Journal of Pure and Applied Algebra 29 (1983) 129–141
North-Holland

A GENERAL FORMULATION OF HOMOTOPY LIMITS

Dominique BOURN and Jean-Marc CORDIER
U.E.R. de Mathématiques, Université de Picardie, 35 rue St. Leu, 80039 Amiens, France

Communicated by A. Heller
Received 5 January 1982
Revised 11 August 1982

The problem of homotopy coherence has occurred recently in two contexts: explicitly in strong shape theory (Edwards–Hastings [11], Dydak–Segal [10] etc.) and implicitly in the simplicial localization of Dwyer–Kan [9]. In both cases, the authors take into consideration the notion of higher order homotopies (or "homotopy coherences").

One of the first places in which these higher order homotopy coherences have been used is in the study of homotopy limits (in a restricted sense in Bousfield–Kan [6] and Edwards–Hastings [11] and more generally Vogt [19] and Porter [15]). These homotopy limits then appear naturally in strong shape theory.

Coming from another direction and following an article of Thomason [18] which sheds light on possible relations between lax limits, and homotopy limits, Gray [13] has introduced a generalization of homotopy limits (in the sense of Bousfield–Kan – the precise definition will be given later). This latter definition has two imperfections; it cuts off the coherence at level 2 and it does not allow the generalization of the replacement schemes necessary for the development of the analogues of the Bousfield–Kan spectral sequences. Our own work in shape theory [5] and coherence [4, 7, 8] has led us to study a fresh definition which remedies these defects. This general definition is not however entirely new as a particular case of it already appears in Segal's paper [16].

Like Gray we feel that the best presentations of homotopy limits are made in terms of indexed limits. However for us, the natural context for these indexed limits is that of profunctors (sometimes also called distributors) [14, 1] (it is in these terms, in fact, that the replacement scheme seems most natural). The first section recalls some necessary facts about Bousfield–Kan homotopy limits and indexed limits. From a careful inspection of the coherence of a homotopy cone, we introduce in Section 2 a general notion of homotopy limits for a simplicial category. We next show that the replacement scheme [6] holds in this situation and exhibit an indexing for this notion of limit. We give general conditions of existence and study the cases of the two important simplicial categories Cat and Top. In particular we show that lax limits and a construction of Segal are particular cases of homotopy limits.

In Section 3, we compare this notion with the notion of Gray.

We would like to thank the referee for his helpful comments on an earlier version of this note.

1. The Bousfield–Kan homotopy limits and the simplicial enriched profunctors

Let us recall the notion of B–K-homotopy limit in order to illustrate what we mean.

Let Δ be the category of the ordered sets $[n]=\{0,1,\dots,n\}$, $n\in\mathbb{N}$, and non decreasing maps. Let $\mathcal{S}=\mathrm{Set}^{\Delta^{\mathrm{op}}}$ be the category of simplicial sets, which is cartesian closed like every presheaf category.

Let I be an ordinary small category, X and Y two functors $:I\to\mathcal{S}$. Then Bousfield–Kan [6] define $\mathrm{Hom}(X,Y)$ as the kernel (often denoted by the formula $\int_i [X_i, Y_i]$) of the following diagram:

$$\prod_{i\in I}[X_i,Y_i] \overset{a}{\underset{b}{\rightrightarrows}} \prod_{i\to i'\in I}[X_i,Y_{i'}]$$

where $[S,S']$, for any $S,S'\in\mathcal{S}$, is the function space that is the enriched hom in $\mathcal{S}$, and a and b are respectively induced by:

$$[X_i,Y_i]\xrightarrow{Y_j}[X_i,Y_{i'}],\qquad [X_{i'},Y_{i'}]\xrightarrow{X_j}[X_i,Y_{i'}].$$

Therefore this $\mathrm{Hom}(X,Y)$ is nothing but the enriched $\mathrm{Nat}(X,Y)$ in $\mathcal{S}$, where I,X,Y are trivially enriched in $\mathcal{S}$ (where $I(i,i')$ is considered as a constant simplicial set).

There is a canonical functor $I/-:I\to\mathrm{Cat}$ which associates the category I/i over i to each object i of I. A category C being a particular simplicial set (often noted $\mathrm{Ner}\,C$, but here we shall forget, as in [6] the notation Ner), the functor $I/-$ can be considered $:I\to\mathcal{S}$.

Definition (Bousfield–Kan). Let $F:I\to\mathcal{S}$ be a functor, then

$$\mathrm{holim}\,F=\mathrm{Hom}(I/-,F)=\int_i [I/i, Fi].$$

Let us show that this notion of limit is a particular case of the notion of indexed limit in the context of enriched categories in the category of simplicial sets $\mathcal{S}$ [13].

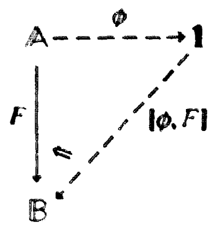
Let $\mathbb{V}$ be a symmetric monoidal closed category, complete and cocomplete. Let $\mathbf{1}$ be the $\mathbb{V}$-category with only one object $*$ and such that $\mathbf{1}(*,*)$ is the unit object of $\mathbb{V}$. If $\mathbb{A}$ is a $\mathbb{V}$-category, there is no more a canonical $\mathbb{V}$-functor: $\mathbb{A}\to\mathbf{1}$ and so no canonical notion of limit. Let $\phi:\mathbb{A}\dashrightarrow\mathbf{1}$ a $\mathbb{V}$-profunctor (i.e. a $\mathbb{V}$-functor $:\mathbb{A}\to\mathbb{V}$) and $F:\mathbb{A}\to\mathbb{B}$ a $\mathbb{V}$-functor. Then we recall:

Definition. The projective ϕ-indexed cone functor over F (noted $[\![\phi,F]\!]:\mathbb{B}^{\mathrm{op}}\to\mathbb{V}$, that is a profunctor : $\mathbf{1} \to \mathbb{B}$) is the right Kan extension of F along ϕ given by the formula

$$[\![\phi, F]\!](B) = \int_A [\phi(A), \mathbb{B}(B, FA)]$$

where for X, Y in $\mathbb{V}$, the object $[X, Y]$ of $\mathbb{V}$ is the enriched hom(X, Y).

Therefore we have the following situation:



The $\mathbb{V}$-functor F admits a projective (or inverse) ϕ-indexed limit [2], if $[\![\phi, F]\!]$ is representable, in other words if there exists an object of $\mathbb{B}$ (denoted by ϕ-lim F) such that

$$[\![\phi, F]\!](B) = \mathbb{B}(B, \phi\text{-lim } F).$$

Likewise a $\mathbb{V}$-functor $L : \mathbb{A}^{op} \to \mathbb{B}$ has an inductive (or direct) ϕ-indexed limit (denoted by ϕ-colim F), if $L^{op} : \mathbb{A} \to \mathbb{B}^{op}$ has a ϕ-indexed limit.

We shall call ϕ the indexing of the limit.

Example. (1) It is clear that ordinary limits are indexed limits: Set $\mathbb{V} = \mathrm{Set}$ and let $\mathbb{A}$ and $\mathbb{B}$ be two ordinary categories, $\mathbf{1}$ the category with only one object $*$, $F : \mathbb{A} \to \mathbb{B}$ a functor. If we consider $f : \mathbb{A} \to \mathbf{1}$ the unique possible functor, then $[\![f, F]\!](B) = \{\text{cones from } B \text{ to } F\}$.

(2) In the case $\mathbb{V} = \mathcal{S}$ and $\phi = I/-$, then holim $F = (I/-)$-lim F since, following the definition of indexed limits, we have for any S in $\mathcal{S}$:

$$[\![I/-, F]\!](S) = \int_i [I/i, [S, F_i]] = \left[S, \int_i [I/i, F_i] \right] = [S, \mathrm{holim}\, F].$$

(3) We have the Yoneda embedding $\mathrm{Yo} : \Delta \to \mathrm{Set}^{\Delta^{op}} = \mathcal{S}$. If $X : \Delta \to \mathcal{S}$ is a cosimplicial object in $\mathcal{S}$, then Bousfield–Kan define $\mathrm{Tot}\, X = \mathrm{Hom}(\mathrm{Yo}, X)$. It is clear too that $\mathrm{Tot}\, X = \mathrm{Yo}\text{-lim}\, X$ and that Yo is the indexing for the total objects.

Similarly, if $X : \Delta^{op} \to \mathcal{S}$ is a simplicial object in $\mathcal{S}$, then Yo-colim $X = \mathrm{Diag}\, X$.

The replacement scheme of Bousfield–Kan allows us to describe, in more detail, this notion of indexed cone. Recall that if $S \in \mathcal{S}$, then $S = \int^n S_n \times [n]$, since $\mathcal{S}$ is a presheaf category and therefore every object S of $\mathcal{S}$ is a direct limit of representables ($= [n]$, for some n). Whence

$$[\![I/-, F]\!](S) = \int_n [[n] \times S_n, \textstyle\prod^n F] = \mathrm{Tot}[S, \textstyle\prod^{\cdot} F]$$

where $[S, \prod^{\cdot} F]$ is the following cosimplicial space in $\mathcal{S}$:

$$\prod_i [S, F_i] \rightrightarrows \prod_{i_0 \to i_1} [S, Fi_1] \cdots \prod_{i_0 \to i_1 \cdots \to i_n} [S, Fi_n] \cdots$$

Then a projective $(I/-)$-indexed cone is given by

- for each object $i \in I$: a morphism $\tau_i : S \to F_i$,
- for each morphism $f : i_0 \to i_1$: a homotopy

$$\begin{array}{ccc} \tau_f : S & \longrightarrow & Fi_0 \\ & \searrow \;\Swarrow & \downarrow Ff \\ & & Fi_1 \end{array}$$

$$\tau f : S \times [1] \to Fi_1 \qquad \text{with} \quad d_1(\tau f) = Ff \cdot \tau i_0, \quad d_0(\tau f) = \tau i_1.$$

- for each pair: $i_0 \xrightarrow{f} i_1 \xrightarrow{g} i_2$: a 2-homotopy $\tau(g, f)$:

$$\begin{array}{ccc} & Fg \cdot Ff \cdot \tau i_0 & \\ {\scriptstyle Fg \cdot \tau f} \Swarrow & \tau(g,f) & \Searrow {\scriptstyle \tau(g \cdot f)} \\ Fg \cdot \tau i_1 & \xRightarrow[\tau g]{} & \tau i_2 \end{array}$$

$$\text{with} \quad d_0(\tau(g, f)) = \tau g, \quad d_2(\tau(g, f)) = Fg \cdot \tau f, \quad d_1(\tau(g, f)) = \tau(g \cdot f).$$

- and so on, for higher n-simplexes: $(i_0 \to i_1 \to \cdots \to i_n)$.

It is clear that the higher homotopies of $[\![I/-, F]\!](S)$ determine a notion of higher homotopy between projective $(I/-)$-indexed cones.

2. A general formula for the homotopy limits

In the previous case, only ordinary categories are involved. But there are diagrams where higher coherences are involved. This kind of diagrams has been studied by Vogt [19]. Our own works on shape theory led the second named author to study a simplicial description of a coherent diagram: that is a simplicial functor (i.e. enriched in $\mathcal{S}$) from the category $S(I)$ of Dwyer-Kan to Top the category of topological spaces and continuous maps, but considered as enriched in $\mathcal{S}$ with the usual function complex given by

$$\mathrm{Top}(X, Y)_n = \mathrm{Hom}(X \times |[n]|, Y).$$

So, if $\mathbb{A}$ is a simplicial category (i.e. enriched in $\mathcal{S}$), the geometric intuition of the coherences of a homotopy cone from $\mathbb{A}$, is the same data as in the previous case (I an ordinary category) plus higher coherence data concerning higher data in the complex $\mathbb{A}(A, A')$ for every A, A' in $\mathbb{A}$.

Given $\mathbb{A}$ and $\mathbb{B}$ two simplicial categories and $F:\mathbb{A}\to\mathbb{B}$ a simplicial functor we can describe it as an object of the total space (denoted by hocone(B,F)) of the following cosimplicial space $\prod^{\cdot}(B,F)$:

$$\prod_{A}\mathbb{B}(B,FA)\rightrightarrows\prod_{A_0,A_1}[\mathbb{A}(A_0,A_1),\mathbb{B}(B,FA_1)]\cdots$$

$$\cdots\prod_{A_0,A_1,\ldots,A_n}[\mathbb{A}(A_0,A_1)\times\cdots\times\mathbb{A}(A_{n-1},A_n),\mathbb{B}(B,FA_n)]\cdots$$

with cofaces and codegeneracies

$$\prod\nolimits^{n-1}(B,F)\underset{s_i}{\overset{d_i}{\rightleftarrows}}\prod\nolimits^{n}(B,F)$$

defined in the following way, let us denote by $P(A_0,\ldots,A_n)$ the canonical projection

$$\prod\nolimits^{n}(B,F)\to[\mathbb{A}(A_0,A_1)\times\cdots\times\mathbb{A}(A_{n-1},A_n),\mathbb{B}(B,FA_n)].$$

Let us denote by

$$k_0:\mathbb{A}(A_0,A_1)\times\mathbb{A}(A_1,A_2)\to\mathbb{A}(A_1,A_2)$$

the canonical projection, by

$$k_i:\mathbb{A}(A_{i-1},A_i)\times\mathbb{A}(A_i,A_{i+1})\to\mathbb{A}(A_{i-1},A_{i+1}),\quad 0<i<n,$$

the composition law, and by

$$k_n:\mathbb{B}(B,FA_{n-1})\to[\mathbb{A}(A_{n-1},A_n),\mathbb{B}(B,FA_n)]$$

the morphism given by

$$\mathbb{B}(B,F-):\mathbb{A}(A_{n-1},A_n)\to[\mathbb{B}(B,FA_{n-1}),\mathbb{B}(B,FA_n)].$$

Then

$$P(A_0,\ldots,A_n)d_i=$$
$$[\mathrm{id}\times\cdots\times k_i\times\cdots\times\mathrm{id},\mathbb{B}(B,FA_n)]P(A_0,\ldots,A_{i-1},A_{i+1},\ldots,A_n)$$

for $0\le i<n$, and

$$P(A_0,\ldots,A_n)d_n=[\mathbb{A}(A_0,A_1)\times\cdots\times\mathbb{A}(A_{n-2},A_{n-1}),k_n]P(A_0,\ldots,A_{n-1}).$$

Let us denote by $h_i:[0]\to\mathbb{A}(A_i,A_i)$ the morphism given by 1_{A_i}. Then

$$P(A_0,\ldots,A_{n-1})s_i=$$
$$[\mathrm{id}\times\cdots\times h_i\times\cdots\times\mathrm{id},\mathbb{B}(B,FA_{n-1})]P(A_0,\ldots,A_{i-1},A_i,A_i,A_{i+1},\ldots,A_{n-1}).$$

Definition. The homotopy limit of F is a representative of the functor

$$\mathrm{hocone}(-,F)=\mathrm{Tot}\prod\nolimits^{\cdot}(-,F):\mathbb{B}^{\mathrm{op}}\to\mathscr{S}\qquad\text{(i.e. a profunctor }:1\to\mathbb{B}).$$

So we have $\mathbb{B}(B,\mathrm{holim}\,F)=\mathrm{hocone}(B,F)$.

The replacement scheme

Proposition. *If F takes its value in $\mathcal{S}$, then* holim F *is the total space of* $\prod^{\cdot} F$:

$$\prod_A FA \rightrightarrows \prod_{A_0, A_1} [A(A_0, A_1), FA_1] \cdots$$
$$\cdots \prod_{A_0, \dots, A_n} [A(A_0, A_1) \times \cdots \times A(A_{n-1}, A_n), F(A_n)] \cdots$$

Proof. For every object S of $\mathcal{S}$, $[S, \mathrm{Tot} \prod^{\cdot} F] = \mathrm{Tot}[S, \prod^{\cdot} F]$ with

$$[S, \textstyle\prod^n F] = \left[S, \prod_{A_0, \dots, A_n} [\mathbb{A}(A_0, A_1) \times \cdots \times \mathbb{A}(A_{n-1}, A_n), FA_n] \right]$$
$$= \prod_{A_0, \dots, A_n} [S, [\mathbb{A}(A_0, A_1) \times \cdots \times \mathbb{A}(A_{n-1}, A_n), FA_n]]$$
$$= \prod_{A_0, \dots, A_n} [\mathbb{A}(A_0, A_1) \times \cdots \times \mathbb{A}(A_{n-1}, A_n), [S, FA_n]] = \textstyle\prod^n(S, F)$$

whence $[S, \mathrm{Tot} \prod^{\cdot} F] = \mathrm{hocone}(S, F)$.

Remarks. It is clear that, if $\mathbb{A} = I$ is an ordinary category, then $I(i, i')$ being a trivial simplicial set:

$$\prod_{i_0, \dots, i_n} [I(i_0, i_n) \times \cdots \times I(i_{n-1}, i_n), Fi_n] = \prod_{i_0 \to i_1 \cdots \to i_n} Fi_n$$

and therefore, by the B-K replacement scheme, this notion of homotopy limit coincides with the Bousfield-Kan homotopy limit.

Homotopy limits as indexed limits

As in the Bousfield-Kan case, we are going to show that there exists an indexing for this notion of homotopy limits.

By analogy with the trivial case ($A = I$), let us define $\mathbb{A}/C$ as the following simplicial object in $\mathcal{S}$, for every simplicial category $\mathbb{A}$ and every object C of $\mathbb{A}$:

$$\coprod_A \mathbb{A}(A, C) \rightrightarrows \coprod_{A_0, A_1} \mathbb{A}(A_0, A_1) \times \mathbb{A}(A_1, C)$$
$$\cdots \coprod_{A_0, \dots, A_n} \mathbb{A}(A_0, A_1) \times \cdots \times \mathbb{A}(A_{n-1}, A_n) \times \mathbb{A}(A_n, C) \cdots$$

with faces and degeneracies given by the following formulae

$$(\mathbb{A}/C)_{n-1} \underset{s_i}{\overset{d_i}{\leftrightarrows}} (\mathbb{A}/C)_n,$$

$$d_0(z_1, \dots, z_n, g) = (z_2, \dots, z_n, g),$$

$$d_i(z_1, \dots, z_i, z_{i+1}, \dots, g) = (z_1, \dots, z_{i+1} \cdot z_i, \dots, g), \quad 0 < i < n,$$

$$d_n(z_1, \dots, z_n, g) = (z_1, \dots, g \cdot z_n),$$

$$s_i(z_1, \dots, z_i, z_{i+1}, \dots, z_{n-1}, g) = (z_1, \dots, z_i, 1_{A_i}, z_{i+1}, \dots, z_{n-1}, g).$$

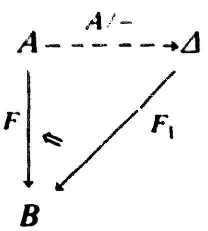
This construction defines a simplicial bifunctor:

$$\mathbb{A}/- : \Delta^{\mathrm{op}} \times \mathbb{A} \to \mathscr{S} \qquad (\Delta \text{ trivially enriched in } \mathscr{S})$$

that is a $\mathscr{S}$-profunctor : $\mathbb{A} \dashrightarrow \Delta$.

Proposition. *The $\mathscr{S}$-profunctor $\prod^{\cdot}(-, F) : \Delta \dashrightarrow \mathbb{B}$ is the right Kan extension of F along $\mathbb{A}/-$.*

Proof. We are in the following situation:



Then the right Kan extension F_1 of F along $\mathbb{A}/-$ is given by

$$\begin{aligned} F_1(B, n) &= \int_A [(\mathbb{A}/A)_n, \mathbb{B}(B, FA)] = \mathrm{Nat}((\mathbb{A}/-)_n, \mathbb{B}(B, F-)) \\ &= \mathrm{Nat}\Big(\coprod_{A_0, \dots, A_n} \mathbb{A}(A_0, A_1) \times \cdots \times \mathbb{A}(A_{n-1}, A_n) \times \mathbb{A}(A_n, -),\ \mathbb{B}(B, F-) \Big) \\ &= \prod_{A_0, \dots, A_n} \mathrm{Nat}(\mathbb{A}(A_0, A_1) \times \cdots \times \mathbb{A}(A_{n-1}, A_n) \times \mathbb{A}(A_n, -), \mathbb{B}(B, F-)) \\ &= \prod_{A_0, \dots, A_n} \mathrm{Nat}(\mathbb{A}(A_n, -), [\mathbb{A}(A_0, A_1) \times \cdots \times \mathbb{A}(A_{n-1}, A_n), \mathbb{B}(B, F-)]) \\ &= \prod_{A_0, \dots, A_n} [\mathbb{A}(A_0, A_1) \times \cdots \times \mathbb{A}(A_{n-1}, A_n), \mathbb{B}(B, FA_n)] = \prod^n(B, F). \end{aligned}$$

We have already an indexing $\mathrm{Yo} : \Delta \dashrightarrow \mathbf{1}$ for the total objects. Furthermore, there is a composition $\otimes$ for $\mathscr{S}$-profunctors [1, 14].

Recall that if $\phi : \mathbb{A} \dashrightarrow \mathbb{B}$ and $\psi : \mathbb{B} \dashrightarrow \mathbb{C}$ are two $\mathbb{V}$-profunctors between the $\mathbb{V}$-categories $\mathbb{A}, \mathbb{B}, \mathbb{C}$ (i.e. $\mathbb{V}$-functors $\phi : \mathbb{B}^{\mathrm{op}} \times \mathbb{A} \to \mathbb{V}$ and $\psi : \mathbb{C}^{\mathrm{op}} \times \mathbb{B} \to \mathbb{V}$) the composite $\psi \otimes \phi$ is defined by

$$\psi \otimes \phi(C, A) = \int^B \psi(C, B) \otimes \phi(B, A),$$

that is the kernel of the maps d_0, d_1:

$$\coprod_{B, B'} \psi(C, B) \otimes \mathbb{B}(B, B') \otimes \phi(B', A) \underset{d_1}{\overset{d_0}{\rightrightarrows}} \coprod_B \psi(C, B) \otimes \phi(B, A)$$

where d_0 and d_1 are respectively induced by

$$\psi_{BB'}\otimes\phi(B',A):\psi(C,B)\otimes\mathbb{B}(B,B')\otimes\phi(B',A)\to\psi(C,B')\otimes\phi(B',A),$$

$$\psi(C,B)\otimes\phi_{BB'}:\psi(C,B)\otimes\mathbb{B}(B,B')\otimes\phi(B',A)\to\psi(C,B)\otimes\phi(B,A),$$

the maps $\phi_{BB'}$ and $\psi_{BB'}$, being respectively the actions of $\mathbb{B}(B,B')$ on ϕ and ϕ.

This composition can clearly be extended to the natural transformations, and is associative up to isomorphism. The unit associated to $\mathbb{A}$ being the $\mathbb{V}$-functor $\mathbb{A}(-,-):\mathbb{A}^{op}\times\mathbb{A}\to\mathbb{V}$.

Let us denote by $H_{\mathbb{A}}$ the composite

$$\mathbb{A}\xrightarrow{\ \mathbb{A}/-\ }\Delta\xrightarrow{\ \mathrm{Yo}\ }\mathbf{1}$$

given by $H_{\mathbb{A}}(C)=\mathrm{Yo}\otimes\mathbb{A}/-(C)=\mathrm{Diag}(\mathbb{A}/C)$.

Proposition. *If A is a simplicial category, $H_{\mathbb{A}}$ is the indexing for the homotopy limits.*

Proof. Let $F:\mathbb{A}\to\mathbb{B}$ be a simplicial functor. We are in the following situation:

$$\begin{array}{ccccc}\mathbb{A} & \xrightarrow{\ \mathbb{A}/-\ } & \Delta & \xrightarrow{\ \mathrm{Yo}\ } & \mathbf{1}\\ {\scriptstyle F}\downarrow & & & & \\ \mathbb{B} & & & & \end{array}$$

The right Kan extension of F along $H_{\mathbb{A}}$ can be calculated in two steps: the right Kan extension F_1 of F along $\mathbb{A}/-$ and the right Kan extension F_2 of F_1 along Yo.

Now by the previous proposition $F_1=\prod^{\cdot}(-,F)$, and Yo is the indexing for total spaces, so $F_2=\mathrm{Tot}\prod^{\cdot}(-,F)=\mathrm{hocone}(-,F)$.

Existence of homotopy limits

If $\mathbb{B}$ is a simplicial category with ordinary limits and cotensor products (i.e. is complete) (we shall denote by $[S,B]$ the cotensor product of an object B of $\mathbb{B}$ by an object S of $\mathscr{S}$), then, following the general theorems on the existence of indexed limits [2], $\mathbb{B}$ admits homotopy limits by the formula

$$\mathrm{holim}\,F=\int_A[H_{\mathbb{A}}(A),FA].$$

Remark. If $\mathbb{B}$ admits ordinary limits and only cotensor products by the $[n]$'s, then cotensor product by any $S\in\mathscr{S}$ exists, every S being a direct limit of $[n]$'s.

$H_{\mathbb{A}}$ being a composite, we have the following result which is the general replacement scheme.

Proposition. *If $F:\mathbb{A}\to\mathbb{B}$ is a simplicial functor where $\mathbb{B}$ is a complete simplicial category (then total objects exist in $\mathbb{B}$), we have* $\operatorname{holim} F=\operatorname{Tot}\prod^{\cdot} F$ *where $\prod^{\cdot} F$ is the cosimplicial object:*

$$\prod_A FA \rightrightarrows \prod_{A_0,A_1} [\mathbb{A}(A_0,A_1),FA_1]$$
$$\cdots \prod_{A_0,\dots,A_n} [\mathbb{A}(A_0,A_1)\times\cdots\times\mathbb{A}(A_{n-1},A_n),FA_n]\cdots$$

Proof. $\mathbb{B}(B,\operatorname{Tot}\prod^{\cdot} F)=\operatorname{Tot}\mathbb{B}(B,\prod^{\cdot} F)$ and

$$\mathbb{B}(B,\textstyle\prod^n F)=\mathbb{B}\Big(B,\displaystyle\prod_{A_0,\dots,A_n}[\mathbb{A}(A_0,A_1)\times\cdots\times\mathbb{A}(A_{n-1},A_n),FA_n]\Big)$$
$$=\prod_{A_0,\dots,A_n}\mathbb{B}(B,[\mathbb{A}(A_0,A_1)\times\cdots\times\mathbb{A}(A_{n-1},A_n),FA_n])$$

and by the universal property of cotensor products

$$=\prod_{A_0,\dots,A_n}[\mathbb{A}(A_0,A_1)\times\cdots\times\mathbb{A}(A_{n-1},A_n),\mathbb{B}(B,FA_n)]$$
$$=\operatorname{hocone}(B,F)_n.$$

Similarly, if $\mathbb{B}$ admits ordinary colimits and tensor products (i.e. is cocomplete), then $\mathbb{B}$ admits homotopy colimits with:

$$\operatorname{hocolim} F=\int^A H_{\mathbb{A}^{op}}(A)\otimes FA$$

Furthermore, when $\mathbb{B}$ is cocomplete, it is clear that hocolim F is just the diagonal in $\mathbb{B}$ of the following simplicial object in $\mathbb{B}$:

$$\coprod_A FA \rightleftarrows \coprod_{A_0,A_1} FA_1\otimes\mathbb{A}(A_1,A_0)$$
$$\cdots\coprod_{A_0,\dots,A_n} FA_n\otimes\mathbb{A}(A_n,A_{n-1})\otimes\cdots\otimes A(\mathbb{A}_1,A_0)\cdots$$

Examples (1) $\operatorname{hocolim}\mathbb{A}(-,A)=H_{\mathbb{A}}(A)$ since

$$\operatorname{hocolim}\mathbb{A}(-,A)=\int^{A'} H_{\mathbb{A}}(A')\times\mathbb{A}(A',A)=H_{\mathbb{A}}\otimes\bar{A}=H_{\mathbb{A}}(A)$$

where $\bar{A}:1\to\mathbb{A}$ is associated to A.

Excluding $\mathscr{S}$, there are two particularly important cases of simplicial categories, the category Cat of categories and functors, and the category Top of topological spaces and continuous maps. So we shall study these two examples.

(2) Let us recall that we have an embedding $N:\mathrm{Cat}\hookrightarrow\mathscr{S}$ which is actually a simplicial embedding. It admits [13] a simplicial left adjoint K. So N preserves homotopy limits and K homotopy colimits. Furthermore the notion of total object exists in Cat and is preserved by N.

If X is a cosimplicial object in Cat, we have

$$\operatorname{Tot} X = \int_{p\leq 2} [[p], X^p]$$

(a more detailed description is given in [4]) where $[p]$ is the obvious category associated to the ordered set $[p]$.

On the other hand, a category being a particular simplicial set (via N), a 2-category (a category enriched in Cat) is a particular simplicial category.

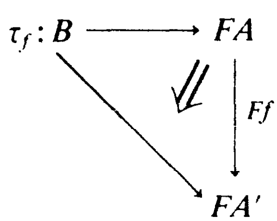
The lax limits of Gray

There are several ways to describe the lax limits. From lax transformations [3, 12] which are generalized natural transformations, or as indexed limits [17]. We shall describe the lax cones.

Let $\mathbb{A}$ and $\mathbb{B}$ be two 2-categories, F a 2-functor: $\mathbb{A}\to\mathbb{B}$. A lax cone from an object B of $\mathbb{B}$ to F is the following data:

for each $A\in\mathbb{A}$: a morphism $\tau_A : B\to FA$,

for each $f: A\to A'$: a 2-morphism



satisfying obvious coherence conditions concerning the composition of 1-morphisms and the compatibility with the 2-morphisms.

These coherences are such that a lax cone is exactly an object of the total category of the following diagram [4]:

$$\prod_A \mathbb{B}(X,FA) \rightleftarrows \prod_{A_0,A_1} [\mathbb{A}(A_0,A_1), \mathbb{B}(X,FA_1)] \cdots$$

$$\cdots \overset{\rightarrow}{\rightrightarrows} \prod_{A_0,A_1,A_2} [A(A_0,A_1)\times\mathbb{A}(A_1,A_2), \mathbb{B}(X,FA_2)].$$

Proposition. *Let* $F:\mathbb{A}\to\mathbb{B}$ *be a 2-functor between two 2-categories, then* holim F = lax lim F *as soon as one of these two terms is defined.*

Proof. Obvious, since N is a simplicial embedding preserving the total objects.

Example. (3) The category Top has a structure of a simplicial category given by the functions complexes. Furthermore Top having cotensor products by $[n]$, has cotensor products by every S of $\mathcal{S}$, so that homotopy limits exist in Top. Similarly Top admits homotopy colimits. The realisation functor Real: $\mathcal{S}\to$Top and the singular functor Sin: Top$\to\mathcal{S}$ are simplicial functors and determine a simplicial adjoint pair, so that Sin preserves homotopy limits and Real preserves homotopy colimits.

A construction by Segal

Therefore if we consider a simplicial functor $F:\mathbb{A}\to\text{Top}$ ($\mathbb{A}$ a simplicial category), we have the following formula:

$$\operatorname{hocolim} F=\int^{n} |[n]|\times \coprod_{A_0,\dots,A_n} FA_0\times \text{Real}(\mathbb{A}(A_0,A_1)\times\cdots\times\mathbb{A}(A_{n-1},A_n))$$

which is nothing but the construction $\prod^{*}F$ in the appendix B of the paper [16] by Segal, when K is the topological category obtained by the realisation of a simplicial category $\mathbb{A}$ and $\prod:K\to 1$.

3. Comparison with the Gray homotopy limits

In his paper [13] Gray generalizes the notion of homotopy limit given by Bousfield-Kan, by means of indexed limits. In order to explain his description, we must recall that if $\mathbb{A}$ is a 2-category, then the indexing $L_{\mathbb{A}}$ of the lax limits is given by [17]

$$L_{\mathbb{A}}(A)=\text{lax colim}\,\mathbb{A}(-,A).$$

On the other hand, the simplicial adjoint K to the simplicial embedding $N:\text{Cat}\hookrightarrow\mathcal{S}$ preserves the products and so determines an adjunction between 2-categories and simplicial categories. Let us denote by $K(\mathbb{A})$ the 2-category associated to the simplicial category $\mathbb{A}$ and by $\eta(\mathbb{A})$ the simplicial functor $\mathbb{A}\to K(\mathbb{A})$.

The Gray indexing is the composite:

$$G_{\mathbb{A}}=\mathbb{A}\xrightarrow{\ \eta(\mathbb{A})\ }K(\mathbb{A})\overset{L_{K(\mathbb{A})}}{-\,-\,-\,-\!\!\rightarrow} 1$$

Indeed, $L_{K(\mathbb{A})}$ being a 2-functor is a simplicial functor via the previous embedding.

This definition has two imperfections. In the case of a simplicial category $\mathbb{A}$ which is not a 2-category, the indexing factorizing through $K(\mathbb{A})$ breaks the simplicial coherences beyond the dimension 2. On the other hand, it does not allow one to generalize the replacement scheme, which is so important for obtaining spectral sequences. The obstruction is that, though $L_{K(\mathbb{A})}$ can be factorized as $\text{Yo}\otimes K(\mathbb{A})/-$ as a 2-profunctor, this is longer the case as a simplicial profunctor: the simplicial embedding N does not preserve colimits and therefore does not preserve the composition of profunctors.

Nevertheless we have a natural comparison between these two notions of homotopy limits given by the following result:

Proposition. *If $\mathbb{A}$ is a simplicial category, there is a natural transformation between $H_{\mathbb{A}}$ and $G_{\mathbb{A}}$.*

Proof. (1) In general, for every simplicial functor $F:\mathbb{A}\to\mathbb{B}$, we have a natural transformation:

$$H_F:\quad \begin{array}{ccc} \mathbb{A} & \xrightarrow{\ H_{\mathbb{A}}\ } & \mathbf{1} \\ {\scriptstyle F}\downarrow & \Downarrow & \nearrow {\scriptstyle H_{\mathbb{B}}} \\ \mathbb{B} & & \end{array}$$

given by the following diagram:

$$\begin{array}{ccccc} H_{\mathbb{B}}(FA) & \longleftarrow & \coprod_{B_0} \mathbb{B}(B_0,FA) & \leftleftarrows & \coprod_{B_0,B_1} \mathbb{B}(B_0,B_1)\times\mathbb{B}(B_1,FA)\cdots \\ \uparrow {\scriptstyle H_F(A)} & & \uparrow {\scriptstyle \tau_0} & & \uparrow {\scriptstyle \tau_1} \\ H_{\mathbb{A}}(A) & \longleftarrow & \coprod_{A_0} \mathbb{A}(A_0,A) & \leftleftarrows & \coprod_{A_0,A_1} \mathbb{A}(A_0,A_1)\times\mathbb{A}(A_1,A)\cdots \end{array}$$

where

$$\tau_n: \coprod_{A_0,\ldots,A_n} \mathbb{A}(A_0,A_1)\times\cdots\times\mathbb{A}(A_n,A)\to \coprod_{B_0,\ldots,B_n} \mathbb{B}(B_0,B_1)\times\cdots\times\mathbb{B}(B_n,FA)$$

is defined in the following way. Let us denote by $j(A_0,\ldots,A_n)$ the inclusion:

$$\mathbb{A}(A_0,A_1)\times\cdots\times\mathbb{A}(A_n,A)\to \coprod_{A_0,\ldots,A_n} \mathbb{A}(A_0,A_1)\times\cdots\times\mathbb{A}(A_n,A)$$

then $\tau_n\cdot j(A_0,\ldots,A_n)=j(FA_0,\ldots,FA_n)\cdot h$ where h is the obvious map:

$$\mathbb{A}(A_0,A_1)\times\cdots\times\mathbb{A}(A_n,A)\to\mathbb{B}(FA_0,FA_1)\times\cdots\times\mathbb{B}(FA_n,FA).$$

(2) Furthermore for every 2-category $\mathbb{D}$, there is a natural transformation:

$$\mathbb{D}\ \begin{array}{c} \xrightarrow{\ H_{\mathbb{D}}\ } \\ \Downarrow \\ \xrightarrow[\ L_{\mathbb{D}}\ ]{} \end{array}\ \mathbf{1}$$

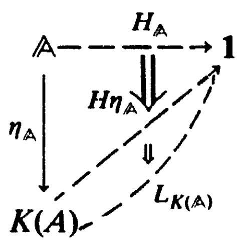
since $L_{\mathbb{D}}$ is nothing but $N\otimes K\otimes H_{\mathbb{D}}$. Indeed $L_{\mathbb{D}}(D)=\operatorname{lax\,colim}\mathbb{D}(-,D)$ and $H_{\mathbb{D}}(D)=\operatorname{hocolim}\mathbb{D}(-,D)$. The functor K preserves the direct homotopy limits and lax limits between two 2-categories are homotopy limits so that

$$K(H_{\mathbb{D}}(D))=L_{\mathbb{D}}(D).$$

The requested natural transformation is then:

$$\begin{array}{ccccc} \mathbb{D} & \xrightarrow{\ H_{\mathbb{D}}\ } & \mathcal{S} & \xrightarrow{\ \mathrm{Id}\ } & \mathcal{S} \\ & & {\scriptstyle K}\searrow & \Downarrow & \nearrow{\scriptstyle N} \\ & & & \mathrm{Cat} & \end{array}$$

Whence for every simplicial category $\mathbb{A}$, we have the following natural transformation:



Corollary. *Let* $\mathbb{B}$ *be a complete* (*resp. cocomplete*) *simplicial category and* $F:\mathbb{A}\to\mathbb{B}$ *a simplicial functor, then we have a comparison morphism*

$$\textit{holim}\, F \leftarrow \text{Gray holim}\, F \qquad (\textit{resp.}\ \text{hocolim}\, F \to \text{Gray hocolim}\, F).$$

References

[1] J. Benabou, Les distributeurs, Inst. Math. Pures and Appl., Unv. Louvain la Neuve, Rapport No. 33 (1973).

[2] F. Borceux and G.M. Kelly, A notion of limit for enriched categories, Bull. Australian Math. Soc. 12 (1975) 49–72.

[3] D. Bourn, Natural anadeses and catadeses, Cahiers Topologie et Géom. Diff. 14 (1974) 371–480.

[4] D. Bourn, A canonical action on indexed limits: An application to coherent homotopy, Lect. Notes in Math. 962 (Springer, Berlin, 1982) 23–32.

[5] D. Bourn and J.M. Cordier, Distributeurs et théorie de la forme, Cahiers Topologie et Géom. Diff. 21 (1980) 161–189.

[6] A.K. Bousfield and D.M. Kan, Homotopy limits, completions and localizations, Lect. Notes in Math. 304 (Springer, Berlin, 1972).

[7] J.M. Cordier, Sur la notion de diagramme homotopiquement cohérent, Proc. 3éme Colloque sur les Catégories, Amiens (1980), Cahiers Topologie et Géom. Diff. 23 (1982) 93–112.

[8] J.M. Cordier, Sur les limites homotopiques de diagrammes homotopiquement cohérents, to appear.

[9] W.G. Dwyer and D.M. Kan, Simplicial localizations of categories, J. Pure Apl. Algebra 17 (1980) 267–284.

[10] J. Dydak and J. Segal, Strong shape theory, Dissertationes Math. 1972 (1981).

[11] D.A. Edwards and H.M. Hastings, Čech and Steenrod homotopy theoretics with applications to geometric topology, Lect. Notes in Math. 542 (Springer, Berlin, 1976).

[12] J.W. Gray, Formal category theory, Lect. Notes in Math. 391 (Springer, Berlin, 1974).

[13] J.W. Gray, Closed categories, lax limits and homotopy limits, J. Pure Appl. Algebra 19 (1980) 127–158.

[14] F.W. Lawvere, Theoria delle categorie sopra un topos di base, Mimeographed Notes, Perugia (1973).

[15] T. Porter, Stability results for topological spaces, Math. Z. 140 (1974) 1–21.

[16] G.B. Segal, Categories and cohomology theories, Topology 13 (1974) 293–312.

[17] R. Street, Limits indexed by category valued 2-functors, J. Pure Appl. Algebra 8 (1976) 149–181.

[18] R.W. Thomason, Homotopy colimits in the category of small categories, Math. Proc. Camb. Phil. Soc. 85 (1979) 91–109.

[19] R. Vogt, Homotopy limits and colimits, Math. Z. 134 (1973) 11–52.