb{C}[T]$ and the conductor $\mathfrak{c}_B$ is a maximal ideal in B, but it is the product of two maximal ideals in $\bar{B}$, hence $v(B)=1$.

Corollary 2.5. *Let $f: A \to B$ be an étale covering.*

(a) *If B is* QN, *then A is* QN.

(b) *If the irreducible components of* Spec(A) *are rational curves, then A is* QN *if and only if B is* QN. *In this case B is a trivial covering.*

Proof. The homomorphism f is faithfully flat. Then A is SN if and only if B is SN. The conclusion follows from Proposition 2.1 and Corollary 2.4.

References

[1] M. Artin, On the join of Hensel rings, Advances in Math. 7 (1971) 282–296.

[2] H. Bass and P. Murty, Grothendieck groups and Picard groups of abelian group rings, Ann. of Math. 86 (1967) 16–73.

[3] C. Berge, Théorie des Graphes et ses Applications (Dunod, Paris, 1958).

[4] E. Davis, On the geometric interpretation of seminormality, Proc. AMS 68 (1978) 1–5.

[5] S. Greco, Seminormality and quasinormality of group rings, J. Pure Appl. Algebra 18 (1980) 129–142.

[6] S. Greco and C. Traverso, On seminormal schemes, Compositio Math. 40 (1980) 325–365.

[7] A. Grothendieck, Revetements Etale et Groupe Fondamental (SGA I), Lecture Notes in Math. 224 (Springer, Berlin–New York, 1971).

[8] A. Grothendieck and J. Dieudonné, Elements de Géométrie Algébrique, IV (Publ. Math. Paris 1964/67).

[9] C. Pedrini, Incollamenti di ideali primi e gruppi di Picard, Rend. Sem. Mat. Univ. Padova 48 (1973) 39–66.

[10] M. Raynaud, Annaux Locaux Henseliens, Lectures Notes in Math. 169 (Springer, Berlin–New York, 1970).

[11] L. Roberts, The K-theory of some reducible affine varieties, J. Algebra 35 (1975) 516–527.

[12] L. Roberts, The K-theory of some reducible affine curves: A combinatorial approach, in: Algebraic K-Theory, Lectures Notes in Math. 551 (Springer, Berlin–New York, 1976).

[13] C. Traverso, Seminormality and Picard group, Ann. Sc. Norm. Sup. Pisa 24 (1970) 585–595.

[14] A. Vistoli, Incollamento di punti chiusi e gruppo fondamentale algebrico e topologico, Rend. Sem. Mat. Univ. Padova, to appear.