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A GENERALIZATION TO LOCALLY COMPACT ABELIAN GROUPS OF A SPECTRAL PROBLEM FOR COMMUTING PARTIAL DIFFERENTIAL OPERATORS

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Let G be a locally compact abelian group, and let Ω be an open relatively compact subset of positive Haar measure. Let Λ be a subset of the dual group $\hat{G}$ such that the restriction to Ω of $\lambda(\cdot)$ for $\lambda \in \Lambda$ constitute an orthonormal basis for $L_2(\Omega)$ with normalized measure. We show that the pair Ω, Λ can be characterized completely in terms of group theory and the geometry of fundamental domains for discrete subgroups. Proofs are only sketched. The relationship to partial differential operators is pointed out.

1. As in the abstract above, G denotes a, generally non-compact, locally compact abelian group with Haar measure m, and dual group $\hat{G}$. Back ground references are [3, 9]. We consider an open relatively compact subset $\Omega \subset G$, $0 < m(\Omega)$, and let $L_2(\Omega)$ be the corresponding Hilbert space constructed from the normalized measure:

$$(u, v) = m(\Omega)^{-1} \int_\Omega u(x)\bar{v}(x)\, \mathrm{d}m(x), \quad u, v \in L_2(\Omega).$$

Let Λ be a subset of $\hat{G}$ and assume that the functions $\{\lambda(\cdot) \colon \lambda \in \Lambda\}$, when restricted to Ω, form an orthonormal basis for $L_2(\Omega)$. We say that the pair Ω, Λ is a *spectral pair;* it follows from the assumptions that Λ is necessarily discrete; and we have the generalized Fourier series for functions u in $L_2(\Omega)$: $u = \sum_\Lambda \hat{u}(\lambda)\lambda(\cdot)$ where $\hat{u}(\lambda) = (u, \lambda)$. Let e_λ be the restriction to Ω of $\lambda(\cdot)$, as a function on G. Then, clearly, there is a unique strongly continuous unitary representation U of G on $L_2(\Omega)$ such that $U_t e_\lambda = \lambda(t) e_\lambda$ for $t \in G$, $\lambda \in \Lambda$, and Λ is the spectrum of this representation.

2. If $\Omega \subset \mathbb{R}^n$ is open, $0 < m(\Omega) < \infty$, then Fuglede showed in [3] that the infinitesimal generators $H_1, \dots, H_n$ of the unitary representation U form a commuting family of self-adjoint extensions of the differential operators $i(\partial/\partial x_k)$, $1 \le k \le n$, regarded as symmetric operators in $L_2(\Omega)$ with domain $C_0^\infty(\Omega)$. With an added regularity assumption on Ω (Nikodym region) he also showed that every commuting family of selfadjoint extensions gives rise to a unique spectral pair Ω, Λ where Λ is the joint spectrum.

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3. In recent papers [5, 6] we obtained a complete characterization of spectral pairs Ω, Λ in $\mathbb{R}^n$. This note arises from the observation that the spectral theoretic components of the problem are independent of the special structure of $\mathbb{R}^n$. Moreover, restrictive and technical side assumptions were imposed in earlier work, so the present results are stronger even for the special case $G=\mathbb{R}^n$. We sketch here, in the abstract setting, how the technical side conditions can be seen to be automatically satisfied. Complete details will be published elsewhere.

4. Let Ω, Λ be an arbitrary spectral pair, and consider the following pair of subgroups $K \subset A$ of G defined as follows:

$$K=\{t\in G\text{: the function } \lambda \to \lambda(t) \text{ is constant on } \Lambda\},$$

$$A=\{t\in G\text{: } U_t \text{ implements an automorphism of the } L_\infty(\Omega)\text{-multiplication algebra}\}.$$

For $\lambda_0 \in \Lambda$ we may consider instead $\Lambda - \lambda_0$ so that, without loss of generality, the added condition $0\in\Lambda$ may be assumed. In this special case ($0\in\Lambda$) we have

$$K=\Lambda^0=\{t\in G\text{: } \lambda(t)=1, \lambda\in\Lambda\}$$

and

$$A=\{t\in G\text{: } U_t \text{ acts multiplicatively}\}.$$

Moreover, the maximal covariant subalgebra $\mathcal{A}$ of the $L^\infty(\Omega)$-multiplication algebra $\mathcal{M}$ is then given by

$$\mathcal{A}=\{f\in\mathcal{M}\text{: } U_t(fu)=U_t(f)U_t(u), t\in G, u\in L_2\}.$$

If $\alpha_t = U_t \cdot U_t^*$ denotes the corresponding automorphism of $\mathcal{A}$, then it is well known [1] that the corresponding spectrum S is a subgroup of $\hat{G}$. A direct calculation shows that $S=\{\sigma\in\Lambda\text{: } \sigma+\Lambda\subset\Lambda, \Lambda-\sigma\subset\Lambda\}$. Finally, we set

$$D=S^0=\{t\in G\text{: } \sigma(t)=1, \sigma\in S\}.$$

Theorem. *Let Ω, Λ be a spectral pair with $\emptyset \neq \Omega (\subset G)$ open and relatively compact. Then one of the following two cases must occur:*

(i) *$A=G$, and Λ is a coset of a discrete subgroup Λ_1 with $\hat{G}/\Lambda_1$ compact. Moreover Ω is equivalent to a fundamental domain for the subgroup Λ_1^0.*

(ii) *$A\neq G$. In this case $K\subset A\subset D$, D is discrete, and both of the quotient groups D/A and A/K are finite. There are finite sets of representatives R_Λ, R_Γ, and R_Ω, each containing 0, R_Λ for A/K in K^0, R_Ω for A/K in A, and R_Γ for D/A in D. There is a fundamental domain $\mathcal{D}$ for D in G such that:* (a) *$\Omega=\bigcup\{\mathcal{D}+a\text{: } a\in R_\Omega\}$ disjoint union after correction on a null set,* (b) *Λ is a translate of $S\oplus R_\Lambda$, and* (c) *Ω is a direct summand with translation set $\Gamma=K\oplus R_\Gamma$.*

5. **Proof sketch.** Case (i) follows directly from known results and we refer the reader to [3, 5] for details. As for case (ii), we assume $A\neq G$, and begin by showing

that K must necessarily be discrete.

The following three lemmas apply:

(A) Let $x, t \in G$. Assume that the points x and $x+t$ both belong to Ω. Let W be a neighbourhood of x such that $W \cup (W+t) \subset \Omega$. Then for any $u \in L^2(\Omega)$, we have the identity

$$U_t u(\cdot) = u(\cdot + t) \quad \text{almost surely in } W, \tag{1}$$

(i.e. everywhere on W after correction on a set of zero Haar measure).

(B) Let C be the Cartesian product T^{Λ} of functions from Λ to the circle group $\{|z|=1\}$. Then $\varphi(t) = (\lambda(t))_{\lambda \in \Lambda}$ defines a homomorphism of G into C, and there is a null subset $N \subset \bar{\Omega}$ such that the restriction of φ to $\bar{\Omega} \backslash N$ is $1-1$.

(C) The translates of Ω by elements in K overlap on subsets of zero measure.

Comments on proofs for (A) *through* (C):

(A): Since e_λ is the restriction of λ to the subset Ω, we have $U_t e_\lambda(x) = \lambda(t) e_\lambda(x) = \lambda(t)\lambda(x) = \lambda(x+t)$. If it is further known that $x+t \in \Omega$, then $\lambda(x+t) = e_\lambda(x+t)$. By linearity we get $U_t u(x) = u(x+t)$ for all $u \in \operatorname{span}\{e_\lambda\}$. For $u \in L^2(\Omega)$, we have

$$U_t u = \sum_{\lambda \in \Lambda} \lambda(t) \hat{u}(\lambda) e_\lambda. \tag{2}$$

The convergence is in mean of order two on Ω, and we may approximate $U_t u$ pointwise (a.s.) with finite partial sums of the series on the right hand side of (2).

Since $U_t u_j = u_j(\cdot + t)$, everywhere on W for the approximating functions $u_j \in \operatorname{span}\{e_\lambda\}$, $(u_j \to u)$, we get the identity (1), a.s. on W, in the limit.

Suppose $W \cap (W+t) = \emptyset$. Then we have the following immediate corollary:

Any continuous function u on Ω, supported in W, and satisfying $U_t u = u$, must necessarily be $u = 0$.

Proof: Clearly $u(\cdot + t)\bar{u} \equiv 0$. Hence,

$$\int_\Omega u(x+t)\overline{u(x)}\, dx = (U_t u, u) = (u, u) = \| u \|^2_{L^2(\Omega)} = 0,$$

and it follows that $u = 0$.

(B): May be derived from a result in ergodic theory [4, Lemma 7], and we also refer to [5] for further details.

(C): May be generalized directly from [5, §3], or with some modifications from [3, §6].

We sketch the details below for the readers' convenience. Assume $\gamma \in K \backslash \{0\}$. Set $E = \Omega \cap (\Omega - \gamma)$, and $F = E \backslash (E + \gamma)$. Then we have $F \cup (F+\gamma) \subset \Omega$, and $F \cap (F+\gamma) = \emptyset$. Define $u = \chi_F$ (the indicator function of the subset F). By (A) then, $u(\cdot + \gamma) = U_\gamma u = u$ (locally a.s.). But this can only happen if $u = 0$ (a.s.), or equivalently $m(E) = 0$.

We finally note that (A) implies discreteness of K by the corollary to (A); and that (B) implies compactness of the quotient G/K (since the closure of the image of φ in C is compact by Tychonoff). It follows by the third lemma (C) that some fundamental domain $\tilde{\Omega}$ for K in G exists satisfying $\Omega \subset \tilde{\Omega}$, again after correction on a null set.

6. By the construction of D we note the existence of an isomorphism $\pi : L_\infty(G/D) \to \mathcal{A}$ given as follows $\pi^{-1}(f) = h$, $f \in \mathcal{A}$, where $h(t) = U_t(f)(x_0)$ and x_0 is suitable chosen (fixed) point in Ω. (The reader is refered to [10, Theorem 2] for details.) We therefore have an imprimitivity system $G, D, \mathcal{A}, L_2(\Omega), U, \pi$ in the sense of Mackey [7], i.e.

$$U_t \pi(f) U_t^* = \pi(f(\cdot + t)), \quad t \in G,\ f \in L_\infty(G/D),$$

and it follows that U is induced from a multiplicity-free representation L of the closed subgroup D. Hence a decomposition of L will induce a decomposition of U by general theory [7].

7. By construction A/K acts as a group of automorphisms [1] of the W^*-algebra $\mathcal{M} \sim L_\infty(\Omega)$. Under this action $\mathcal{M}$, (and therefore, by lemma (A), Ω) decomposes according to the dual group $(A/K)\hat{}$, which by Pontryagin [9] is K^0/A^0, and therefore, by compactness of G/K, discrete. It follows by an application of lemma (A) that Ω is a translate of $\mathcal{S} + R_\Omega$ where $\mathcal{S}$ and R_Ω are as in the statement of the theorem. Moreover the sets $\mathcal{S} + a$, $a \in R_\Omega$, overlap on sets of zero measure. Finiteness of $R_\Omega \sim A/K$ follows by translation invariance of Haar measure. The choice of representers R_Ω *in* A is possible by self-duality for finite groups [2], viz. A/K.

Consider the measure $f \to (\pi(f)u, v)$ for fixed $u, v \in C(\tilde{\Omega})$. A direct calculation gives

$$m(\Omega)(\pi(f)u, v) = \int_{\mathcal{S}} f(x) h_{u,v}(x)\, \mathrm{d}m(x)$$

where

$$h_{u,v}(x) = \sum_{a \in R_\Omega} u\bar{v}(x + a).$$

By [8, eqs. (4)–(6)] and inner product is defined on

$$V = C(\tilde{\Omega}) / \{u\colon u(a) = 0 \text{ for all } a \in R_\Omega\}$$

by $\langle \underline{u}, \underline{v} \rangle = h_{u,v}(0)$, and the representation space of L is the completion of V in this Hilbert space inner product. This argument shows that V is in fact finite-dimensional, $\dim V = \#(R_\Omega) = (A : K)$. The two isomorphic commutant algebras [7, 8] $(U, \pi)'$ and $(L)'$ may be computed directly, the first one being isomorphic to the functions ψ on Λ satisfying $\psi(\lambda + \sigma) = \psi(\lambda)$, $\lambda \in \Lambda$, $\sigma \in S$; while the algebra $(L)'$ is isomorphic to the functions on $R_\Omega \sim A/K$.

8. By construction $\mathcal{A} \sim L_\infty(G/D)$ is the spectral subspace corresponding to the identity element in $(A/K)\hat{}$ under the action of A/K on $\mathcal{M}$. Hence $U_a f U_a^* = f$ for $f \in \mathcal{A}$. Applying this to $\pi(e_\sigma) = f$, $\sigma \in S$, we get $\sigma(a) = 1$, which is to say $a \in S^0 = D$. Since $K \subset D$ and $R_\Omega \oplus K = A$, the inclusion $A \subset D$ follows.

The group D/K may now be considered: (i) a finite set R of representatives may be chosen, (ii) $\mathcal{Q} \oplus R = \tilde{\Omega}$ where $\mathcal{Q}$ is a fundamental domain for D obtained from the decomposition of Ω in Section 7, and (iii) $\tilde{\Omega}$ is the fundamental domain for K, chosen in Section 5 such that $\Omega \subset \tilde{\Omega}$. Since $K \oplus R_\Omega = A$, a set of representatives R_Γ for D/A in D may be chosen such that $R = R_\Gamma \oplus R_\Omega$. If $\Gamma = K + R_\Gamma$ we therefore have $\Omega \oplus \Gamma = \mathcal{Q} \oplus R_\Omega \oplus K \oplus R_\Gamma = \mathcal{Q} \oplus A \oplus R_\Gamma = \mathcal{Q} \oplus D = G$.

9. Let M be the finite square matrix with entries $\{\varrho(a)\}_{\varrho \in R_\Lambda, a \in R_\Omega}$. Then totality of the family of functions $\{e_\lambda : \lambda \in \Lambda\} \subset L_2(\Omega)$, $\Lambda = S + R_\Lambda$ can be shown to be equivalent to the condition, $\det(M) \neq 0$. This is a block Vandermonde determinant, so it follows by totality of $\{e_\lambda\}_\Lambda$ that the inhomogeneous part R_Λ must be a *full* set of representatives.

Since A/K is finite we may choose representatives for $\chi \in (A/K)\hat{}$ *in A* (finite groups are self-dual). Since $\Lambda \subset K^0$ the identity $\sum_{a \in R_\Omega} (\varrho - \varrho')(a) = 0$ for $\varrho \neq \varrho'$ in R_Λ follows from the assumed orthogonality of $\{e_\lambda\}_\Lambda$. The fullness of the set R_Ω of representatives, in turn, follows from this and a well known result on finite groups [2].

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