A General Theorem on Dominant-Diagonal Matrices

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1. DEFINITIONS

Let $A = (a_{ij})$ be an $n \times n$ complex matrix, where $n \geqslant 2$. For convenience we define $\Lambda_i(A) = \sum_{j \neq i} |a_{ij}|$.

The matrix A is said to have a *dominant diagonal* if

$$|a_{ii}| \geqslant \Lambda_i(A) \qquad \text{for} \quad i = 1, 2, \ldots, n. \tag{1}$$

The matrix A is said to have a *dominant nonzero diagonal* if it has a dominant diagonal and none of its diagonal elements is zero. The matrix A is said to have a *strictly dominant diagonal* if strict inequality holds in (1) for all values of i.

2. THE BASIC THEOREM

We shall now prove a general theorem about such matrices. From it will follow some well-known results and also several new results, including a stronger form of the Gershgorin circle theorem. The theorem and proof are essentially a variant of [4] with special attention to the consequences of equality in (1).

THEOREM 1. *Let $A = (a_{ij})$ be an $n \times n$ complex matrix ($n \geqslant 2$) which has a dominant nonzero diagonal, and let k be the number of values of i for which equality holds in (1). Then A is singular if and only if it can be reduced, by the same permutation of its rows and columns, to the form*

$$\begin{pmatrix} Q & 0 \\ R & S \end{pmatrix}, \tag{2}$$

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where 0 *consists of zeros and* $Q = (q_{ij})$ *is* $m \times m$, *where* $2 \leqslant m \leqslant k$ *(the case* $n = k = m$, *in which* $Q = A$, *is not necessarily excluded); and where there is a vector* $y = (y_1, y_2, \ldots, y_m)^T$ *for which*

$$Qy = 0, \qquad |y_1| = |y_2| = \cdots = |y_m| = 1, \tag{3}$$

and

$$|q_{ii}| = \Lambda_i(Q), \qquad i = 1, 2, \ldots, m. \tag{4}$$

Proof. If A can be reduced to such a form, in which Q is singular, then clearly A is singular.

Now assume A is singular, and let $x = (x_1, x_2, \ldots, x_n)^T$ be a vector such that $Ax = 0$ and $\max_{1 \leqslant r \leqslant n} |x_r| = 1$.

If i is such that $|a_{ii}| > \Lambda_i(A)$, then

$$0 = \sum_j a_{ij} x_j = a_{ii} x_i + \sum_{j \neq i} a_{ij} x_j,$$

$$\therefore \quad |a_{ii}|\,|x_i| \leqslant \sum_{j \neq i} |a_{ij}|\,|x_j| \leqslant \Lambda_i(A) < |a_{ii}|;$$

hence

$$|a_{ii}| > \Lambda_i(A) \Rightarrow |x_i| < 1. \tag{5}$$

Define m to be the number of values of i for which $|x_i| = 1$. By (5) and hypothesis, $m \leqslant k$. By the normalization of x, $1 \leqslant m$. (The case $m = 1$ will be excluded later.)

We shall now construct the form (2). Apply a permutation to the coordinates of x, obtaining a vector $\hat{x}$ for which

$$|\hat{x}_i| = 1 \quad \text{for} \quad i \leqslant m, \qquad |\hat{x}_i| < 1 \quad \text{for} \quad i > m. \tag{6}$$

Notice that if $m = n$, the identity permutation is used. Now apply the same permutation to the rows and columns of A, obtaining the matrix $\hat{A} = (\hat{a}_{ij})$. Then the $\hat{a}_{ii}$ and $\Lambda_i(\hat{A})$ are the same as the a_{ii} and $\Lambda_i(A)$, but in a different order. Hence $\hat{A}$ also has a dominant nonzero diagonal, and from (5) and (6) we obtain

$$i \leqslant m \Rightarrow |\hat{x}_i| = 1 \Rightarrow |\hat{a}_{ii}| = \Lambda_i(\hat{A}). \tag{7}$$

Since $\hat{A}\hat{x} = 0$, we proceed as before to obtain, for $i \leqslant m$,

$$\sum_{j \neq i} |\hat{a}_{ij}|\,|\hat{x}_j| \geqslant |\hat{a}_{ii}|\,|\hat{x}_i| = |\hat{a}_{ii}|$$
$$= \Lambda_i(\hat{A}) = \sum_{j \neq i} |\hat{a}_{ij}|,$$

which is false unless $\hat{a}_{ij} = 0$ for all values of j for which $|\hat{x}_j| < 1$, i.e., for $j > m$. Therefore $\hat{A}$ is of the form (2), where Q is the $m \times m$ submatrix in the upper left corner.

To prove the relations (3), let us set $y_i = \hat{x}_i$ for $i = 1, 2, \ldots, m$. Then $\hat{A}\hat{x} = 0 \Rightarrow Qy = 0$, and $|y_i| = 1$ for $i = 1, 2, \ldots, m$ by (7). Since $Qy = 0$, we have, for $i = 1, 2, \ldots, m$,

$$0 = \sum_j q_{ij} y_j = q_{ii} y_i + \sum_{j \neq i} q_{ij} y_j,$$

$$\therefore \quad |q_{ii}| = \sum_{j \neq i} |q_{ij}| = \Lambda_i(Q),$$

which proves (4).

The case $m = 1$ can now be excluded because if Q were a nonzero 1×1 matrix it could not be singular as required by (3).

3. BASIC RESULTS

The following well-known theorem (see [5] and the references given there) about matrices having strictly dominant diagonals follows directly from Theorem 1.

COROLLARY A. *An $n \times n$ complex matrix having a strictly dominant diagonal is nonsingular.*

Proof. If the matrix were singular, then we would have $2 \leqslant m \leqslant 0$ in Theorem 1.

Clearly this corollary is not in its strongest form. The stronger form is

COROLLARY B. *If an $n \times n$ complex matrix has a dominant nonzero diagonal, with strict inequality in* (1) *for all rows except possibly one, then it is nonsingular.*

By applying this corollary to the characteristic determinant of a square matrix, we can obtain a new and slightly stronger form of the Gershgorin circle theorem. A boundary point of the union of the Gershgorin circles can be a characteristic root only if it lies on two or more circles, or if it is a degenerate circle. Thus much of the boundary can be eliminated immediately, without assuming irreducibility.

COROLLARY C. *Let the $n \times n$ complex matrix A have a dominant nonzero diagonal with exactly k rows where equality holds in* (1). *If every such row (except possibly one) has more than k nonzero elements, then A is nonsingular.*

Proof. The matrix A cannot be brought into the form (2), so it must be nonsingular.

COROLLARY D. *Let the $n \times n$ complex matrix A have a dominant nonzero diagonal with equality in* (1) *only for $i = r$ and $i = s$, where $r \neq s$. Then A is singular if and only if the rth and sth rows contain no nonzero elements except a_{rr}, a_{rs}, a_{sr}, and a_{ss}, and for some real δ, $a_{rs} = a_{rr}e^{i\delta}$ and $a_{sr} = a_{ss}e^{-i\delta}$.*

Proof. In Theorem 1 we have

$$Q = \begin{pmatrix} a_{rr} & a_{rs} \\ a_{sr} & a_{ss} \end{pmatrix},$$

and the desired results follow from (3).

Further results for three or more cases of equality can be derived in the same way.

4. RESULTS FOR IRREDUCIBLE MATRICES

An $n \times n$ complex matrix is said to be *reducible* if it can be brought into the form

$$\begin{pmatrix} Q & 0 \\ R & S \end{pmatrix}$$

by the same permutation of its rows and columns, where Q and S are square matrices no larger than $n - 1 \times n - 1$ and 0 consists of zeros; otherwise the matrix is said to be *irreducible.*

COROLLARY E [5]. *Let A be an irreducible $n \times n$ complex matrix having a dominant diagonal with at least one case of strict inequality in* (1). *Then A is nonsingular.*

Proof. Since A is irreducible, it cannot have a row of zeros. Therefore A has a dominant nonzero diagonal. If A were singular, then by Theorem 1 it would be reducible, or else $Q = A$, violating the requirement that there be at least one case of strict inequality in (1).

COROLLARY F [3]. *Let A be an irreducible $n \times n$ real matrix having a dominant diagonal, and suppose that $a_{ii} \geqslant 0$ and $a_{ij} \leqslant 0$ for all $i \neq j$. Then A is singular if and only if $\sum_j a_{ij} = 0$ for $i = 1, 2, \ldots, n$.*

Proof. As in Corollary E, the matrix A must have a dominant nonzero diagonal. If $\sum_j a_{ij} = 0$, then clearly A is singular. Now assume that A is singular and apply Theorem 1. Since A is irreducible, we must have $Q = A$, and the desired result follows from (4).

The following result can be proved more easily with the use of Corollary B.

THEOREM 2 [1]. *If $A = (a_{ij})$ is an $n \times n$ complex matrix $(n \geqslant 2)$ for which*

$$|a_{ii}|\,|a_{kk}| > \Lambda_i(A)\Lambda_k(A) \qquad \text{for} \quad i, k = 1, 2, \ldots, n, \quad i \neq k, \tag{8}$$

then A is nonsingular.

Proof. The relations (8) imply $a_{ii} \neq 0$ for all i and $|a_{ii}| > \Lambda_i(A)$ for all but possibly one value r of i. If $|a_{rr}| \geqslant \Lambda_r(A)$, then A is nonsingular by Corollary B.

Now assume that $|a_{rr}| < \Lambda_r(A)$. No generality is lost by assuming that $r = 1$ and $a_{rr} = 1$ so that (8) implies

$$1 < \Lambda_1(A),$$

$$|a_{ii}| > \Lambda_1(A)\Lambda_i(A) \qquad \text{for} \quad i = 2, 3, \ldots, n.$$

Multiply the first column of A by $\Lambda_1(A)$, obtaining a new matrix $\bar{A}$ for which

$$|\hat{a}_{11}| = \Lambda_1(\hat{A}),$$

$$|\hat{a}_{ii}| > \Lambda_i(\hat{A}) \qquad \text{for} \quad i = 2, 3, \ldots, n.$$

By Corollary B, the matrix $\hat{A}$ is nonsingular; therefore A is also nonsingular.

5. PARTITIONED MATRICES

Let $\|\cdots\|$ be a vector norm on the space of n-dimensional complex column vectors, and let A be an $n \times n$ complex matrix. We define the two quantities

$$\operatorname{lub} A = \sup_{x \neq 0} \frac{\|Ax\|}{\|x\|},$$

$$\operatorname{glb} A = \inf_{x \neq 0} \frac{\|Ax\|}{\|x\|}.$$

Let $A = (A_{ij})$ be an $N \times N$ block matrix whose elements are $n \times n$ complex matrices. Then A is said to have a *dominant block diagonal* if

$$\operatorname{glb} A_{ii} \geqslant \sum_{j \neq i} \operatorname{lub} A_{ij} \qquad \text{for} \quad i = 1, 2, \ldots, N. \tag{9}$$

The matrix A is said to have a *dominant nonsingular block diagonal* if, in addition, the diagonal elements A_{ii} are all nonsingular, i.e., $\operatorname{glb} A_{ii} \neq 0$. The matrix A is said to have a *strictly dominant block diagonal* if strict inequality holds in (9) for all values of i.

The following theorem is analogous to Theorem 1. Its proof is exactly analogous to the proof of Theorem 1.

THEOREM 3. *Let A be a block matrix having a dominant nonsingular block diagonal, as defined above, with $N \geqslant 2$. Let K be the number of values of i for which equality holds in* (9). *Then A is singular if and only if it can be reduced, by the same permutation of its rows and columns* (*moving the elements A_{ij} as units*), *to the form*

$$\begin{pmatrix} Q & 0 \\ R & S \end{pmatrix},$$

where 0 *consists of zeros and $Q = (Q_{ij})$ is an $M \times M$ block matrix for which $2 \leqslant M \leqslant K$* (*the case $N = K = M$, in which $Q = A$, is not necessarily* *excluded*); *and where there exists a partitioned vector* $y = (y_1, y_2, \ldots, y_M)^T$ *for which*

$$Qy = 0,$$

$$||y_1|| = ||y_2|| = \cdots = ||y_M|| = 1,$$

and

$$\operatorname{glb} Q_{ii} = \sum_{j \neq i} \operatorname{lub} Q_{ij} \qquad \textit{for} \quad i = 1, 2, \ldots, M.$$

Results for block matrices which are analogous to Corollaries A–E and to Theorem 2 can easily be stated and proved. Feingold and Varga [2] have stated and proved some of these results by more direct methods. Their results do not require that all the submatrices be of the same size.

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