Journal of Pure and Applied Algebra 25 (1982) 1–20
North-Holland Publishing Company

A GEOMETRIC INVARIANT FOR NILPOTENT-BY-ABELIAN-BY-FINITE GROUPS

Robert BIERI
Mathematisches Seminar der Johann Wolfgang Goethe Universität, Frankfurt am Main, West Germany

and

Ralph STREBEL
Department of Mathematics, Cornell University, Ithaca, NY 14853, USA

Communicated by K.W. Gruenberg
Received 16 March 1981

We attach to every finitely generated nilpotent-by-Abelian-by-finite group G a closed subset $\sigma(G)$ of a sphere and demonstrate that this geometric invariant measures certain finiteness properties of G, such as being finitely presented, constructible, coherent, or polycyclic-by-finite.

Introduction

Finitely generated soluble groups occurring in applications are often nilpotent-by-Abelian-by-finite, that is, they contain normal subgroups $N \lhd H \lhd G$ such that N is nilpotent, H/N Abelian, and G/H finite; in symbols $G \in \mathfrak{NAF}$. This is, in particular, the case when G is a finitely generated soluble group with a faithful linear representation. In [4] we constructed for every finitely generated $\mathbb{Z}Q$-module A, where Q denotes a finitely generated Abelian group of torsion-free rank n, a subset Σ_A of the unit sphere $S^{n-1} \subseteq \mathbb{R}^n$. This construction is shown here to lead to an invariant $\sigma(G)$ for all finitely generated groups G in $\mathfrak{NAF}$. The invariant $\sigma(G)$ will be a subset of a certain sphere $S(G)$.

A soluble-by-finite group G is *coherent* if all its finitely generated subgroups are finitely related; it is *constructible* (in the sense of Baumslag–Bieri [1]) if it can be obtained from the trivial group by applying finitely often the processes of forming an ascending HNN-extension over a base group already obtained, and of passing to a supergroup of finite index of a group already obtained. Our main application characterizes poly-cyclic-by-finite and finitely generated soluble-by-finite, coherent respectively constructible, groups in terms of $\sigma(?)$. One should note that these finitely generated groups are of finite Prüfer rank and hence automatically in $\mathfrak{NAF}$.

Theorem A. *Let G be a finitely generated group in* $\mathfrak{N}\mathfrak{A}\mathfrak{F}$. *Then:*

(i) *G is polycyclic-by-finite if and only if* $\sigma(G)$ *is empty.*

(ii) *G is coherent if and only if* $\sigma(G)$ *contains at most one point.*

(iii) *G is constructible if and only if* $\sigma(G)$ *is contained in an open hemisphere.*

Corollary B. *If G is a finitely generated group in* $\mathfrak{N}\mathfrak{A}$ *then G is constructible (resp. coherent, resp. polycyclic) if and only if its metabelian top* G/G'' *is so.*

The statement of Corollary B becomes false if 'finitely presented' is added to the list: it is true that the metabelian top G/G'' of a finitely presented soluble group is again finitely presented [4, Corollary 5.6], but a finitely generated group G in $\mathfrak{N}_2\mathfrak{A}$ need not be finitely related if G/G'' is so; see, e.g., [4, Subsection 5.4].

1. The invariant Σ_A^c

In this section we recall the definition of Σ_A^c and some of its properties; we also analyse the behaviour of Σ_A^c with respect to a change of groups $\varphi: Q_1 \to Q$. The main sources are [4, Section 2] and [5, Sections 1 and 2].

1.1. Let Q be a finitely generated Abelian group of torsion-free rank n. By a *valuation of* Q we mean a homomorphism $v: Q \to \mathbb{R}$ of Q into the additive group of the reals. Two valuations v, v' are called equivalent if they are positive scalar multiples of one another; the set $S(Q)$ of all equivalence classes $[v]$ of non-zero valuations v will be called the *valuation sphere of* Q. This sphere carries the structure inherited from $\mathrm{Hom}(Q, \mathbb{R}) \cong \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}^n, \mathbb{R})$. First of all it is, as a topological space, homeomorphic to the unit sphere $S^{n-1} \subsetneq \mathbb{R}^n$. Also, we can talk of subspheres, and of open and closed hemispheres of $S(Q)$; and there is the antipodal map $S(Q) \to S(Q)$ sending $[v]$ to $-[v] = [-v]$. Moreover, one has the notion of the (convex) sum of subsets of $S(Q)$: if Σ_1 and Σ_2 are subsets of $S(Q)$ we define their sum by

$$\Sigma_1 + \Sigma_2 = \{[v] \mid v = \varrho_1 v_1 + \varrho_2 v_2 \neq 0 \text{ and } 0 \leq \varrho_i \in \mathbb{R}\}. \tag{1.1}$$

Next let $Q_0 \leq Q$ be a subgroup of Q. Define $S(Q, Q_0)$ to be the subsphere of $S(Q)$ made up of all those valuation classes $[v]$ whose representatives vanish on Q_0, in other words

$$S(Q, Q_0) = \{[v] \in S(Q) \mid v|_{Q_0} = 0\}.$$

Let $S(Q, Q_0)^c$ denote the set-theoretic complement of this subsphere. A homomorphism $\varphi: Q_1 \to Q$ of finitely generated Abelian groups will then induce a continuous map

$$\varphi^*: S(Q, \operatorname{im} \varphi)^c \to S(Q_1), \quad [v] \mapsto [v \circ \varphi].$$

Note that $S(Q, \operatorname{im}\varphi)$ is empty if φ has finite cokernel; then $\varphi^*: S(Q) \rightarrowtail S(Q_1)$ is an embedding whose image is the subsphere $S(Q_1, \ker\varphi)$ of $S(Q_1)$.If, on the other hand, φ is injective φ^* is surjective.

1.2. There is an explicit model of $S(Q)$ which is often useful: every surjective homomorphism $\vartheta: Q \twoheadrightarrow \mathbb{Z}^n$ onto the standard lattice $\mathbb{Z}^n$ in $\mathbb{R}^n$ gives rise to a homeomorphism

$$\vartheta^*: S^{n-1} = \{u \in \mathbb{R}^n \mid \langle u, u\rangle = 1\} \xrightarrow{\sim} S(Q), \quad u \mapsto [v_u],$$

where $v_u(q) = \langle u, \vartheta q\rangle$ and where, as always, n denotes the torsion-free rank of Q. If $\varphi: Q_1 \to Q$ is a homomorphism and the maps $\vartheta_1: Q_1 \twoheadrightarrow \mathbb{Z}^{n_1}$ and $\vartheta: Q \twoheadrightarrow \mathbb{Z}^n$ are as before, there exists a unique homomorphism $\bar\varphi: \mathbb{Z}^{n_1} \to \mathbb{Z}^n$ rendering the square

$$\begin{array}{rccc}
\varphi: & Q_1 & \longrightarrow & Q \\
 & \downarrow{\scriptstyle \vartheta_1} & & \downarrow{\scriptstyle \vartheta} \\
\bar\varphi: & \mathbb{Z}^{n_1} & \longrightarrow & \mathbb{Z}^n
\end{array}$$

commutative. This square, in turn, gives rise to a new square

$$\begin{array}{rccc}
\varphi^*: & S(Q, \operatorname{im}\varphi)^c & \longrightarrow & S(Q_1) \\
 & \uparrow{\scriptstyle \vartheta_1^*} & & \uparrow{\scriptstyle \vartheta^*} \\
\bar\varphi^*: & \{u \in S^{n-1} \mid \bar\varphi^{tr} u \neq 0\} & \longrightarrow & S^{n_1-1}, \quad u \mapsto \dfrac{\bar\varphi^{tr} u}{\|\bar\varphi^{tr} u\|}\,.
\end{array}$$

In it $\bar\varphi^{tr}$ denotes the transposed (or adjoint) of the linear extension $\mathbb{R}^{n_1} \to \mathbb{R}^n$ of the homomorphism $\bar\varphi: \mathbb{Z}^{n_1} \to \mathbb{Z}^n$.

1.3. Consider next an arbitrary $\mathbb{Z}Q$-module A; let $I = \operatorname{Ann}_{\mathbb{Z}Q}(A) \lhd \mathbb{Z}Q$ be its annihilator ideal and $C(A) = 1 + I$ its centralizer in $\mathbb{Z}Q$. We attach to A a subset Σ_A of $S(Q)$ by putting

$$\Sigma_A = \bigcup_{\lambda \in C(A)} \{[v] \mid v(q) > 0 \text{ for all } q \in \operatorname{supp}(\lambda)\}, \tag{1.2}$$

where $\operatorname{supp}(\lambda) \subset Q$ consists of all elements q with non-zero coefficient in λ. One infers from (1.2) that Σ_A is open in $S(Q)$ and that $\Sigma_A = \Sigma_{\mathbb{Z}Q/\operatorname{Ann}(A)}$. The set-theoretic complement $\Sigma_A^c = S(Q) \setminus \Sigma_A$ is therefore compact. For the proof of the following lemma see [4] and [5]; but note that the Q-modules A', A and A'' are not assumed to be finitely generated, so that one of our former arguments needs a slight adjustment which we leave as an exercise.

Lemma 1.1. (a) *If A, A' are Q-modules and* $\operatorname{Ann}_{\mathbb{Z}Q}(A) \subsetneq \operatorname{Ann}_{\mathbb{Z}Q}(A')$ *then* $\Sigma_A^c \supseteq \Sigma_{A'}^c$.

(b) *If A is a Q-module, I its annihilator,* $\sqrt{I} = \{\lambda \in \mathbb{Z}Q \mid \lambda^m \in I$ *for some* $m \geq 1\}$ *the* *radical of I and* $P_1, \ldots, P_m$ *are ideals with* $P_1 \cap P_2 \cap \cdots \cap P_m = \sqrt{I}$, *then*

$$\Sigma_A^c = \Sigma_{\mathbb{Z}Q/I}^c = \Sigma_{\mathbb{Z}Q/\sqrt{I}}^c = \Sigma_{\mathbb{Z}Q/P_1}^c \cup \cdots \cup \Sigma_{\mathbb{Z}Q/P_m}^c.$$

(c) *If* $A' \rightarrowtail A \twoheadrightarrow A''$ *is a short exact sequence of Q-modules then* $\Sigma_A^c = \Sigma_{A'}^c \cup \Sigma_{A''}^c$. □

Remark. The original definition of Σ_A, as given in [4, Subsection 2.2], works only for finitely generated Q-modules and is then equivalent to the one adopted above [4, Proposition 2.1]. Since in the sequel we want to be able to view a $\mathbb{Z}Q$-module A over the group ring $\mathbb{Z}Q_0$ of a subgroup $Q_0 \leq Q$, whereby A might very well be finitely generated over $\mathbb{Z}Q$, but infinitely generated over $\mathbb{Z}Q_0$, we allow infinitely generated modules from the outset.

1.4. *A conjecture.* Let $\varphi: Q_1 \to Q$ be a homomorphism of finitely generated Abelian groups, A a Q-module and A_1 the Q_1-module obtained from A by pulling back the action via φ. Then each point $[v] \in \Sigma_A^c \subseteq S(Q)$ with $v \circ \varphi \neq 0$ gives rise to a point $[v \circ \varphi] \in \Sigma_{A_1}^c \subseteq S(Q_1)$; for a centralizer $\lambda_1 \in \mathbb{Z}Q_1$ showing that $[v \circ \varphi]$ belonged by (1.2) to Σ_{A_1}, would map under the canonical ring homomorphism $\varphi_*: \mathbb{Z}Q_1 \to \mathbb{Z}Q$ to a centralizer $\lambda = \varphi_*(\lambda_1)$ exhibiting that $[v]$ belonged to Σ_A, a contradiction. Bringing the map $\varphi^*: S(Q, \operatorname{im}\varphi)^c \to S(Q_1)$, introduced in 1.1, into play this observation can be stated as

$$\varphi^*(\Sigma_A^c \cap S(Q, \operatorname{im}\varphi)^c) \subseteq \Sigma_{A_1}^c. \tag{1.3}$$

We conjecture inclusion (1.3) *is always an equality.* This is true when φ has finite cokernel (apply Proposition 2.3 of [4] to $\mathbb{Z}Q/\operatorname{Ann}_{\mathbb{Z}Q}(A)$). Next the canonical factorization $\iota \circ \pi: Q_1 \twoheadrightarrow \operatorname{im}\varphi \hookrightarrow Q$ of φ gives rise to a canonical factorization of φ^*, namely

$$\begin{array}{ccccc} S(Q, \operatorname{im}\varphi)^c & \xrightarrow{\iota^*} & S(\operatorname{im}\varphi) & \xrightarrow{\pi^*} & S(Q_1) \\ \cup & & \cup & & \cup \\ \Sigma_A^c \cap S(Q, \operatorname{im}\varphi)^c & \longrightarrow & \Sigma_{A_0}^c & \xrightarrow{\sim} & \Sigma_{A_1}^c \end{array}$$

(cf. the end of 1.1). In the above A_0 denotes A with action restricted to $\operatorname{im}\varphi$. The diagram shows that we can concentrate on inclusions $\varphi = \iota: Q_1 \to Q$; moreover, by passing to subgroups of finite indices we can assume, if need be, that Q_1 is a direct factor of a free Abelian group Q.

1.5. To see better where the problem now lies, consider the special case where Q is free of rank 3 and Q_1 is a direct factor of rank 2; choose ϑ_1 and ϑ so that $\varphi: \mathbb{Z}^2 \to \mathbb{Z}^3$ takes (x_1, x_2) to $(x_1, x_2, 0)$, and replace $S(Q_1)$ and $S(Q)$ by their explicit models (see 1.2). Then

$$\varphi^*: \quad S^2 \setminus \{(0,0,1),(0,0,-1)\} \to S^1,$$
$$(u_1, u_2, u_3) \mapsto (u_1, u_2)/(u_1^2 + u_2^2)^{\frac{1}{2}}.$$

Let $A=\mathbb{Z}Q/I$ be a (cyclic) Q-module and set $A_1=\mathbb{Z}Q_1/(\mathbb{Z}Q_1\cap I)$. The problem is to show that $(u_1,u_2)\in S^1$ must belong to Σ_{A_1} if the open meridian $(\varphi^*)^{-1}\{(u_1,u_2)\}\subseteq S^2$ lies in Σ_A. Phrased more algebraically: if for all reals $r\in]-1,1[$ there exists $\lambda_r\in I$ such that

$$\langle (u_1(1-r^2)^{\frac{1}{2}}, u_2(1-r^2)^{\frac{1}{2}}, r), \vartheta_q\rangle > 0$$

for all $q\in \operatorname{supp}(1+\lambda_r)$, then $\mathbb{Z}Q_1\cap I$ must contain an element λ such that

$$\langle (u_1,u_2), \vartheta_1 q\rangle > 0$$

for all $q\in\operatorname{supp}(1+\lambda)\subseteq Q_1$. This reformulation of the problem reveals the heart of the matter to be a problem in elimination theory.

1.6. *A partial solution.* Resorting to the connection with valuations of a field, discussed in [5, Section 2], the problem can be shown to have an affirmative answer in important special cases. Let $\varphi: Q_1\to Q$ be an inclusion, A a Q-module and A_1 the Q_1-module obtained from A via φ. Set $I=\operatorname{Ann}_{\mathbb{Z}Q}(A)$. Since $\mathbb{Z}Q$ is Noetherian there exist finitely many prime ideals $P_1,P_2,\dots,P_m$ whose intersection equals the radical $\sqrt{I}=\{\lambda\in\mathbb{Z}Q\mid \lambda^m\in I \text{ for some } m\geq 1\}$ of I. It follows that $\sqrt{\mathbb{Z}Q_1\cap I}$, the radical being taken in $\mathbb{Z}Q_1$, coincides with the intersection of the prime ideals $\mathbb{Z}Q_1\cap P_j$. Since by Lemma 1.1 (b) we have

$$\Sigma_A^{\mathrm{c}}=\Sigma^{\mathrm{c}}_{\mathbb{Z}Q/\sqrt{I}}=\Sigma^{\mathrm{c}}_{\mathbb{Z}Q/P_1}\cup\cdots\cup\Sigma^{\mathrm{c}}_{\mathbb{Z}Q/P_m},$$

and similarly for $\Sigma^{\mathrm{c}}_{A_1}$, we can concentrate attention on a domain $A=\mathbb{Z}Q/P$ and the subdomain $A_1=(\mathbb{Z}Q_1+P)/P$ generated by the canonical image of Q_1.

1.7. Suppose now $[v_1]$ belongs to $\Sigma^{\mathrm{c}}_{A_1}$. By [5, Theorem 2.1] there exists an ordered Abelian group Γ_1, a (ring theoretic) valuation $w_1: A_1\to\Gamma_1\cup\{\infty\}$ and an increasing embedding

$$\tau_1:\operatorname{im}\left(Q_1\xrightarrow{\kappa_1}A_1\setminus\{0\}\xrightarrow{w_1}\Gamma_1\right)\rightarrowtail\mathbb{R},$$

all in such a way that the composite

$$Q_1\twoheadrightarrow\operatorname{im}(w_1\circ\kappa_1)\rightarrowtail\mathbb{R}$$

equals v_1. In the above, κ_1 takes $q\in Q_1$ to $q+P\in A_1$. Choose elements $q_1,\dots,q_k$ of Q whose canonical images in the field of fraction $\operatorname{Quot}(A)$ of A form a transcendence basis of $\operatorname{Quot}(A)$ over $\operatorname{Quot}(A_1)$. Let Q_0 be the subgroup of Q generated by $Q_1\cup\{q_1,\dots,q_k\}$ and let A_0 denote the subdomain of A generated by the canonical image of Q_0 in A. The valuation $w_1: A_1\to\Gamma_1\cup\{\infty\}$ extends uniquely to a valuation $w_0: A\to\Gamma_1\cup\{\infty\}$ taking each of $q_1,\dots,q_k$ to 0; see, e.g., [6,VI, §10.1]. Next extend w_0 to a valuation $w': A\to\Gamma\cup\{\infty\}$ where the value group Γ contains Γ_1 as a subgroup of finite index. Since Q is finitely generated there exists a largest isolated subgroup $\Delta\leq\Gamma$ which does not contain all of $w'(\kappa(Q))$, where $\kappa: Q\to A$ takes $q\in Q$ to

$q+P\in A$. Let w be the composite

$$w: A \to \Gamma \cup \{\infty\} \twoheadrightarrow \Gamma/\Delta \cup \{\infty\}.$$

This is again a valuation and $\mathrm{im}(w \circ \kappa)$ will be an Archimedean subgroup of Γ/Δ, so that we can find an increasing homomorphism $\tau: \mathrm{im}(w\circ\kappa) \twoheadrightarrow \mathbb{R}$. Let $v: Q \to \mathbb{R}$ be the composite

$$v: Q \twoheadrightarrow \mathrm{im}(w\circ\kappa) \xrightarrow{\tau} \mathbb{R}.$$

By [5, Theorem 2.1] the point $[v]$ belongs to Σ_A^c. Two cases arise:

Case (A): $Q_1 \not\subseteq \ker v$. As $\mathrm{im}(w' \circ \kappa(Q_1)) = \mathrm{im}(w_1 \circ \kappa_1(Q_1))$ is Archimedean, this means that $\Delta \cap \mathrm{im}(w'\circ\kappa(Q_1)) = \{0\}$. It follows that $[v\circ\varphi] = [v_1]$, where $\varphi: Q_1 \to Q$ is the inclusion, and thus $[v] \in \Sigma_A^c$ is the desired preimage of $[v_1] \in \Sigma_{A_1}^c$ under the map $\varphi^*: S(Q, Q_1)^c \twoheadrightarrow S(Q_1)$.

Case (B): $Q_1 \subseteq \ker v$. Then, first of all, $\Delta \neq 0$ and so Γ, and hence Γ_1, are not Archimedean. In addition, A cannot be a finitely generated A_0-module. To see this, let $q \in Q$ be an element with $v(q) > 0$. Then $q^{-1} + P$ is not integral over A_0. Otherwise there would exist a monic polynomial $f(X) \in \mathbb{Z}Q_0[X]$ of degree m, say, such that $f(q^{-1}) \in I = \mathrm{Ann}_{\mathbb{Z}Q}(A)$. The element

$$\lambda = 1 - q^m f(q^{-1}) \in q \cdot \mathbb{Z}Q_0[q]$$

would centralize A, and since $Q_0 = qp(Q_1, q_1, \ldots, q_k) \subseteq \ker v$ we would clearly have $v(q') > 0$ for every $q' \in \mathrm{supp}\,\lambda$. By (1.2) this would force $[v]$ to lie in Σ_A, contradicting our previous statement that $[v] \in \Sigma_A^c$.

1.8. We add a comment. Call a point $[v] \in S(Q)$ which can be represented by a valuation $v: Q \twoheadrightarrow \mathbb{Z} \hookrightarrow \mathbb{R}$, a *discrete-valuation point.* If Σ is any subset of $S(Q)$ let $\mathrm{dis}\,\Sigma$ denote the set of all discrete-valuation points in Σ. For every homomorphism $\varphi: Q_1 \to Q$ the induced mapping $\varphi^*: S(Q, \mathrm{im}\,\varphi)^c \to S(Q_1)$ sends discrete-valuation points to discrete-valuation points and so restricts to a mapping $\varphi^*: \mathrm{dis}\, S(Q, \mathrm{im}\,\varphi)^c \to \mathrm{dis}\, S(Q_1)$. Finally if, in the set-up of 1.7, $[v_1]$ belongs to $\mathrm{dis}\,\Sigma_{A_1}^c$ the valuation w_1 can be chosen to be a discrete valuation $w_1: A_1 \to \mathbb{Z} \cup \{\infty\}$; see [5, Theorem 2.2]. Hence $w: A \to \Gamma \cup \{\infty\}$ will also be a discrete valuation and only Case (A) will arise.

1.9. The next proposition summarizes part of the preceding discussion:

Proposition 1.2. *Let $\varphi: Q_1 \to Q$ be a homomorphism of finitely generated Abelian groups, A a Q-module and A_1 the Q_1-module obtained from A via φ. Then $\varphi^*: S(Q, \mathrm{im}\,\varphi)^c \to S(Q_1)$ induces a map*

$$\varphi^*: \Sigma_A^c \cap S(Q, \mathrm{im}\,\varphi)^c \to \Sigma_{A_1}^c \tag{1.4}$$

and a surjection

$$\varphi^*: (\operatorname{dis} \Sigma_A^c) \cap S(Q, \operatorname{im} \varphi)^c \twoheadrightarrow \operatorname{dis} \Sigma_{A_1}^c.$$

If, in addition, $\mathbb{Z}Q/\operatorname{Ann}_{\mathbb{Z}Q}(A)$ *is a finitely generated* $\mathbb{Z}Q_1$*-module, then* $\Sigma_A^c \subsetneq S(Q, \operatorname{im} \varphi)^c$ *and* (1.4) *is also a surjection.* □

1.10. As an application of Proposition 1.2 we obtain information on the qualitative behaviour of $\operatorname{dis} \Sigma_A^c$ with respect to tensor products.

Theorem 1.3. *If* A, B *are* $\mathbb{Z}Q$*-modules and the tensor product* $A \otimes B$ *over the integers is regarded as a* $\mathbb{Z}Q$*-module with diagonal action, then*

$$\operatorname{dis} \Sigma_{A\otimes B}^c \subsetneq \operatorname{dis} \Sigma_A^c + \operatorname{dis} \Sigma_B^c.$$

Proof. We consider the direct product $Q \times Q$ with the projections $\pi_i: Q \times Q \twoheadrightarrow Q$ and the diagonal embedding $\Delta: Q \rightarrowtail Q \times Q$. Let M denote the Abelian group $A \otimes B$ considered as a $\mathbb{Z}(Q \times Q)$-module with full action, $(q, p)(a \otimes b) = qa \otimes pb$, where $q, p \in Q$ and $a \in A, b \in B$. Let $v: Q \to \mathbb{Z}$ be a discrete valuation with $[v] \in \Sigma_{A\otimes B}^c$. By Proposition 1.2 there is a discrete valuation $v': Q \times Q \to \mathbb{Z}$ with $[v'] \in \Sigma_M^c$ and such that $v' \circ \Delta$ is equivalent to v. Now v' can be written as $v' = v_1\pi_1 + v_2\pi_2$, where $v_i: Q \to \mathbb{Z}$. Then $v' \circ \Delta = v_1 + v_2$ and so $[v] = [v_1 + v_2]$. We claim that $[v_1] \in \Sigma_A^c \cup \{[0]\}$ and $[v_2] \in \Sigma_B^c \cup \{[0]\}$.

Assume $v_1 \neq 0$ and let $\lambda \in \mathbb{Z}Q$ be a centralizer of A. Then $\lambda \otimes 1 \in \mathbb{Z}(Q \times Q)$ centralizes M. As $v_1(q) = v'((q, 1))$ for every $q \in Q$, and since $[v'] \in \Sigma_M^c$ there exists by (1.2) an element $q \in \operatorname{supp}(\lambda)$ with $v_1(q) = v'((q, 1)) \leq 0$. Since this is true for all $\lambda \in C(A)$ it follows, again by (1.2), that $[v_1] \in \Sigma_A^c$. Similarly, either $v_2 = 0$ or $[v_2] \in \Sigma_B^c$. □

1.11. *Illustration.* Let Q be free Abelian of rank 2, choose an isomorphism $\vartheta: Q \xrightarrow{\sim} \mathbb{Z}^2$ and denote the preimages of $(1,0) \in \mathbb{Z}^2$ and $(0,1) \in \mathbb{Z}^2$ by q_1 and q_2. Let Q act on the additive group of the rationals $\mathbb{Q}$ via a homomorphism $\kappa: Q \to \mathbb{Q}^\times$ and denote the resulting module by A. If the $\kappa(q_i)$ have the prime factorization

$$\kappa(q_i) = \pm \prod \{p^{m_{ip}} \mid p > 0 \text{ prime}\} \quad (i = 1, 2),$$

and $\mathscr{P}$ denotes the set of primes actually occurring in either $\kappa(q_1)$ or $\kappa(q_2)$, then $(\vartheta^*)^{-1}(\Sigma_A^c)$ is given by

$$\Sigma_A^c \cong \left\{ \frac{(m_{1p}, m_{2p})}{(m_{1p}^2 + m_{2p}^2)^{\frac{1}{2}}} \,\middle|\, p \in \mathscr{P} \right\} \subsetneq S^1$$

(see [5, example 2.6]).

Now let A and A' be given by the following specific choices of actions:

$$\kappa_A(q_1) = p_1^{m_{11}} \cdots p_l^{m_{1l}}, \qquad \kappa_A(q_2) = 1,$$
$$\kappa_{A'}(q_1) = 1, \qquad \kappa_{A'}(q_2) = p_1^{m_{21}} \cdots p_l^{m_{2l}},$$

where $p_1, p_2, \ldots, p_l$ are distinct primes and m_{1j}, m_{2j} are non-negative integers with $m_{1j}+m_{2j}>0$ for all j. Then $\Sigma_A^c \cong \{(1,0)\}$ and $\Sigma_{A'}^c \cong \{(0,1)\}$, while

$$\Sigma_{A\otimes A'}^c \cong \left\{ \frac{(m_{1j}, m_{2j})}{(m_{1j}^2+m_{2j}^2)^{\frac{1}{2}}} \;\middle|\; 1 \le j \le l \right\}.$$

A more drastic case is this: Set $A=\mathbb{Z}Q/\mathbb{Z}Q\cdot(1-q_2)$ and $A'=\mathbb{Z}Q/\mathbb{Z}Q\cdot(1-q_1)$. Then $\Sigma_A^c \cong \{\pm(1,0)\}$ and $\Sigma_{A'}^c = \{\pm(0,1)\}$, as can be seen directly from definition (1.2); see [5, Theorem 5.2] for a general discussion. The tensor product $A\otimes A'$, however, is Q-isomorphic to $\mathbb{Z}Q$ whence $\Sigma_{A\otimes A'}^c \cong S^1$.

1.12. Let Q be a finitely generated Abelian group and A a Q-module. The subset $\operatorname{dis}\Sigma_A^c$ of Σ_A^c has a further useful property: it determines when an element $q\in Q$ *is integral with respect to* A, i.e. when there exists a monic integral polynomial $f\in\mathbb{Z}[X]$ such that $f(q)$ annihilates A. To be specific, let

$$\bar{H}_q = \{[v] \mid v(q) \ge 0\} \subseteq S(Q)$$

be the closed 'hemisphere' defined by q. (We allow q to be of finite order and hence $\bar{H}_q$ to be all of $S(Q)$.) Then the proof of Theorem 4.8 in [5] shows that *the following statements are equivalent:*

(i) $\Sigma_A^c \subseteq \bar{H}_q$,
(ii) $\operatorname{dis}\Sigma_A^c \subseteq \bar{H}_q$, and
(iii) *q is integral with respect to A.*

By the rational convex closure of a set $\Sigma \subseteq S(Q)$ we mean the intersection $\hat{\Sigma}$ of all closed hemispheres of the form $\bar{H}_q$ containing Σ

$$\hat{\Sigma} = \bigcap \{\bar{H}_q \mid q\in Q, \Sigma \subseteq \bar{H}_q\}.$$

The equivalence of the statements (i) and (ii) above yields immediately.

Proposition 1.4. *For every Q-module A the rational convex closure of Σ_A^c and* $\operatorname{dis}\Sigma_A^c$ *coincide.*

1.13. Another interesting geometric invariant is the *smallest* subsphere (of the form $S(Q,P)$, $P\le Q$) containing Σ_A^c. Clearly P is given by

$$\begin{aligned} P &= \{q\in Q \mid v(q)=0 \text{ for all } [v]\in\Sigma_A^c\} \\ &= \{q\in Q \mid \Sigma_A^c \subseteq \bar{H}_q \cap \bar{H}_{q^{-1}}\} \\ &= \{q\in Q \mid \operatorname{dis}\Sigma_A^c \subseteq \bar{H}_q \cap \bar{H}_{q^{-1}}\}. \end{aligned}$$

The last inequality is justified by the equivalence of (i) and (ii). By the equivalence of (ii) and (iii) the subgroup P can also be described as consisting of all elements $q\in Q$ with both q and q^{-1} integral with respect to A. Put differently, P is made up of all elements $q\in Q$ for which there exists a polynomial $f(X)\in\mathbb{Z}[X]$ with both leading and trailing coefficient 1, and such that $f(q)$ annihilates A. This subgroup P will play a principal role in the next section.

2. Nilpotent-by-Abelian-by-finite groups

2.1. Throughout this section G will denote a finitely generated group containing a pair (N, H) of normal subgroups $N \lhd H \lhd G$ such that N is nilpotent, $Q = H/N$ is Abelian and G/H is finite; note that Q will automatically be finitely generated. Put differently, G is a finitely generated group in the class $\mathfrak{N}\mathfrak{A}\mathfrak{F}$ of all nilpotent-by-Abelian-by-finite groups. We call (N, H) an *admissible* pair (of subgroups) of G. As usual $\gamma_i N$ will denote the i-th term of the lower central series of N; that is

$$\gamma_1 N = N \quad \text{and} \quad \gamma_{i+1} N = [\gamma_i N, N] \quad \text{for } i \geq 1.$$

We shall use the abbreviation $N_{ab} = N/\gamma_2 N$. Note that N_{ab} is a (left) H/N-module via conjugation.

For every admissible pair (N, H) of G we consider the subset $\Sigma^c_{N_{ab}} \subseteq S(H/N)$. The aim of this section is to show that this subset $\Sigma^c_{N_{ab}}$ depends, in a sense to be made precise, only on G.

2.2. *Definition of* $S(G)$. Let (N, H) be an admissible pair of G. As in Subsection 1.13, let P denote the largest subgroup of H, containing N and such that

$$\Sigma^c_{N_{ab}} \subseteq S(H/N, P/N) = \{[v] \mid v : H/N \to \mathbb{R} \text{ and } v|_{P/N} = 0\}.$$

There is another characterization of P which will turn out to be very useful.

Proposition 2.1. *The subgroup P is the largest locally polycyclic normal subgroup of H; in particular, P is a normal subgroup of G and it contains every nilpotent normal subgroup of H.*

Proof. First we show that P is locally polycyclic. For each $q \in P/N$ there exists a polynomial $f(X) \in \mathbb{Z}[X]$ having both leading and trailing coefficient equal to 1 and such that $f(q)$ annihilates N_{ab}. This implies that each finitely generated P/N-submodule of N_{ab} is finitely generated qua $\mathbb{Z}$-module, a property that is inherited by all tensor powers $\bigotimes^i N_{ab}$ of N_{ab}, the action being diagonal. But each factor $\gamma_i N/\gamma_{i+1} N$ is a $\mathbb{Z}(H/N)$-homomorphic image of the i-th tensor power $\bigotimes^i N_{ab}$. Using the fact that polycyclic groups are finitely presented the claim is easily completed by induction.

Second, we prove that P is the largest normal subgroup of H which is locally polycyclic. Let $P_1 \lhd H$ be another normal subgroup which is locally polycyclic. Then so is their product $P_1 \cdot P$ by a result of Baer's (see, e.g., [8, Theorem 2.31]). It follows that H contains a largest locally polycyclic normal subgroup, say P_2. Choose a finite subset $\{a_1, a_2, \ldots, a_m\} \subseteq N_{ab}$ which generates N_{ab} qua H/N-module;

such a subset exists since G and H are finitely generated groups and H/N is finitely presented. Let $S \leq P_2/\gamma_2 N$ be a finitely generated subgroup containing $\{a_1, \ldots, a_m\}$ and such that $SN_{ab} = P_2/\gamma_2 N$. Then S will be polycyclic and hence

$$\mathbb{Z}(P_2/N) \cdot a_1 + \cdots + \mathbb{Z}(P_2/N) \cdot a_m \subseteq S \cap N_{ab}$$

finitely generated qua $\mathbb{Z}$-module. This implies that every $q \in P_2/N$ is integral with respect to $N_{ab} = \mathbb{Z}(H/N)a_1 + \cdots + \mathbb{Z}(H/N)a_m$, whence $P_2 \subseteq P$ by the definition of P. The remaining assertions are now plain. □

If X is any group, the result of Baer's quoted above implies that X contains a largest locally polycyclic normal subgroup $P(X) \lhd X$. If $Y \lhd X$ is a normal subgroup of X then $P(Y) = Y \cap P(X)$; for $Y \cap P(X) \subseteq P(Y)$ by the maximality of $P(Y)$ and $P(Y) \subseteq P(X)$ since $P(Y)$ is normal in X, being characteristic in Y and Y being normal in X. Going back to an admissible pair (N, H) of our group G we see, in particular, that $P = P(H)$ depends only on H, and not on N as one might expect from the original definition in 1.13; that $N \leq P(H)$ and that $P(H) = H \cap P(G)$.

If (N, H) and (N', H') are admissible pairs of G, the new pair $(N_1 = N \cap N', H_1 = H \cap H')$ is also admissible (see Lemma 2.2 below). By the preceding remarks

$$P(H_1) = H \cap H' \cap P(G) = H \cap P(H') = P(H) \cap H'.$$

The inclusions $\iota: H_1 \to H$ and $\iota': H_1 \to H'$ induce therefore injections $\iota'_*: H_1/P(H_1) \rightarrowtail H/P(H)$ and $\iota'_*: H_1/P(H_1) \rightarrowtail H'/P(H')$; as their cokernels are finite, they give in turn rise to homeomorphisms $(\iota_*)^*: S(H/P(H)) \xrightarrow{\sim} S(H_1/P(H_1))$ and $(\iota_*)^*: S(H'/P(H')) \xrightarrow{\sim} S(H_1/P(H_1))$. In particular, we obtain a canonical homeomorphism

$$\kappa_{(H,H')}: S(H/P(H)) \xrightarrow{\sim} S(H_1/P(H_1)) \xleftarrow{\sim} S(H'/P(H')). \tag{2.1}$$

As $\kappa_{(H,H')}$ is induced by inclusions the family of those normal subgroups $H \lhd G$ for which there exists a nilpotent normal subgroup N such that (N, H) is admissible, leads to a compatible system of homeomorphisms $\kappa_{(H,H')}$ between the family of valuation spheres $S(H/P(H))$ so that we can identify them to obtain a single sphere, denoted $S(G)$.

Lemma 2.2. *Let (N, H) be an admissible pair and $N' \lhd H' \lhd G$ normal subgroups of G. Assume H'/N' is Abelian and G/H' finite. Then $(N \cap N', H \cap H')$ is admissible.* □

2.3. *The Comparison Theorem.* If (N, H) is an admissible pair of G, let $\pi: H/N \to H/P(H)$ denote the canonical projection. It induces an injection

$$\pi^*: S(H/P(H)) \xrightarrow{\sim} S(H/N, P(H)/N) \subseteq S(H/N).$$

By the original definition of $P = P(H)$ we know that $\Sigma^c_{N_{ab}} \subseteq S(H/N, P/N)$ and so

$$\sigma(N, H, G) = (\pi^*)^{-1}(\Sigma^c_{N_{ab}}) \tag{2.2}$$

is an isomorphic copy of $\Sigma^c_{N_{ab}}$. The inclusion $\sigma(N,H,G) \subseteq S(H/N,P/N)$ depends essentially only on G, as can be seen from the next result.

Theorem 2.3. *Let (N,H) and (N',H') be admissible pairs of a finitely generated group G in $\mathfrak{NA}\mathfrak{F}$. Then the canonical homeomorphism $\kappa_{(H,H')}: S(H/P(H)) \xrightarrow{\sim} S(H'/P(H'))$ maps $\sigma(N,H,G)$ onto $\sigma(N',H',G)$.*

Proof. Set $N_1 = N \cap N'$, $H_1 = H \cap H'$ as well as $Q = H/N$, $Q' = H'/N'$ and $Q_1 = H_1/N_1$. Note that the canonical images of Q_1 in Q respectively Q' have finite index. The following diagram summarizes the situation:

$$\begin{array}{ccccc} S(H/P(H)) & \xrightarrow{\sim} & S(H_1/P(H_1)) & \xleftarrow{\sim} & S(H'/P(H')) \\ \wr\big\downarrow \pi^* & & \wr\big\downarrow \pi_1^* & & \wr\big\downarrow \pi'^* \\ S(Q,P(H)/N) & \xrightarrow{\iota^* \sim} & S(Q_1,P(H_1)/N_1) & \xleftarrow{\sim} & S(Q',P(H')/N') \\ \cup & & \cup & & \cup \\ \Sigma^c_{N_{ab}} & & \Sigma^c_{(N_1)_{ab}} & & \Sigma^c_{N'_{ab}} \end{array}$$

Both squares commute. If we denote by $(N_{ab})_1$ the Q-module N_{ab} viewed as Q_1-module, then $\iota^*(\Sigma^c_{N_{ab}}) = \Sigma^c_{(N_{ab})_1} \subseteq S(Q_1)$ by Subsection 1.4, and a similar statement holds for $\Sigma^c_{N'_{ab}}$. It will therefore suffice to prove that the subsets $\Sigma^c_{(N_1)_{ab}}$ and $\Sigma^c_{(N_{ab})_1}$ of $S(Q_1)$ coincide. The inclusion $N_1 \hookrightarrow N$ induces a Q_1-module homomorphism factoring canonically like this:

$$(N_1)_{ab} = N_1/[N_1,N_1] \xrightarrow{\alpha} N_1/[N_1,N] \xrightarrow{\beta} N/[N,N] = (N_{ab})_1;$$

we shall prove that both α and β induce equalities

$$\Sigma^c_{(N_1)_{ab}} = \Sigma^c_{N_1/[N_1,N]} = \Sigma^c_{(N_{ab})_1}.$$

As to β, consider the exact sequence of Q_1-modules

$$H_2(N/N_1;\mathbb{Z}) \to N_1/[N_1,N] \xrightarrow{\beta} (N_{ab})_1 \to (N/N_1)_{ab} \to 0$$

associated to the extension $N_1 \rightarrowtail N \twoheadrightarrow N/N_1$. Since N/N_1 is contained in G/N_1, which is a finitely generated, Abelian-by-finite group, it is finitely related and thus $H_2(N/N_1;\mathbb{Z})$ is a finitely generated Abelian group, as is $(N/N_1)_{ab}$. Now Σ^c_B is empty whenever B is a module whose additive group if finitely generated (see, e.g., [4, Proposition 2.1]). This fact, in conjunction with Lemma 1.1 (c), implies that $\Sigma^c_{N_1/[N_1,N]} = \Sigma^c_{(N_{ab})_1}$.

Concerning α, note first that

$$N_1/[N_1,N] \cong (N_1)_{ab}/(\mathrm{I}K)\cdot(N_1)_{ab},$$

where IK denotes the augmentation ideal of $K = N/N_1$. As $N/H_1 \cap N$ is finite the centralizer $H_2 = C_{H_1}(K) = C_{H_1}(N/N_1)$ has finite index in H_1 and it contains, of course, N_1. Set $Q_2 = H_2/N_1$ and view $(N_1)_{ab}$ and $N_1/[N_1, N]$ as Q_2-modules. The inclusion $Q_2 \hookrightarrow Q_1$ has finite cokernel and so induces a homeomorphism $S(Q_1) \xrightarrow{\sim} S(Q_2)$; it will suffice to establish that $\Sigma^c_{(N_1)_{ab}} = \Sigma^c_{N_1/[N_1,N]}$ as subsets of $S(Q_2)$; see Subsection 1.4. Lemma 1.1 (b) tells us that this goal will be reached by proving that the radicals of the annihilators $\mathrm{Ann}_{\mathbb{Z}Q_2}(N_1)_{ab}$ and $\mathrm{Ann}_{\mathbb{Z}Q_2} N_1/[N_1, N]$ coincide. Clearly $\mathrm{Ann}(N_1)_{ab} \subseteq \mathrm{Ann}\, N_1/[N_1, N]$. Conversely, let $\lambda \in \mathbb{Z}Q_2$ annihilate $N_1/[N_1, N]$. Since N is nilpotent there exists a natural number m such that $(\mathrm{IK})^m \cdot (N_1)_{ab} = 0$; here $K = N/N_1$. But $\lambda \cdot (N_1)_{ab} \subseteq (\mathrm{IK}) \cdot (N_1)_{ab}$ and hence

$$\begin{aligned}\lambda^m (N_1)_{ab} &\subseteq \lambda^{m-1}(\mathrm{IK}) \cdot (N_1)_{ab} = (\mathrm{IK}) \cdot \lambda^{m-1}(N_1)_{ab} \\ &\subseteq \cdots \subseteq (\mathrm{IK})^m (N_1)_{ab} = 0.\end{aligned}$$

(The displayed equality holds since H_2 centralizes K). □

2.4. *Definition of $\sigma(G)$.* The canonical homeomorphisms make it possible to define the valuation sphere $S(G)$ by identifying the valuation spheres $S(H/P(H))$, where H runs over the normal subgroups of G for which there exists an admissible pair (N, H). The Comparison Theorem 2.3 allows to identify the subsets $\sigma(N, H, G)$, defined in (2.2), in a parallel manner. The resulting subset of $S(G)$ will be denoted by $\sigma(G)$.

3. Homomorphic images and subgroups

3.1. Let G_1 and G be finitely generated groups in $\mathfrak{NA}\mathfrak{F}$ and let $\psi: G_1 \to G$ be a homomorphism. If (N_1', H_1') and (N, H) are admissible pairs of G_1 and G, respectively, consider the new pair $(N_1, H_1) = (N_1' \cap \psi^{-1}(N),\ H_1' \cap \psi^{-1}(H))$ of G_1. It is admissible by Lemma 2.2 and clearly $\psi(N_1) \subseteq N$ and $\psi(H_1) \subseteq H$, so that we have an induced homomorphism $\varphi = \psi_*: Q_1 = H_1/N_1 \to Q = H/N$. However, in spite of the existence of admissible pairs compatible with ψ, one cannot define $\varphi^*: S(G) \dashrightarrow S(G_1)$ taking $\sigma(G)$ into $\sigma(G_1)$ in the obvious way. For as im ψ need not be normal in G, $\psi(P(H_1))$ need not be contained in $P(H)$, whence $\varphi = \psi_*: H_1/P(H_1) \dashrightarrow H/P(H)$ cannot be defined in the obvious way. In addition, the subsets $\Sigma^c_{N_{ab}} \subseteq S(Q)$ and $\Sigma^c_{(N_1)_{ab}} \subseteq S(Q_1)$ are related in the following complicated way (use Lemma 1.1 (a)):

$$\begin{array}{ccccc}
\Sigma^c_{(\psi(N_1)\cdot\gamma_2 N/\gamma_2 N)} \cap S(Q, \mathrm{im}\,\varphi)^c & \subseteq & \Sigma^c_{N_{ab}} \cap S(Q, \mathrm{im}\,\varphi)^c & \subseteq & S(Q) \\
\downarrow \varphi^* & & \downarrow & & \\
\Sigma^c_{((\psi(N_1)\cdot\gamma_2 N)/\gamma_2 N)_1} & \subseteq & \Sigma^c_{(N_1)_{ab}} & \subseteq & S(Q_1).
\end{array}$$

We describe below three situations in which useful results can be obtained.

3.2. Epimorphisms behave nicely. With the obvious definitions we have:

Proposition 3.1. *If $\pi : G_1 \twoheadrightarrow G$ is an epimorphism of finitely generated groups in $\mathfrak{N}\mathfrak{A}\mathfrak{F}$ then π induces an embedding $\varphi^* = (\pi_*)^* : S(G) \rightarrowtail S(G_1)$ mapping $\sigma(G)$ into $\sigma(G_1)$.*

Proof. If (N_1, H_1) is an admissible pair of G_1, set $N = \pi(N_1)$ and $H = \pi(H_1)$. Then $\pi(P(H_1)) \subseteq P(H)$ and consequently $\varphi^* = (\pi_*)^* : S(G) \to S(G_1)$ exists and is injective. Note that $S(H/N, \operatorname{im} \varphi)$ is empty. Moreover,

$$\varphi^*(\Sigma^c_{N_{ab}}) = \Sigma^c_{(N_{ab})_1} \subseteq \Sigma^c_{(N_1)_{ab}}$$

by 1.4 and Lemma 1.1 (a). □

3.3. Next we consider inclusions $\iota : G_1 \hookrightarrow G$ where G_1 contains the nilpotent normal subgroup N of a suitable admissible pair (N, H) of G. Admittedly this case is rather special, but it leads to some instructive examples. The result below could be phrased in terms of $S(G_1), S(G)$ and $\sigma(G_1), \sigma(G)$, but we refrain from doing so lest we have to introduce an analog of $S(H/N, \operatorname{im} \varphi)^c$.

Proposition 3.2. *Let G be a finitely generated group in $\mathfrak{N}\mathfrak{A}\mathfrak{F}$, let (N, H) be an admissible pair of G and consider a finitely generated subgroup $G_1 \leq G$ containing N. Set $H_1 = G_1 \cap H$, and let $\varphi : Q_1 = H_1/N \to Q = H/N$ be the canonical map. Then $\Sigma^c_{N_{ab}}$ is contained in $S(Q, \operatorname{im} \varphi)^c$ and the projection $\varphi^* : S(Q, \operatorname{im} \varphi)^c \twoheadrightarrow S(Q_1)$ maps $\Sigma^c_{N_{ab}}$ onto $\Sigma^c_{(N_{ab})_1}$.*

Proof. The assumption that G_1 be finitely generated implies that N_{ab} is finitely generated when viewed as a $\mathbb{Z}Q_1$-module. For every $q \in Q$ there exists therefore a monic polynomial $f(X) \in \mathbb{Z}Q_1[X]$ such that $f(q) \in \operatorname{Ann}_{\mathbb{Z}Q}(N_{ab})$. As Q is finitely generated this forces $\mathbb{Z}Q/\operatorname{Ann}_{\mathbb{Z}Q}(N_{ab})$ to be finitely generated qua $\mathbb{Z}Q_1$-module. The claim now follows from Proposition 1.2. □

Example. (Cf. 1.11.) Let Q be free of rank 3 and let $Q_1 < Q$ be a direct factor of rank 2. Choose a basis $\{q_1, q_2, q_3\}$ of Q with q_1, q_2 contained in Q_1. Let Q_1 act on $\mathbb{Q}$ via a homomorphism $\kappa : Q_1 \to \mathbb{Q}^\times$ and set $A_1 = \mathbb{Z}Q_1 \cdot 1 \subset \mathbb{Q}$. The additive group of A_1 is then $\mathbb{Z}[1/q_1 \cdots q_l]$, where $p_1, \ldots, p_l$ are the primes occurring in either $\kappa(q_1)$ or $\kappa(q_2)$, and so multiplication by their product $q_1 \cdots q_l$ induces an automorphism. Let q_3 act on A_1 by this automorphism and denote the resulting Q-module by A. Then the split extension $G_1 = A_1 \rtimes Q_1$ is in an obvious way a (normal) subgroup of $G = A \rtimes Q$. By the preceding proposition $\varphi^*(\Sigma^c_A) = \Sigma^c_{A_1}$. Indeed we can be quite explicit (see 1.2 and 1.11): if $\kappa(q_i), i = 1$ or 2, has the prime factorization $\pm \prod p^{m_{ij}}$ then

$$\Sigma^c_A = \left\{ \frac{(m_{1j}, m_{2j}, 1)}{(m_{1j}^2 + m_{2j}^2 + 1)^{\frac{1}{2}}} \;\middle|\; 1 \leq j \leq l \right\}$$

and

$$\Sigma_{A_1}^c \cong \left\{ \frac{(m_{1j}, m_{2j})}{(m_{1j}^2 + m_{2j}^2)^{\frac{1}{2}}} \;\middle|\; 1 \le j \le l \right\}, \tag{3.1}$$

and φ^* is the obvious projection. It becomes apparent from (3.1) that $\varphi^*: \Sigma_A^c \twoheadrightarrow \Sigma_{A_1}^c$ need not be injective. More importantly, one sees that Σ_A^c is contained in the upper hemisphere, whereas $\Sigma_{A_1}^c$ need not be contained in any hemisphere whatsoever. As we shall estabish in Section 5 this corresponds to the fact that G is constructible (this can readily be verified), whereas G_1 need not be.

3.4. For our third result we need the notion of a *k-fold sum* of a subset Σ of the valuation sphere $S(Q)$, defined inductively by

$$\Sigma^{(1)} = \Sigma, \quad \Sigma^{(2)} = \Sigma + \Sigma \quad \text{and} \quad \Sigma^{(k+1)} = \Sigma^{(k)} + \Sigma,$$

the sum $\Sigma_1 + \Sigma_2$ of the two subsets being defined by (1.1).

Proposition 3.3. *Let G be a finitely generated group in $\mathfrak{NAF}$ and G_1 a finitely generated subgroup. Let (N, H) be an admissible pair of G, set $(N_1, H_1) = (G \cap N, G_1 \cap H)$ and let $\varphi: Q_1 = H_1/N_1 \to Q = H/N$ be induced by the inclusion $G_1 \to G$. If k is a natural number with $\gamma_{k+1} N \subseteq \gamma_2 N_1$ then*

$$\{\varphi^*(\operatorname{dis} \Sigma_{N_{ab}}^c \cap S(Q, \operatorname{im} \varphi)^c)\}^{(k)} \supseteq \operatorname{dis} \Sigma_{(N_1)_{ab}}^c.$$

Proof. The Q_1-module $(N_1)_{ab}$ has a series whose factors are sections of the lower central factors $A_i = \gamma_i N / \gamma_{i+1} N$, where $1 \le i \le k$ and the A_i are viewed as Q_1-modules. Therefore Lemma 1.1 (c) yields that

$$\Sigma_{(N_1)_{ab}}^c \subseteq \bigcup_{1 \le i \le k} \Sigma_{A_i}^c \subseteq S(Q_1).$$

The i-th lower central factor $A_i = \gamma_i N / \gamma_{i+1} N$ is a homomorphic image of the i-fold tensor power of $(N_{ab})_1$ over $\mathbb{Z}$, with diagonal Q_1-action. Here $(N_{ab})_1$ denotes N_{ab} viewed as a Q_1-module. Theorem 1.2 and Lemma 1.1 (a) then give inductively

$$\operatorname{dis} \Sigma_{A_i}^c \subseteq \{\operatorname{dis} \Sigma_{(N_{ab})_1}^c\}^{(i)} \subseteq \{\operatorname{dis} \Sigma_{(N_{ab})_1}^c\}^{(k)}$$

for all $1 \le i \le k$. Consequently

$$\operatorname{dis} \Sigma_{(N_1)_{ab}}^c \subseteq \{\operatorname{dis} \Sigma_{(N_{ab})_1}^c\}^{(k)} = \{\varphi^*(\operatorname{dis} \Sigma_{N_{ab}}^c \cap S(Q, \operatorname{im} \varphi)^c\}^{(k)},$$

the equality being covered by Proposition 1.2.

3.5. For later use we record an immediate consequence:

Corollary 3.4. *If G is a finitely generated group in $\mathfrak{NAF}$ whose set $\sigma(G)$ contains at most one point then the same holds for every finitely generated subgroup G_1 of G.*

Proof. With the notation of Proposition 3.3 take k to be the nilpotency class of N. Then the proposition shows that dis $\Sigma^c_{(N_1)_{ab}}$ has at most one discrete-valuation point. Hence dis $\Sigma^c_{(N_1)_{ab}}$ coincides with its rational convex closure, and dis $\Sigma^c_{(N_1)_{ab}} = \Sigma^c_{(N_1)_{ab}}$ by Proposition 1.4. This shows, in particular, that $\Sigma^c_{(N_1)_{ab}}$ has at most one point. □

4. Ascending HNN-extensions

We consider a group $G \in \mathfrak{NAF}$ which is an HNN-extension over a *finitely generated* base group $G_1 \leq G$. Of course, G is then also finitely generated. As G is soluble-by-finite one of the associated subgroups coincides with the base group G_1, and such an HNN-extension will be said to be *ascending*. Thus there is an element $t \in G$ and an injective endomorphism $\alpha : G_1 \rightarrowtail G_1$ such that G has a presentation

$$G \cong \langle G_1, t;\ tgt^{-1} = \alpha(g) \text{ for all } g \in G_1 \rangle.$$

Note that the normal closure $\mathrm{gp}_G(G_1)$ of G_1 in G is an ascending union $\bigcup\{t^{-i}G_1t^i \mid i \in \mathbb{N}\}$ of conjugates of G_1 and that G is the split extension of $\mathrm{gp}_G(G_1)$ by the infinite cyclic group $\mathrm{gp}(t)$. We aim at giving an upper bound for $\sigma(G)$ in terms of $\sigma(G_1)$.

Proposition 4.1. *Let G in $\mathfrak{NAF}$ be an ascending* HNN-*extension over a finitely generated base group G_1 with stable letter t. Let (N, H) be an admissible pair with $N \leq \mathrm{gp}_G(G_1)$; set $(N_1, H_1) = (G_1 \cap N, G_1 \cap H)$, and let $\varphi : Q_1 = H_1/N_1 \to Q = H/N$ be induced by the inclusion $G_1 \to G$. Then*

$$\Sigma^c_{N_{ab}} \subseteq (\varphi^*)^{-1}(\Sigma^c_{(N_1)_{ab}}) \cup \{[v_0]\}.$$

Here φ^* is the projection $S(Q, \mathrm{im}\,\varphi)^c \twoheadrightarrow S(Q_1)$ and $v_0 : Q \to \mathbb{R}$ is the composite

$$Q = H/N \hookrightarrow G/N \twoheadrightarrow G/\mathrm{gp}_G(G_1) \xrightarrow{\ \sigma\ } \mathbb{R},$$

the homomorphism σ taking $t \cdot \mathrm{gp}_G(G_1)$ to $1 \in \mathbb{R}$.

Proof. Let A_1 denote the canonical image of the Q_1-module $(N_1)_{ab}$ in N_{ab}. Since the normal closure $\mathrm{gp}_G(G_1)$ of G_1 is the ascending union $\bigcup\{t^{-i}G_1t^i \mid i \in \mathbb{N}\}$, the abelian group N_{ab}, viewed as a Q_1-module, is the ascending union $\bigcup\{t^{-i}A_1t^i \mid i \in \mathbb{N}\}$ of Q_1-modules. Let d be the index of $H \cdot \mathrm{gp}_G(G_1)$ in G and put $s = t^d \cdot N \in Q$. Then $N_{ab} = \bigcup\{s^{-i}A_1 \mid i \in \mathbb{N}\}$. As Q is commutative this implies that

$$\mathrm{Ann}_{\mathbb{Z}Q_1}(N_{ab}) = \mathrm{Ann}_{\mathbb{Z}Q_1}(A_1) \supseteq \mathrm{Ann}_{\mathbb{Z}Q_1}((N_1)_{ab}).$$

By formula (1.3) and Lemma 1.1 (a) this gives

$$\varphi^*(\Sigma^c_{N_{ab}} \cap S(Q, \mathrm{im}\,\varphi)^c) \subseteq \Sigma^c_{A_1} \subseteq \Sigma^c_{(N_1)_{ab}},$$

or

$$\Sigma^{c}_{N_{ab}} \subsetneq (\varphi^*)^{-1}(\Sigma^{c}_{(N_1)_{ab}}) \cup S(Q, \operatorname{im}\varphi). \tag{4.1}$$

We show next that $\operatorname{im}\varphi$ and $\ker v_0$ coincide. Indeed, since G/N cannot have an infinite strictly ascending chain of subgroups and $\mathrm{gp}_G(G_1)$ contains N by assumption, one has

$$\operatorname{im}\varphi = (G_1 \cap H)\cdot N/N = (G_1\cdot N)\cap H/N = (\bigcup t^{-i}G_1t^{i}\cdot N)\cap H/N$$
$$= \mathrm{gp}_G(G_1)\cap H/N = \ker(Q = H/N \hookrightarrow G/N \twoheadrightarrow G/\mathrm{gp}_G(G_1)).$$

Hence $S(Q, \operatorname{im}\varphi) = \{[v'] \mid v'\circ\varphi = 0\} = \{[v_0], [-v_0]\}$, revealing that (4.1) will imply our contention once we have shown that $[-v_0] \notin \Sigma^{c}_{N_{ab}}$. Now G_1 is finitely generated by assumption, and so $(N_1)_{ab}$ and its homomorphic image $A_1 \leq N_{ab}$ are finitely generated Q_1-modules. From the fact that A_1 is stable under the action of s we infer that there is a monic polynomial $f(X) \in \mathbb{Z}Q_1[X]$ of degree m, say, which annihilates A_1 and hence all of N_{ab}. But then $\lambda = 1 - f(s)\cdot s^{-m} \in \mathbb{Z}Q$ centralizes N_{ab} and $v_0(q) < 0$ for all $q \in \operatorname{supp}(\lambda)$. We conclude from (1.2) that $[-v_0] \in \Sigma_{N_{ab}}$, as desired. □

Corollary 4.2. *Let G be a group in $\mathfrak{NAF}$ which is an* HNN*-extension over a finitely generated base group $G_1 \leq G$. If $\sigma(G_1)$ is contained in an open hemisphere then so is $\sigma(G)$.*

Proof. Let N, H, N_1, H_1, $\varphi: Q_1 \to Q$ and s be as in the proof of Proposition 4.1. By assumption $\Sigma_1 := \Sigma^{c}_{(N_1)_{ab}} \subsetneq S(Q_1)$ is contained in an open hemisphere. As Σ_1 is closed (see 1.2) and hence compact it follows that Σ_1 is contained in an open hemisphere of the form $H_{q_1} = \{[v_1] \mid v_1(q_1) > 0\}$, where $q_1 \in Q_1$. Hence $(\varphi^*)^{-1}(\Sigma_1) \subsetneq H_{\varphi(q_1)}$. Using the compactness of $\Sigma^{c}_{N_{ab}}$ and the fact that $[-v_0]$ is not in $\Sigma^{c}_{N_{ab}}$, one concludes by an easy geometric argument that $\Sigma^{c}_{N_{ab}} \subsetneq H_q$, where q is obtained from $\varphi(q_1)$ by 'tilting $\varphi(q_1)$ slightly in the direction of s', i.e. where $q = \varphi(q_1)^m \cdot s$ with m a sufficiently large natural number. □

5. Some applications

5.1. *Finitely presented groups.* We start by recalling a main result of [4], formulated in terms of the invariant $\sigma(?)$.

Theorem 5.1. *If G is a finitely presented group in $\mathfrak{NAF}$ then $\sigma(G)$ does not contain a pair of antipodal points. If G is a finitely generated metabelian-by-finite group and $\sigma(G)$ does not contain a pair of antipodal points then G is finitely presented.* □

5.2. *Constructible groups.* In [1] Baumslag–Bieri study groups which, roughly speaking, are built up from the trivial group by using finite extensions, amalgamated products and finite-rank HNN-extensions, where both the free factors (respectively the base group) as well as the amalgamated subgroup (respectively the associated subgroups) are groups already obtained. These groups are called *constructible*; they contain in general non-abelian free subgroups. For *soluble-by-finite, constructible* groups the definition can be recast and reads then like this [1, Subsection 3.2]: a soluble-by-finite group G is called *constructible* if it can be obtained from the trivial group by applying finitely many times the processes of performing an ascending HNN-extension over a group already obtained, or passing to a supergroup of finite index of a group already obtained. The group G is necessarily finitely generated and of finite Prüfer rank, and so in $\mathfrak{N}\mathfrak{A}\mathfrak{F}$ (see, e.g., [8, Part (a) of the proof of Theorem 10.38] for the last assertion).

From the fact that $\sigma(G_1)$ and $\sigma(G)$ coincide if G_1 has finite index in G, and Proposition 4.2 it is immediate that $\sigma(G)$ is contained in an open hemisphere if G is a constructible group in $\mathfrak{N}\mathfrak{A}\mathfrak{F}$. Here we prove the converse:

Theorem 5.2. *A finitely generated, soluble-by-finite group G is constructible if and only if $G\in\mathfrak{N}\mathfrak{A}\mathfrak{F}$ and $\sigma(G)$, (or, equivalently,* $\operatorname{dis}\sigma(G)$*) is contained in an open hemisphere. If so, $\sigma(G)$ is finite and consists entirely of discrete-valuation points.*

Proof. The remark concerning $\operatorname{dis}\sigma(G)$ follows from Proposition 1.4 and the compactness of $\sigma(G)$. Next let (N,H) be an admissible pair for G, and set $Q=H/N$. Since $\Sigma^c_{N_{ab}}\subseteq S(Q)$ is contained in an open hemisphere, it is by [5, Corollary 4.7] finite and made up of discrete-valuation points. Also there exist by [5, Theorem 4.6] a maximal $\mathbb{Z}$-linearly independent subset $\{q_1,\dots,q_n\}\subseteq Q$ and a finitely generated subgroup $B_0\le N_{ab}$ such that

$$q_jB_0\subsetneq B_0 \quad \text{for } 1\le j\le n, \tag{5.1}$$

and

$$N_{ab}=\bigcup_{i\ge 1}(q_1\cdots q_n)^{-i}B_0. \tag{5.2}$$

Let $x_1,\dots,x_n$ be preimages in H of $q_1,\dots,q_n$. We are going to prove by induction on $l\ge 1$ that there exist finitely generated subgroups $B_l\le N/\gamma_{l+1}N$ such that with $\bar{x}_j=x_j\cdot\gamma_{l+1}N$

$$\bar{x}_jB_l\bar{x}_j^{-1}\subseteq B_l \quad \text{for } 1\le j\le n, \tag{5.3}$$

$$B_l \text{ generates } N/\gamma_{l+1}N \text{ as a } \operatorname{gp}(\bar{x}_1,\dots,\bar{x}_n)\text{-operator group,} \tag{5.4}$$

and

$$B_l \text{ contains the commutators } [\bar{x}_j,\bar{x}_k] \text{ for } 1\le j<k\le n. \tag{5.5}$$

If $l=1$, choose $i\in\mathbb{N}$ so large that all commutators $[\bar{x}_j,\bar{x}_k]$ are contained in

$(q_1\cdots q_n)^{-i}B_0$ and set $B_1=(q_1\cdots q_n)^{-i}B_0$. Then clearly (5.3)–(5.5) hold. If $l>1$, let $\tilde{B}_{l-1}$ be a finitely generated subgroup of $\tilde{N}=N/\gamma_{l+1}N$ whose canonical image in $N/\gamma_l N$ coincides with B_{l-1}. Then there exists a *finitely generated,* central subgroup $A\subseteq\gamma_l N/\gamma_{l+1}N$ such that

$$\tilde{x}_j\tilde{B}_{l-1}\tilde{x}_j^{-1}\subseteq A\cdot\tilde{B}_{l-1}\quad\text{for } 1\le j\le n,$$

and

$$A\cdot\tilde{B}_{l-1}\text{ contains all the commutators }[\tilde{x}_j,\tilde{x}_k].$$

On the other hand, commutation induces a Q-module epimorphism

$$\underbrace{N_{\mathrm{ab}}\otimes N_{\mathrm{ab}}\otimes\cdots\otimes N_{\mathrm{ab}}}_{l\text{-factors}}\twoheadrightarrow\gamma_l N/\gamma_{l+1}N,$$

the tensor power having diagonal Q-action. The image A_0 of $B_0\otimes B_0\otimes\cdots\otimes B_0$ in $\gamma_l N/\gamma_{l+1}N$ is then a finitely generated Abelian subgroup having properties analogous to (5.1) and (5.2). In particular, there exists a natural number i such that the finitely generated Abelian group $A\le\gamma_l N/\gamma_{l+1}N$ is contained in $(q_1\cdots q_n)^{-i}A_0$. Then

$$B_l=((q_1\cdots q_n)^{-i}A_0)\cdot\tilde{B}_{l-1}\le\tilde{N}$$

is a finitely generated subgroup enjoying the desired properties (5.3)–(5.5).

Let c be the nilpotency class of N, and set $H_0=B_c\le N=N/\gamma_{c+1}N$, and inductively $H_j=\mathrm{gp}(H_{j-1},x_j)$. We claim that H_j is an HNN-extension with base group H_{j-1}. Firstly, $x_jH_{j-1}x_j^{-1}\subseteq H_{j-1}$; for H_0 has this property and it contains the commutators $[x_k,x_j]$ for all $1\le k<j$. Let $\alpha_j:H_{j-1}\rightarrowtail H_{j-1}$ denote the injective endomorphism induced by conjugation by x_j and consider the HNN-extension

$$E=\langle H_{j-1},t;\ tht^{-1}=\alpha_j(h)\text{ for all }h\in H_{j-1}\rangle.$$

The inclusion $H_{j-1}\le H_j$ and the assignment $t\mapsto x_j$ induce an epimorphism $\varphi:E\twoheadrightarrow B_j$. Since φ induces an isomorphism of the infinite cyclic groups $E/\mathrm{gp}_E(H_{j-1})$ and $H_j/\mathrm{gp}_{H_j}(H_{j-1})$, one has $\ker\varphi\subseteq\mathrm{gp}_E(H_{j-1})$. But the restriction of φ to H_{j-1} is injective and $\mathrm{gp}_E(H_{j-1})$ is the ascending union of conjugates of H_{j-1}, hence $\ker\varphi=\{1\}$. It follows that H_n is a constructible group containing N and preimages of a maximal $\mathbb{Z}$-linearly independent subset of $Q=H/N$. So H_n has finite index in H, hence in G and G is constructible. □

5.3. Remark. Every soluble-by-finite, constructible group is (torsion-free)-by-finite [1, Subsection 3.3, Remark (2)]. This fact and the proof of the preceding theorem imply the following reduction result:

Every constructible, soluble-by-finite group contains a (soluble) group of finite index which can be obtained from the trivial group by a finite sequence of HNN-*extensions.*

5.4. *Coherent groups.* A group is said to be coherent if every finitely generated subgroup is finitely related. The soluble coherent groups have been determined in [3] and [7]: they are locally either polycyclic or ascending HNN-extensions over polycyclic groups. Here we prove (without using this earlier result):

Theorem 5.3. A finitely generated, soluble-by-finite group G is coherent if and only if $G \in \mathfrak{N}\mathfrak{A}\mathfrak{F}$ and $\sigma(G)$ (or, equivalently, $\operatorname{dis}\sigma(G)$) consists of at most one point.

Proof. The remark concerning $\operatorname{dis}\sigma(G)$ is immediate from Proposition 1.4. Assume first G is in $\mathfrak{N}\mathfrak{A}\mathfrak{F}$ and $\sigma(G)$ consists of at most one point. Then the same is true by Corollary 3.4 for every finitely generated subgroup $G_1 \leq G$. But a subset Σ_1 of a sphere consisting of at most one point is certainly contained in an open hemisphere. Hence every finitely generated subgroup $G_1 \leq G$ is constructible by Theorem 5.1, and so, in particular, finitely presented. This proves that G is coherent.

Conversely, assume G is a finitely generated soluble-by-finite, coherent group. Then G is of finite Prüfer rank. To establish this we can assume without loss that G is soluble and then use induction on the derived length of G. Let K be a subgroup of finite index in G with $[G,G] \leq K \leq G$ and such that $K/[G,G]$ is free Abelian of finite rank m. Then K is still finitely generated and hence by assumption finitely presented. By repeatedly using the key result of [2] and the coherence of G we obtain a chain of finitely generated subgroups

$$K = K_0 \geq K_1 \geq K_2 \geq \cdots \geq K_m$$

such that K_i is an ascending HNN-extension over K_{i+1} for $0 \leq i < m$ and $K_m \leq [G,G]$. By induction K_m is of finite Prüfer rank, hence so is K and G.

Now if G is a finitely generated soluble-by-finite group of finite Prüfer rank we infer, on the one hand, that $G \in \mathfrak{N}\mathfrak{A}\mathfrak{F}$ (see, e.g., [8], Part (a) of the proof of Theorem 10.38) and, on the other hand, that $\sigma(G)$ is finite and coincides with $\operatorname{dis}\sigma(G)$ [5, Proposition 2.5]. Let (N,H) be an admissible pair of G and assume $\Sigma^c_{N_{ab}} \subsetneq S(Q)$ contains two points $[v] \neq [w]$. If $[v]$ and $[w]$ are not antipodal, then $v + w : Q \to \mathbb{Z}$ is a non-trivial homomorphism. By the key result of [2] G is an HNN-group over a finitely generated base group $G_1 \leq \pi^{-1}(\ker v) \leq H$; here $\pi : H \twoheadrightarrow H/N = Q$ denotes the canonical projection. Let φ be the canonical embedding $Q_1 = (G_1 \cap H)/(G_1 \cap N) \to Q = H/N$. Then as in the proof of Proposition 4.1 $\varphi(Q_1) = \ker(v + w)$ and hence $[v \circ \varphi]$ and $[w \circ \varphi]$ are by construction two antipodal points of $S(Q_1)$. Moreover, they belong by formula (1.3) to $\Sigma^c_{(G_1 \cap N)_{ab}}$. Thus either $\Sigma^c_{N_{ab}}$ or $\Sigma^c_{(G_1 \cap N)_{ab}}$ contains a pair of antipodal points, whence by Theorem 5.1 either G or G_1 cannot be finitely presented, and G is not coherent. □

5.5. *Polycyclic groups.* Similarly we have:

Theorem 5.4. *A finitely generated soluble-by-finite group G is polycyclic-by-finite if and only if $G \in \mathfrak{N}\mathfrak{A}\mathfrak{F}$ and $\sigma(G) = \emptyset$, (or, equivalently* $\operatorname{dis}\sigma(G) = \emptyset$).

Proof. Polycyclic-by-finite groups are finitely generated and of finite Prüfer rank and hence in $\mathfrak{N}\mathfrak{A}\mathfrak{F}$. Let (N, H) be an admissible pair of G. By [4, Theorem 2.4] $\Sigma^c_{N_{ab}}$ is empty if and only if N_{ab} is finitely generated qua $\mathbb{Z}$-module and this, in turn, is equivalent with N being finitely generated nilpotent and hence with G being polycyclic-by-finite. □

References

[1] G. Baumslag and R. Bieri, Constructible solvable groups, Math. Z. 151 (1976) 249–257.

[2] R. Bieri and R. Strebel, Almost finitely presented soluble groups, Comment. Math. Helvetici 53 (1978) 258–278.

[3] R. Bieri and R. Strebel, Soluble groups with coherent group rings, in: C.T.C. Wall, Ed., Homological Group Theory, London Math. Soc. Lecture Note Series 36 (Cambridge University Press, Cambridge, 1979) 235–240.

[4] R. Bieri and R. Strebel, Valuations and finitely presented metabelian groups, Proc. London Math. Soc. 41 (3) (1980) 439–464.

[5] R. Bieri and R. Strebel, A geometric invariant for modules over an Abelian group, J. Reine Angew. Math. 322 (1981) 170–189.

[6] N. Bourbaki, Commutative Algebra (Hermann, Paris and Addison-Wesley, Reading, MA, 1972).

[7] J.R.J. Groves, Soluble groups in which every finitely generated subgroup is finitely presented, J. Australian Math. Soc. Ser. A 26 (1978) 115–125.

[8] D.J.S. Robinson, Finiteness conditions for soluble groups I and II, Ergebnisse der Math. und ihrer Grenzgebiete 62 and 63 (Springer, Berlin–Heidelberg–New York, 1972).