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A geometric example of non-trivially mixed Hodge structures

Andreas Matuschke*

Institut für Mathematik, Humboldt-Universität zu Berlin, Raum 303, Ziegelstr. 13a, D-10099 Berlin, Germany

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Abstract

We show that for a general fibre X_s of the Hessian family X of elliptic curves the mixed Hodge structure on the cohomology group $H^2(X, X_s)$ is a non-splitting extension of $\mathbb{Z}(-2)^4$ by $H^1(X_s)$. © 1998 Elsevier Science B.V.

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This paper is the object of the author's *test problem* within the courses of the Master Class 1994/1995 at the Mathematical Research Institute in the Netherlands. This problem was given to me and supervised by J. Steenbrink from the Katholieke Universiteit Nijmegen. As I understood, the question arose to him after being confronted with a lecture of C. Deninger concerning the relation between extensions of mixed motives and higher K-groups.

We choose homogeneous coordinates $(x:y:z)$ on $\mathbb{P}^2_{\mathbb{C}}$ and $(\alpha:\beta)$ on $\mathbb{P}^1_{\mathbb{C}}$ and consider the projective complex surface

$$\bar{X} = \{\beta x^3 + \beta y^3 + \beta z^3 - 3\alpha xyz = 0\} \subset \mathbb{P}^2_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$$

together with the flat morphism $\bar{f}: \bar{X} \to \mathbb{P}^1_{\mathbb{C}}$ induced by the second projection. We have $\bar{X}$ as blowing up of $\mathbb{P}^2_{\mathbb{C}}$ in 9 points due to the first projection, precisely as blowing up with centre in $V_+(x^3 + y^3 + z^3, xyz)$. We think of $\mathbb{C}$ as embedded into $\mathbb{P}^1_{\mathbb{C}}$ by identifying $\lambda \in \mathbb{C}$ with the point $(\lambda : 1) \in \mathbb{P}^1_{\mathbb{C}}(\mathbb{C})$ and we will denote the point $(1:0)$ with ∞. For $s \in \mathbb{P}^1_{\mathbb{C}}$ we denote with X_s the fibre of $\bar{f}$ over s. If ρ is the third root of

* Tel.: 49 30 2093 1440; fax: 49 30 2093 1866; e-mail: matuschk@mathematik.hu-berlin.de.

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Table 1

Subset in $\mathbb{P}^2 \times \mathbb{P}^1$	Coordinates/equations
$U_1 = \{\beta \neq 0, z \neq 0\}$	$x_1 = x/z,\ y_1 = y/z,\ \alpha_1 = \alpha/\beta$ $f_1 = x_1^3 + y_1^3 + 1 - 3\alpha_1 x_1 y_1$ $g_1 = \alpha_1^3 - 1$
$U_2 = \{\beta \neq 0, y \neq 0\}$	$x_2 = x/y,\ y_2 = z/y,\ \alpha_2 = \alpha/\beta$ $f_1 = x_2^3 + y_2^3 + 1 - 3\alpha_2 x_2 y_2$ $g_1 = \alpha_2^3 - 1$
$U_3 = \{\alpha \neq 0, x - z \neq 0\}$	$x_3 = x/(x-z),\ y_3 = y/(x-z),\ \alpha_3 = \beta/\alpha$ $f_3 = 2\alpha_3 x_3^3 - 3\alpha_3 x_3^2 + 3\alpha_3 x_3 - \alpha_3 - 3x_3^2 y_3 + 3x_3 y_3$ $g_3 = \alpha_3^4 - \alpha_3$
$U_4 = \{\alpha \neq 0, x - y \neq 0\}$	$x_4 = x/(x-y),\ y_4 = z/(x-y),\ \alpha_4 = \beta/\alpha$ $f_4 = 2\alpha_4 x_4^3 - 3\alpha_4 x_4^2 + 3\alpha_4 x_4 - \alpha_4 - 3x_4^2 y_4 + 3x_4 y_4$ $g_4 = \alpha_4^4 - \alpha_4$

unity $(-1+\sqrt{-3})/2$, then X_1, X_ρ, X_{ρ^2} and X_∞ are the singular fibres of $\bar{f}$, each of them isomorph to three lines crossing in three different points. For $S = \mathbb{P}^1_{\mathbb{C}} - \{1, \rho, \rho^2, \infty\}$, $D = X_1 \cup X_\rho \cup X_{\rho^2} \cup X_\infty$, $X = \bar{X} - D$ and $f = \bar{f}|_X$, we have $f: X \to S$ as a smooth projective family of elliptic curves, known as Hessian family. This family admits an interpretation as universal family of elliptic curves with weak level-3 structure. The various statements in this second paragraph can be checked easily from an open affine covering $\bar{X} = \bigcup V_i$ given by the Table 1 with $V_i = V(f_i) \subset U_i = \operatorname{Spec}\mathbb{C}[x_i, y_i, \alpha_i]$ and with $D \cap V_i = V(g_i) \subset V_i$.

If we consider for an arbitrary point $s \in S$ the embedding $X_s \hookrightarrow X$, then the long exact sequence of relative cohomology groups

$$\cdots \to H^1(X) \to H^1(X_s) \to H^2(X, X_s) \to H^2(X) \to H^2(X_s) \to \cdots$$

is an exact sequence of the associated mixed Hodge structures by [4, 8.3.9]. For any inclusion $\alpha: \mathbb{Z}(-2) \hookrightarrow H^2(X)$ the image of $\operatorname{im}\alpha$ in $H^2(X_s)$ vanishes, as the weights are different. Since $\bar{X}$ is rational, we have $H^1(\bar{X}) = 0$ and since $W_1 H^1(X)_{\mathbb{Q}} = \operatorname{im}(H^1(\bar{X}, \mathbb{Q}) \to H^1(X, \mathbb{Q}))$, we conclude that the weights occurring in $H^1(X)$ are greater than 1. Therefore, $H^1(X) \to H^1(X_s)$ is the zero map and if we let N_α denote the inverse image of $\operatorname{im}\alpha$ in $H^2(X, X_s)$, then we obtain by the short exact sequence

$$0 \to H^1(X_s) \to N_\alpha \to \operatorname{im}\alpha \to 0$$

an element $\eta_\alpha \in \operatorname{Ext}_{(mH_s)}(H^1(X_s), \mathbb{Z}(-2))$. The question is, if η_α is non-trivial and what is the geometric meaning of these extensions. A first step on this way is

Proposition 1. *For the mixed $\mathbb{Q}$-Hodge structures on the non-vanishing rational cohomology groups of X we have isomorphisms $H^0(X)_{\mathbb{Q}} \cong \mathbb{Q}$, $H^1(X)_{\mathbb{Q}} \cong \mathbb{Q}(-1)^3$, $H^2(X)_{\mathbb{Q}} \cong \mathbb{Q}(-1) \oplus \mathbb{Q}(-2)^4$ and $H^3(X)_{\mathbb{Q}} \cong \mathbb{Q}(-2)^3$.*

Proof. We are to use the weight spectral sequence with respect to the compactification $X \hookrightarrow \bar{X}$ as in [3, Théorème 2.3.5]. For that we recall that $H^0(\bar{X}) = \mathbb{Z}$, $H^4(\bar{X}) = \mathbb{Z}(-2)$, $H^1(\bar{X}) = H^3(\bar{X}) = 0$ and $H^2(\bar{X}) = \mathbb{Z}(-1)^{10}$, where as generators for $H^2(\bar{X})$ we can choose the cohomology classes of $F, E_1, \ldots, E_9$ with F a general line on $\bar{X}$, i.e. coming from $\mathbb{P}^2_{\mathbb{C}}$ and $E_1, \ldots, E_9$ the exceptional lines of the blowing up $\bar{X} \to \mathbb{P}^2_{\mathbb{C}}$. Every line E_i corresponds to the blowing up of $\mathbb{P}^2_{\mathbb{C}}$ in a point e_i and for later computations we fix

$$\begin{aligned} e_1 &= (0:-1:1), & e_4 &= (-1:0:1), & e_7 &= (-1:1:0),\\ e_2 &= (0:-\rho:1), & e_5 &= (-\rho:0:1), & e_8 &= (-\rho:1:0),\\ e_3 &= (0:-\rho^2:1), & e_6 &= (-\rho^2:0:1), & e_9 &= (-\rho^2:1:0). \end{aligned}$$

Let $D(m)$ be the normalisation of all m-fold intersections of components of D. For $i \in \{1, \rho, \rho^2, \infty\}$ we have $X_i = L_{i1} \cup L_{i2} \cup L_{i3}$ with $L_{ij} \cong \mathbb{P}^1_{\mathbb{C}}$ due to the equations

$$\begin{aligned} L_{11} &= V_+(\alpha - \beta, (1-\rho^2)x + (\rho - 1)y + (\rho^2 - \rho)z),\\ L_{\rho 1} &= V_+(\rho^2\alpha - \beta, (\rho^2 - \rho)x + (\rho - 1)y + (\rho - 1)z),\\ L_{12} &= V_+(\alpha - \beta, (1-\rho)x + (\rho^2 - 1)y + (\rho - \rho^2)z),\\ L_{\rho 2} &= V_+(\rho^2\alpha - \beta, (\rho^2 - 1)x + (\rho^2 - 1)y + (1 - \rho)z),\\ L_{13} &= V_+(\alpha - \beta, (\rho - \rho^2)x + (\rho - \rho^2)y + (\rho - \rho^2)z),\\ L_{\rho 3} &= V_+(\rho^2\alpha - \beta, (\rho - 1)x + (\rho^2 - \rho)y + (\rho - 1)z),\\ L_{\rho^2 1} &= V_+(\rho\alpha - \beta, (\rho - 1)x + (\rho - 1)y + (1 - \rho^2)z), & L_{\infty 1} &= V_+(\beta, x),\\ L_{\rho^2 2} &= V_+(\rho\alpha - \beta, (\rho - \rho^2)x + (\rho^2 - 1)y + (\rho^2 - 1)z), & L_{\infty 2} &= V_+(\beta, y),\\ L_{\rho^2 3} &= V_+(\rho\alpha - \beta, (1 - \rho^2)x + (\rho^2 - \rho)y + (1 - \rho^2)z), & L_{\infty 3} &= V_+(\beta, z). \end{aligned}$$

We put $P_{i1} = L_{i3} \cap L_{i1}$, $P_{i2} = L_{i1} \cap L_{i2}$ and $P_{i3} = L_{i2} \cap L_{i3}$ and obtain $D(0) = \bar{X}$, $D(1) = \coprod L_{ij}$, $D(2) = \coprod P_{ij}$ and $D(m) = \emptyset$ for $m > 2$. Thus, the weight spectral sequence

$${}_wE_1^{-m,m+k} = H^{k-m}(D(m), \mathbb{Q})(-m) \Rightarrow Gr^W_{m+k} H^k(X)_{\mathbb{Q}}$$

has the table of non-vanishing entries

$${}_wE_1^{0,0} = H^0(\bar{X}, \mathbb{Q}) \cong \mathbb{Q} \qquad {}_wE_1^{0,2} = H^2(\bar{X}, \mathbb{Q}) \cong \mathbb{Q}(-1)^{10}$$

$$\uparrow d^{-1,2}$$

$${}_wE_1^{-1,2} = H^0(D(1), \mathbb{Q})(-1) \cong \mathbb{Q}(-1)^{12}$$

$${}_wE_1^{0,4} = H^4(\bar{X}, \mathbb{Q}) \cong \mathbb{Q}(-2)$$

$$\uparrow d^{-1,4}$$

$${}_wE_1^{-1,4} = H^2(D(1), \mathbb{Q})(-1) \cong \mathbb{Q}(-2)^{12}$$

$$\uparrow d^{-2,4}$$

$${}_wE_1^{-2,4} = H^0(D(2), \mathbb{Q})(-2) \cong \mathbb{Q}(-2)^{12}$$

where the maps $d^{-m,m+k}$ correspond to the sum of the Gysin maps associated to the mappings of the components of $D(m)$ into the components of $D(m-1)$.

We have $d^{-1,2}([L_{ij}]) = (L_{ij} \cdot F)[F] + \sum_{k=1}^{9}(L_{ij} \cdot E_k)[E_k] \in H^2(\bar{X}, \mathbb{Q})$ and from the equations for L_{ij} and E_k above we compute the matrix

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}$$

for $d^{-1,2}$ and conclude $rk\, d^{-1,2} = 9$.

For $d^{-2,4}$ we have $d^{-2,4}([P_{ij}]) = \sum_{k=1}^{3}(P_{ij} \cdot L_{ik})[L'_{ik}] = [L'_{ij}] - [L'_{ij-1}] \in H^2(D(1), \mathbb{Q})$, where $i \in \{1, \rho, \rho^2, \infty\}$, $j \in \mathbb{Z}/3\mathbb{Z}$ and $[L'_{ij}]$ is the cohomology class of a point on L_{ij}. Since the order of the three lines L_{ij} can be freely chosen, the equation for $d^{-2,4}$ is only fixed up to sign. However, we obtain $rk\, d^{-2,4} = 8$.

For the single complex ${}_wE_1^{\bullet}$

$$\begin{aligned} &({}_wE_1^0 = H^0(\bar{X}, \mathbb{Q}) \cong \mathbb{Q}) \xrightarrow{0} ({}_wE_1^1 = H^0(D(1), \mathbb{Q})(-1) \cong \mathbb{Q}(-1)^{12}) \\ &\xrightarrow{(d^{-1,2},0)} ({}_wE_1^2 = H^2(\bar{X}, \mathbb{Q}) \oplus H^0(D(2), \mathbb{Q})(-2) \cong \mathbb{Q}(-1)^{10} \oplus \mathbb{Q}(-2)^{12}) \\ &\xrightarrow{0 \oplus d^{-2,4}} ({}_wE_1^3 = H^2(D(1), \mathbb{Q})(-1) \cong \mathbb{Q}(-2)^{12}) \\ &\xrightarrow{d^{-1,4}} ({}_wE_1^4 = H^4(\bar{X}, \mathbb{Q}) \cong \mathbb{Q}(-2)) \to 0, \end{aligned}$$

we have $H^k({}_wE_1^{\bullet}) = Gr_{\bullet}^W H^k(X)_{\mathbb{Q}}$ as mixed Hodge structures over $\mathbb{Q}$ by [3, Théorème 2.3.5]. From $H^4(X, \mathbb{Q}) = 0$ we obtain $rk\, d^{-1,4} = 1$ and, therefore,

$$\begin{aligned} H^0(X, \mathbb{Q}) &= Gr_0^W H^0(X)_{\mathbb{Q}} \cong \mathbb{Q}, \\ H^1(X, \mathbb{Q}) &= Gr_2^W H^1(X)_{\mathbb{Q}} \cong \mathbb{Q}(-1)^3, \\ H^3(X, \mathbb{Q}) &= Gr_2^W H^3(X)_{\mathbb{Q}} \cong \mathbb{Q}(-2)^3 \end{aligned}$$

and

$$Gr_{\bullet}^W H^2(X)_{\mathbb{Q}} = Gr_2^W H^2(X)_{\mathbb{Q}} \oplus Gr_4^W H^2(X)_{\mathbb{Q}} \cong \mathbb{Q}(-1) \oplus \mathbb{Q}(-2)^4.$$

As complex analytical fibration, $f : X \to S$ admits no monodromy in its relative real dimension and we have $R^2 f_* \mathbb{Q}_X \cong \mathbb{Q}_S(-1)$. For any fibre X_s of X we therefore obtain

$$\begin{aligned} H^3(X, X_s; \mathbb{Q}) &\cong H^1(S, \{s\}; R^2 f_* \mathbb{Q}_X) \cong H^1(S, \{s\}; \mathbb{Q}_S(-1)) \\ &\cong H^1(S, \{s\}; \mathbb{Q})(-1) \cong H^1(S, \mathbb{Q})(-1) \cong \mathbb{Q}(-2)^3, \end{aligned}$$

where the first isomorphism is given by the Leray spectral sequence and the last is given by $H^1(S,\mathbb{Q})\cong\mathbb{Q}(-1)^3$, which can be easily checked from the weight spectral sequence corresponding to $S\hookrightarrow\mathbb{P}^1_{\mathbb{C}}$.

If we consider now the relative cohomology sequence

$$\cdots\to H^1(X,\mathbb{Q})\xrightarrow{i^1=0}H^1(X_s,\mathbb{Q})\xrightarrow{\delta^1}H^2(X,X_s;\mathbb{Q})$$
$$\xrightarrow{p^2}H^2(X,\mathbb{Q})\xrightarrow{i^2}H^2(X_s,\mathbb{Q})\xrightarrow{\delta^2}H^3(X,X_s;\mathbb{Q})\to\cdots$$

for $X_s\hookrightarrow X$ as exact seqence of mixed Hodge structures over $\mathbb{Q}$, then we see that $\delta^2=0$ because of different weights. Thus, the homomorphism i_2 is surjective and $H^2(X_s)\cong\mathbb{Z}(-1)$ implies that i_2 is isomorphic to a projection $H^2(X,\mathbb{Q})\twoheadrightarrow Gr_2^W H^2(X)_{\mathbb{Q}} = W_2H^2(X)_{\mathbb{Q}}$. This means that $W_2H^2(X)_{\mathbb{Q}}$ is a direct summand of $H^2(X)_{\mathbb{Q}}$ as mixed Hodge structure over $\mathbb{Q}$ and, therefore, we obtain that $H^2(X)_{\mathbb{Q}}=Gr_2^W H^2(X)_{\mathbb{Q}}\oplus Gr_4^W H^2(X)_{\mathbb{Q}}\cong\mathbb{Q}(-1)\oplus\mathbb{Q}(-2)^4$. □

With the notation from the last paragraph of the proof we also have $rk\ p^2=4$ and we obtain by

$$0\to H^1(X_s,\mathbb{Q})\xrightarrow{\delta^1}H^2(X,X_s;\mathbb{Q})\xrightarrow{p^2}\operatorname{im} p^2\to 0$$

an element $\eta_s\in\operatorname{Ext}_{(\mathbb{Q}mH_s)}(\mathbb{Q}(-2)^4,H^1(X_s)_{\mathbb{Q}})$. Although it would not hurt much if we deal with mixed Hodge structures over $\mathbb{Q}$, the following Proposition 2 implies that η_s is already defined over $\mathbb{Z}$, i.e. as element in $\operatorname{Ext}_{(mH_s)}(\mathbb{Z}(-2)^4,H^1X_s)$.

Lemma. *$H^2(X,\mathbb{Z})$ is torsion free.*

Proof. For $S=\mathbb{P}^1_{\mathbb{C}}-\{1,\rho,\rho^2,\infty\}\approx S^2-\{\textit{four points}\}$, we fix two open subsets S_1 and S_2 with $S_1\cup S_2=S$, $S_1\cap S_2=S_3\amalg S_4\amalg S_5\amalg S_6$ and $S_1,\dots,S_6$ are all homeomorphic to the open disc B^2. We put $W_i=X|_{S_i}$ and obtain the Mayer–Vietoris sequence

$$\cdots\to H_3X\xrightarrow{\delta_2}H_2W_3\oplus H_2W_4\oplus H_2W_5\oplus H_2W_6\xrightarrow{i_2}H_2W_1\oplus H_2W_2\xrightarrow{j_2}H_2X$$
$$\xrightarrow{\delta_1}H_1W_3\oplus H_1W_4\oplus H_1W_5\oplus H_1W_6\xrightarrow{i_1}H_1W_1\oplus H_1W_2\xrightarrow{j_1}H_1X$$
$$\xrightarrow{\delta_0}H_0W_3\oplus H_0W_4\oplus H_0W_5\oplus H_0W_6\xrightarrow{i_0}H_0W_1\oplus H_0W_2\xrightarrow{j_0}H_0X\to 0.$$

Since $S_i\approx B^2$ are simply connected, the maps $W_i\to S_i$ are orientable fibrations in the sense of [10, Theorem 9.3.17]. Consequently, $H_kW_i\cong H_kT$ for $i=1,\dots,6$, where T denotes the oriented topological torus. If $\sigma_i:T\hookrightarrow W_i$ is a orientation preserving map of T onto one of the fibres of W_i, then $H_2W_i=\mathbb{Z}\cdot[\sigma_iT]$. Since there is no monodromy for $X\to S$ in the relative dimension, we obtain as matrix representation for i_2

$$\begin{pmatrix}1&1&1&1\\1&1&1&1\end{pmatrix}$$

and we have $\operatorname{coker} i_2$, and hence, H^2X torsionfree. □

In the beginning we asked, whether there is an inclusion $\alpha:\mathbb{Z}(-2)\hookrightarrow H^2X$, such that the resulting short exact sequence $0\to H^1X\to N_\alpha\to \operatorname{im}\alpha\to 0$ is a non-splitting extension of mixed Hodge structures. This is now equivalent to the question, whether $\eta_s\neq 0$. To answer this question we gather the following facts.

Proposition 2. *For any $s\in X$ we have $\eta_s\neq 0$ if and only if there are an element $\omega\in F^2H^2(X,X_s)_{\mathbb{C}}$ already defined over $\mathbb{Z}$ and a 2-chain T_s on X with boundary in X_s, such that*

$$\int_\omega T_s\notin\mathbb{Z}.$$

Proof. For describing the general shape of an element $\eta\in\operatorname{Ext}_{(mH_s)}(\mathbb{Z}(-2)^4,H^1X_s)$, we fix for $\mathbb{Z}(-2)^4$ a basis $(t_1,\dots,t_4)$ and for $X_s\cong\mathbb{C}/\mathbb{Z}\tau+\mathbb{Z}$ a basis (a,b) in H^1X_s dual to the generators of the lattice. As in [13, Section 10], we have

$$\begin{aligned}&\operatorname{Ext}_{(mH_s)}(\mathbb{Z}(-2)^4,H^1X_s)\\&\quad=\operatorname{Hom}_{\mathbb{C}}(\mathbb{Z}(-2)^4_{\mathbb{C}},H^1(X_s)_{\mathbb{C}})/(F^0\operatorname{Hom}(\mathbb{Z}(-2)^4,H^1X_s)_{\mathbb{C}}\\&\qquad\qquad+\operatorname{Hom}_{\mathbb{C}}(\mathbb{Z}(-2)^4,H^1X_s))\\&\quad=\operatorname{Hom}_{\mathbb{C}}(\mathbb{Z}(-2)^4_{\mathbb{C}},H^1(X_s)_{\mathbb{C}})/\operatorname{Hom}_{\mathbb{C}}(\mathbb{Z}(-2)^4,H^1X_s)\cong\mathbb{C}^8/\mathbb{Z}^8.\end{aligned}$$

If η_s corresponds under this canonical isomorphism to a matrix (η_s^{ij}), then we have

$$\begin{aligned}&W_0H^2(X,X_s)_{\mathbb{Q}}=0, && F^0H^2(X,X_s)_{\mathbb{C}}=\mathbb{C}a+\mathbb{C}b+\sum\mathbb{C}t_i,\\&W_1H^2(X,X_s)_{\mathbb{Q}}=W_2H^2(X,X_s)_{\mathbb{Q}}, && F^1H^2(X,X_s)_{\mathbb{C}}=\mathbb{C}(\tau a+b)\\&\quad=W_3H^2(X,X_s)_{\mathbb{Q}}=\mathbb{Q}a+\mathbb{Q}b, && \quad+\sum\mathbb{C}(t_i+\eta_s^{1i}a+\eta_s^{2i}b),\\& && F^2H^2(X,X_s)_{\mathbb{C}}=\sum\mathbb{C}(t_i+\eta_s^{1i}a+\eta_s^{2i}b),\\&W_4H^2(X,X_s)_{\mathbb{Q}}=\mathbb{Q}a+\mathbb{Q}b+\sum\mathbb{Q}t_i, && F^3H^2(X,X_s)_{\mathbb{C}}=0\end{aligned}$$

and our claim becomes obvious. □

Proposition 3. *$F^2H^2(X,X_s)_{\mathbb{C}}$ is canonically isomorphic to $H^0(\bar{X},\Omega^2_{\bar{X}}(D))$ via integration.*

Proof. For obtaining the concrete Hodge filtration on $H^2(X,X_s)$, we consider the *relative log-complex* (cf. [7, p. 449]) given by $\Omega^\bullet_{\bar{X}}(\log(D+X_s))(-X_s)=\ker(\Omega^\bullet_{\bar{X}}(\log D)\twoheadrightarrow\Omega^\bullet_{X_s})$. Associated to the short exact sequence of complexes

$$0\to\Omega^\bullet_{\bar{X}}(\log(D+X_s))(-X_s)\to\Omega^\bullet_{\bar{X}}(\log D)\to\Omega^\bullet_{X_s}\to 0$$

we have the commutative diagram

$$\begin{array}{ccccc}
0 & & 0 & & 0 \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{O}_{\bar{X}}(-X_s) & \xrightarrow{d} & \Omega^1_{\bar{X}}(\log(D+X_s))(-X_s) & \xrightarrow{d} & \Omega^2_{\bar{X}}(D) \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{O}_{\bar{X}} & \xrightarrow{d} & \Omega^1_{\bar{X}}(\log D) & \xrightarrow{d} & \Omega^2_{\bar{X}}(D) \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{O}_{X_s} & \xrightarrow{d} & \Omega^1_{X_s} & \xrightarrow{d} & 0 \\
\downarrow & & \downarrow & & \\
0 & & 0 & &
\end{array}$$

with zero lines and exact columns.

We have a canonical isomorphism $H^k(X,X_s;\mathbb{C}) = \mathbb{H}^k(\bar{X},\Omega^\bullet_{\bar{X}}(\log(D+X_s))(-X_s))$. For any complex $K^\bullet$ we denote with $\sigma_{\geq\bullet}K^\bullet$ the obvious (sometimes also called stupid) filtration of $K^\bullet$, i.e. $\sigma_{\geq p}K^i = K^i$ for $i \geq p$ and $\sigma_{\geq p}K^i = 0$ for $i < p$. In general, the Hodge filtration on the cohomology groups of a smooth algebraic variety is induced by the obvious filtration of the log-complex associated to a suitably chosen completion (cf. [3, Section 3]). In our relative situation we obtain $F^pH^2(X,X_s)_{\mathbb{C}}$ as image of $\mathbb{H}^2(\bar{X},\sigma_{\geq p}\Omega^\bullet_{\bar{X}}(\log(D+X_s))(-X_s))$ in $\mathbb{H}^2(\bar{X},\Omega^\bullet_{\bar{X}}(\log(D+X_s))(-X_s))$ analogously as in the proof of [4, Proposition 8.3.9]. To determine this image we consider the spectral sequences of hypercohomology $\bar{E}_1^{p,q} \Rightarrow \mathbb{H}^{p+q}(\bar{X},\Omega^\bullet_{\bar{X}}(\log(D+X_s))(-X_s))$ and $E_1^{p,q} \Rightarrow \mathbb{H}^{p+q}(\bar{X},\Omega^\bullet_{\bar{X}}(\log D))$ with the corresponding tables of non-vanishing entries

$$\begin{array}{l}
H^0(\bar{X},\mathcal{O}_{\bar{X}}(-X_s)) \xrightarrow{\bar{d}_1^{0,0}} H^0(\bar{X},\Omega^1_{\bar{X}}(\log(D+X_s))(-X_s)) \xrightarrow{\bar{d}_1^{1,0}} H^0(\bar{X},\Omega^2_{\bar{X}}(D)),\\
H^1(\bar{X},\mathcal{O}_{\bar{X}}(-X_s)) \xrightarrow{\bar{d}_1^{0,1}} H^1(\bar{X},\Omega^1_{\bar{X}}(\log(D+X_s))(-X_s)) \xrightarrow{\bar{d}_1^{1,1}} H^1(\bar{X},\Omega^2_{\bar{X}}(D)),\\
H^2(\bar{X},\mathcal{O}_{\bar{X}}(-X_s)) \xrightarrow{\bar{d}_1^{0,2}} H^2(\bar{X},\Omega^1_{\bar{X}}(\log(D+X_s))(-X_s)) \xrightarrow{\bar{d}_1^{1,2}} H^2(\bar{X},\Omega^2_{\bar{X}}(D))
\end{array}$$

and

$$\begin{array}{l}
H^0(\bar{X},\mathcal{O}_{\bar{X}}(-X_s)) \xrightarrow{d_1^{0,0}} H^0(\bar{X},\Omega^1_{\bar{X}}(\log D)) \xrightarrow{d_1^{1,0}} H^0(\bar{X},\Omega^2_{\bar{X}}(D)),\\
H^1(\bar{X},\mathcal{O}_{\bar{X}}(-X_s)) \xrightarrow{d_1^{0,1}} H^1(\bar{X},\Omega^1_{\bar{X}}(\log D)) \xrightarrow{d_1^{1,1}} H^1(\bar{X},\Omega^2_{\bar{X}}(D)),\\
H^2(\bar{X},\mathcal{O}_{\bar{X}}(-X_s)) \xrightarrow{d_1^{0,2}} H^2(\bar{X},\Omega^1_{\bar{X}}(\log D)) \xrightarrow{d_1^{1,2}} H^2(\bar{X},\Omega^2_{\bar{X}}(D)).
\end{array}$$

By [3, Théorème 3.2.5] we know that $E_\bullet^{\bullet,\bullet}$ degenerates at $E_1^{\bullet,\bullet}$, i.e. $d_p^{\bullet,\bullet} = 0$ for $p \geq 1$. Actually also $\bar{E}_\bullet^{\bullet,\bullet}$ degenerates at $\bar{E}_1^{\bullet,\bullet}$, which can be seen, for instance, by verifying that the proof for [3, Théorème 3.2.5] also applies in the relative situation as described in [4, Section 6.3]. Another way to see this degeneration in our concrete situation is to notice that for $K = -3[F] + [E_1] + \cdots + [E_9]$ the class of the canonical divisor on $\bar{X}$, we have $[X_s] = -K$ and $[D] = -4K$ in $\mathrm{Pic}\bar{X}$ and then apply Hirzebruch–

Riemann–Roch, Serre duality and our preknowledge about the ranks of $H^\bullet(X,X_s;\mathbb{C})$. But we are only interested in

$$\begin{aligned} F^2H^2(X,X_s)_{\mathbb{C}} &= \operatorname{im}(\mathbb{H}^2(\bar{X},\sigma_{\geq 2}\Omega^\bullet_{\bar{X}}(\log(D+X_s))(-X_s)) \\ &\qquad \to \mathbb{H}^2(\bar{X},\Omega^\bullet_{\bar{X}}(\log(D+X_s))(-X_s))) \\ &= H^2(\bar{X},\Omega^2_{\bar{X}}(D))/(\operatorname{im}\bar{d}_1^{1,0}). \end{aligned}$$

and the commutative diagram

$$\begin{array}{ccc} H^0(\bar{X},\Omega^1_{\bar{X}}(\log(D+X_s))(-X_s)) & \xrightarrow{\bar{d}_1^{1,0}} & H^0(\bar{X},\Omega^2_{\bar{X}}(D)) \\ \downarrow & & \downarrow{=} \\ H^0(\bar{X},\Omega^1_{\bar{X}}(\log D)) & \xrightarrow{d_1^{1,0}=0} & H^0(\bar{X},\Omega^2_{\bar{X}}(D)) \end{array}$$

and the injectivity of the left downarrow tells us that $\operatorname{im}\bar{d}_1^{1,0}=0$ and, thus, we have

$$F^2H^2(X,X_s)_{\mathbb{C}} = H^0(\bar{X},\Omega^2_{\bar{X}}(D)). \qquad \square$$

Proposition 4. *Let x_1, y_1, α_1 be the coordinates on U_1 as in the beginning of this paper and $\zeta_{\alpha_1} = \mathrm{d}x_1/(3y_1^2 - 3\alpha_1 x_1)$ a global holomorphic differential on the elliptic curve X_{α_1}. Then*

$$\omega_\infty = \frac{1}{4\pi^2}\cdot \zeta_{\alpha_1}\wedge \mathrm{d}\alpha_1$$

is an element in $H^0(\bar{X},\Omega^2_{\bar{X}}(D)) = F^2H^2(X,X_s)_{\mathbb{C}}$ already defined over $\mathbb{Z}$.

Proof. Since $Gr_4^W H^2(X)_{\mathbb{Q}}\otimes\mathbb{C} = F^2H^2(X)_{\mathbb{C}} = H^0(\bar{X},\Omega^2_{\bar{X}}(D))$ we have $rk\, H^0(\bar{X}, \Omega^2_{\bar{X}}(D))=4$ and we can choose a basis $(\omega_1,\omega_\rho,\omega_{\rho^2},\omega_\infty)$, such that ω_i has positive pole order exactly at the three components of $X_i = L_{i1}\cup L_{i2}\cup L_{i3}$. Precisely, if l'_{ij} are the homogeneous equations for the L_{ij}, then $l_{ij} = l'_{ij}/l'_{i3}$ are rational functions on $\bar{X}$ and we put

$$\omega_i = \frac{1}{4\pi^2}\cdot\frac{\mathrm{d}l_{i1}\wedge\mathrm{d}l_{i2}}{l_{i1}l_{i2}}.$$

The factor $1/4\pi^2$ is necessary to make sure, that the ω_i are already defined over $\mathbb{Z}$, i.e. if T is some 2-cycle on X, then

$$\omega_i(T) = \int_T \omega_i \in \mathbb{Z}.$$

In particular, we have

$$\omega_\infty = \frac{1}{4\pi^2}\cdot\frac{\mathrm{d}x_1\wedge\mathrm{d}y_1}{x_1y_1}.$$

From the equation

$$x_1^3 + y_1^3 + 1 - 3\alpha_1 x_1 y_1 = 0,$$

we obtain

$$(3x_1^2 - 3\alpha_1 y_1)\,\mathrm{d}x_1 + (3y_1^2 - 3\alpha_1 x_1)\,\mathrm{d}y_1 - 3x_1 y_1 \mathrm{d}\alpha_1 = 0$$

and hence

$$\omega_\infty = \frac{1}{4\pi^2} \cdot \frac{\mathrm{d}x_1}{x_1 y_1} \wedge \left(\frac{3x_1 y_1 \mathrm{d}\alpha_1}{3y_1^2 - 3\alpha_1 x_1} - \frac{(3x_1^2 - 3\alpha_1 y_1)\,\mathrm{d}x_1}{3y_1^2 - 3\alpha_1 x_1} \right)$$

$$= \frac{1}{4\pi^2} \cdot \frac{\mathrm{d}x_1}{3y_1^2 - 3\alpha_1 x_1} \wedge \mathrm{d}\alpha_1. \qquad \square$$

Now, we are ready to formulate our main result:

Theorem. *For a general fibre X_s of the Hessian family X, the mixed Hodge structure on the cohomology group $H^2(X, X_s)$ is a non-splitting extension of $\mathbb{Z}(-2)^4$ by $H^1(X_s)$.*

Proof. We are looking for an $s \in S$, where our $\eta_s \in \mathrm{Ext}_{(mH_s)}(\mathbb{Z}(-2)^4, H^1 X_s)$ does not vanish. By Propositions 2–4 it is sufficient to find a 2-chain T in X, such that ∂T is a 1-cycle in some fibre X_s and

$$\int_T \omega_\infty \notin \mathbb{Z}.$$

Let $s \in S$ be arbitrarily fixed. With ζ_s the global holomorphic differential on X_s as defined in Proposition 4 we can choose an 1-cycle C_s on X_s, such that

$$\int_{C_s} \zeta_s \neq 0.$$

Since $H^1(X, \mathbb{Z}) \to H^1(X_s, \mathbb{Z})$ is the zero map, also $H_1(X_s, \mathbb{Z}) \to H_1(X, \mathbb{Z})$ is of rank zero. Thus, there exist a 2-chain T_s on X and an integer $q > 0$, such that ∂T_s is homologous to qC_s. Since homologous 1-cycles on X_s are homotopic, we may assume that $\partial T_s = qC_s$.

Now, we fix a smooth path $\gamma : [0,1] \to S$ with $\gamma(0) = s$ and vary C_s along this path. Due to this variation we obtain for every $t \in [0,1]$ a 2-chain Q_t on X with $\partial Q_t = C_s - C_t$, where C_t is some 1-cycle on $X_{\gamma(t)}$. By $T_t = T_s - qQ_t$ we obtain a 2-chain on X with $\partial T_t = qC_t$. If we can show, that the continuous function $[0,1] \to \mathbb{C}$ given by

$$t \to \int_{T_t} \omega_\infty$$

has not only integer values, then we are done. But for this it is enough to show that the continuous function

$$f(t) = \int_{Q_t} \omega_\infty$$

is not constantly zero for $t \in [0,1]$.

By the Theorem of Fubini we have

$$f(t)=\frac{1}{4\pi^2}\int_{\gamma_t}\left(\int_{C_{\alpha_1}}\zeta_{\alpha_1}\right)\mathrm{d}\alpha_1,$$

where γ_t is the path given by $\gamma|_{[0,t]}$. Since

$$\int_{C_s}\zeta_s\neq 0,$$

there exists an $\varepsilon>0$, such that for all $t\in(0,\varepsilon)$ we have

$$\int_{\gamma_t}\left(\int_{C_{\alpha_1}}\zeta_{\alpha_1}\right)\mathrm{d}\alpha_1\neq 0,$$

which yields the desired result. We easily see that our statement holds for general fibres. □

References

[1] J.-L. Brylinski, S. Zucker, An overview of recent advances in Hodge theory, in: Encyclopaedia of Math. Sciences 69, Several Complex Variables VI, Springer, Berlin, 1990, pp. 39–142.
[2] P. Deligne, Théorie de Hodge, I, Actes du Congrès International des Mathématiciens Nice 1970, I, Gauthier-Villars, Paris, 1971, pp. 425–430.
[3] P. Deligne, Théorie de Hodge, II, Publications Mathématiques, vol. 40, IHES, 1971, pp. 5–58.
[4] P. Deligne, Théorie de Hodge, III, Publications Mathématiques, vol. 44, IHES, 1974, pp. 5–77.
[5] W. Fulton, Intersection Theory, Ergebnisse der Mathematik und Ihrer Grenzgebiete, 3.Folge, Band 2, Springer, Berlin, 1984.
[6] R. Godement, Topologie Algébrique et Théorie Des Faisceaux, Publications de l'Institut de Mathématique de l'Université de Strasbourg XIII, Hermann, Paris, 1964.
[7] P. Griffiths, J. Harris, Principles of algebraic geometry, Pure and Applied Mathematics, A Wiley-Interscience Series of Texts, Monographs and Tracts, Wiley, New York, 1978.
[8] R. Hartshorne, Algebraic Geometry, Graduate Texts in Mathematics 52, Springer, Berlin, 1978.
[9] F. Hirzebruch, Topological Methods in Algebraic Geometry, 3rd enlarged ed., Grundlehren der mathematischen Wissenschaften, Band 131, Springer, Berlin, 1966.
[10] E.H. Spanier, Algebraic Topology, McGraw-Hill, New York, 1966.
[11] J.H.M. Steenbrink, A summary of mixed Hodge theory, in: Motives, Proc. Symp. in Pure Mathematics 55, Part 1, pp. 31–41, 1993.
[12] J.H.M. Steenbrink, Limits of Hodge structures and intermediate Jacobians, Academisch Proefschrift, 1974.
[13] J.H.M. Steenbrink, Mixed Hodge Theory, Course at the M.R.I., Lecture Notes, 1993.