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A Gröbner basis criterion for birational equivalence of affine varieties

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Abstract

This paper presents a Gröbner basis criterion to determine whether a given rational map of two affine varieties is birational and if so, to compute the inverse. Also, with the help of Gröbner basis, we can compute the image of any rational map between two affine varieties. © 1998 Elsevier Science B.V.

1. Introduction and preliminaries

Throughout the paper, we assume that k is an algebraically closed field in order to deal with our proof easily, though some of our results even hold when k is only an infinite field.

In [4, 7, 8], their authors provided algorithmic solutions with a Gröbner basis criterion for isomorphisms of algebraic varieties. Using the criterion, they could determine whether a given polynomial map of algebraic sets is an isomorphism and, if so, to compute the inverse.

This paper generalizes the result in [4, 7, 8] to a given rational map determined by a rational parametrization from a polynomial map. Let $F: k^m \cdots \to k^n$ be a rational map determined by the following rational parametrization $F = (\frac{f_1(X)}{g_1(X)}, \dots, \frac{f_n(X)}{g_n(X)})$, where $f_i(X), g_i(X) \in k[x_1, \dots, x_m]$, and let $V \subset k^m$ be a non-empty affine variety, $h_1(X), \dots, h_s(X)$ be a system of generators of its corresponding ideal $I(V)$ in $k[x_1, \dots, x_m]$. As in [4], we raise the following two problems:

(1) Compute $I(\overline{F(V_g)})$ – the ideal of the Zariski closure of $F(V_g)$, where $V_g = V \setminus V(g)$, $g = \Pi_{i=1}^n g_i$, $V(g)$ is the algebraic set corresponding to the polynomial g in k^m.

(2) Determine whether the restriction rational map $F|_V : V \cdots \to \overline{F(V_g)}$ is birational; If so, compute the local inverse. (Recall that a rational map is called birational if it is locally bijective and its local inverse is also a rational map.) Any standard book in algebraic geometry will provide this definition (see [3]).

We assume that the reader is familiar with the notations of admissible ordering (Lexicographic order), Gröbner basis and reduced Gröbner basis (see [1, 4, 6]). To

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simplify statements, the empty set will be considered to be a (reduced) Gröbner basis of the zero ideal.

In addition to the notations above, we shall use the following abbreviations $X = (x_1, \dots, x_m)$, $Y = (y_1, \dots, y_n)$ and, consequently, $k[X]$, $k[Y]$ and $k[X, Y]$ are rings of polynomials over k in $m, n, m+n$ variables, respectively, $V(I)$ denotes the algebraic set of the ideal I, *graph* denotes the graph of a rational map.

2. Graphs of rational maps

In the last section, we define the rational map determined by a rational parametrization

$$F: V \cdots \to \overline{F(V_g)}.$$

In fact, F is defined on $V_g = V \setminus V(g)$. Then we can determine the ideal corresponding to the graph of $F|_{V_g}$: (*graph* $F|_{V_g}$).

Lemma 2.1. *With F, V, V_g, $I(V)$, $h_1(X), \dots, h_s(X)$ as above, let J be the ideal*

$$(h_1(X), \dots, h_s(X), g_1(X)y_1 - f_1(X), \dots, g_n(X)y_n - f_n(X), 1 - g(X)x)$$

$$\subset k[x, x_1, \dots, x_m, y_1, \dots, y_n],$$

and let $J' = J \bigcap k[y_1, \dots, y_n]$ be the $(m+1)$st elimination ideal. Then $V(J')$ is the smallest variety in k^n containing $F(V_g)$, i.e. $V(J') = \overline{F(V_g)}$.

Proof. Set the polynomial ring $k[x, x_1, \dots, x_m, y_1, \dots, y_n]$, which gives us the affine space k^{m+n+1}. Then consider the ideal J.

Note that the expression $1 - g(X)x = 0$ means that the denominators

$$g = \prod_{i=1}^n g_i$$

never vanish on $V(J)$. Consider the maps

$$j: V_g \to k^{m+n+1}, \qquad \pi_{m+1}: k^{m+n+1} \to k^n,$$

defined by

$$j(x_1, \dots, x_m) = \left(\frac{1}{g(x_1, \dots, x_m)}, x_1, \dots, x_m, \frac{f_1(X)}{g_1(X)}, \dots, \frac{f_n(X)}{g_n(X)}\right),$$

$$\pi_{m+1}(x, x_1, \dots, x_m, y_1, \dots, y_n) = (y_1, \dots, y_n),$$

respectively. (*Remark*: The notation of the projection map is not consistent with the one used in Proof of Theorem 5.1.).

Then we get the commutative diagram

$$\begin{array}{ccc} & k^{m+n+1} & \\ {}^{j}\nearrow & & \searrow{}^{\pi_{m+1}} \\ V_g & \rightarrow & F(V_g) \subset k^n \end{array}$$

i.e. $F = \pi_{m+1} \circ j$, we claim that $j(V_g) = V(J)$. To see this, note that $j(V_g) \subset V(J)$ follows easily from the definitions of j and J. On the other hand, if

$$(x, x_1, \ldots, x_m, y_1, \ldots, y_n) \in V(J),$$

then $g(x_1, \ldots, x_m)x = 1$ implies that none of the $g_i's$ vanish at $(x_1, \ldots, x_m)$ and thus, $g_i(x_1, \ldots, x_m)y_i = f_i(x_1, \ldots, x_m)$ can be solved for

$$y_i = \frac{f_i(x_1, \ldots, x_m)}{g_i(x_1, \ldots, x_m)}.$$

Since $x = 1/g(x_1, \ldots, x_m)$, it follows that our point lies in $j(V_g)$. This proves $V(j) \subset j(V_g)$.

From $F = \pi_{m+1} \circ j$ and $j(V_g) = V(J)$, we obtain

$$F(V_g) = \pi_{m+1}(j(V_g)) = \pi_{m+1}(V(J)).$$

Thus, the image of the rational map $F|_{V_g}$ is the projection of the variety $V(J)$, and our lemma follows. □

Remark 2.1. Theorem 2 in [2, p. 131] is just a special case of Lemma 2.1 for $V = k^m$.

Lemma 2.2. $I(\mathit{graph}\ F|_{V_g}) = J \bigcap k[x_1, \ldots, x_m, y_1, \ldots, y_n]$.

Proof. Since

$$I(\mathit{graph}\ F|_{V_g}) \supset J \bigcap k[x_1, \ldots, x_m, y_1, \ldots, y_n],$$

we only need to show that

$$I(\mathit{graph}\ F|_{V_g}) \subset J \bigcap k[x_1, \ldots, x_m, y_1, \ldots, y_n].$$

For any $P(X, Y) \in k[X, Y]$, write

$$P = \sum a_{i_1, \ldots, i_n}(X) y_1^{i_1} \cdots y_n^{i_n}.$$

So

$$P = (1 - g(X)x)P + g(X)xP$$
$$= (1 - g(X)x)P + x \sum a_{i_1, \dots, i_n}(X) g_1(X) \cdots g_n(X) y_1^{i_1} \cdots y_n^{i_n}. \tag{2.1}$$

Let $deg(P)_Y$ be the total degree of P with respect to Y. Then we shall use induction on $deg(P)_Y$, to show that $h \in J$ if $P \in I$ ($graph\ F|_{V_g}$).

Suppose that $P \in I(graph\ F|_{V_g})$.

(i) $deg(P)_Y = 0$. In this case, $P \in k[x_1, \dots, x_m]$ and $P = (1 - g(X)x)P + xg(X)P$. Since P is 0 on V_g, $g(X)P \in I(V)$. Hence, P lies in J.

(ii) $deg(P)_Y \neq 0$. In this case, there exists some coefficients $a_{i_1, \dots, i_n}(X) \neq 0$ for some $(i_1, \dots, i_n) \neq 0$. We may assume that $i_1 \neq 0$ in some non-zero coefficient.

Then

$$a_{i_1, \dots, i_n}(X) g(X) y_1^{i_1} \cdots y_n^{i_n}$$
$$= a_{i_1, \dots, i_n}(X) g_2(X) \cdots g_n(X)(g_1(X) y_1) y_1^{i_1 - 1} \cdots y_n^{i_n}$$
$$= a_{i_1, \dots, i_n}(X) g_2(X) \cdots g_n(X)(g_1(X) y_1 - f_1(X)) y_1^{i_1 - 1} \cdots y_n^{i_n}$$
$$+ a_{i_1, \dots, i_n}(X) g_2(X) \cdots g_n(X) f_1(X) y_1^{i_1 - 1} \cdots y_n^{i_n}.$$

Using this method step by step, finally, P can be written as follows;

$$P = P_1 + P_2,$$

were $P_1 \in J$ and $P_2 \in k[x, x_1, \dots, x_m, y_1, \dots, y_n]$ and $deg(P_2)_Y < deg(P)_Y$. Hence, by induction hypothesis, we can easily finish the proof of the lemma. □

Lemma 2.3. *Fix* $i \in \{1, \dots, m\}$ *and let* $G = a_i(Y)/b_i(Y)$, *where* $a_i(Y), b_i(Y) \in k[Y]$, $b_i \neq 0$. *Then*

$$b_i(Y) x_i - a_i(Y) \in I(graph\ F|_{V_g})$$

iff

$$d(X)\left[b_i\left(\frac{f_1(X)}{g_1(X)}, \dots, \frac{f_n(X)}{g_n(X)}\right) x_i - a_i\left(\frac{f_1(X)}{g_1(X)}, \dots, \frac{f_n(X)}{g_n(X)}\right)\right] \in I(V),$$

where for any $d(X) \in k[x_1, \dots, x_m]$
such that $d(X)[b_i(\frac{f_1(X)}{g_1(X)}, \dots, \frac{f_n(X)}{g_n(X)}) x_i - a_i(\frac{f_1(X)}{g_1(X)}, \dots, \frac{f_n(X)}{g_n(X)})] \in k[x_1, \dots, x_m]$.

Proof. $\Rightarrow$ If $b_i(Y)x_i - a_i(Y) \in I(graph\ F|_{V_g})$, then

$$d(X)\left[b_i\left(\frac{f_1(X)}{g_1(X)}, \dots, \frac{f_n(X)}{g_n(X)}\right) x_i - a_i\left(\frac{f_1(X)}{g_1(X)}, \dots, \frac{f_n(X)}{g_n(X)}\right)\right] \in I(V_g).$$

Since, $\bar{V}_g = V$,

$$d(X)\left[b_i\left(\frac{f_1(X)}{g_1(X)}, \dots, \frac{f_n(X)}{g_n(X)}\right)x_i - a_i\left(\frac{f_1(X)}{g_1(X)}, \dots, \frac{f_n(X)}{g_n(X)}\right)\right] \in I(V).$$

$\Leftarrow$ Set $U = V_g \setminus V(d(X))$. So we have

$$\left[b_i\left(\frac{f_1(X)}{g_1(X)}, \dots, \frac{f_n(X)}{g_n(X)}\right)x_i - a_i\left(\frac{f_1(X)}{g_1(X)}, \dots, \frac{f_n(X)}{g_n(X)}\right)\right] = 0$$

on U.

Consider the projection

$$\pi : k^{m+n} \to k^m$$

$$(x_1, \dots, x_m, y_1, \dots, y_n) \mapsto (x_1, \dots, x_m).$$

Then $\pi^{-1}(U) \bigcap (graph\ F|_{V_g})$ is an open subset of $graph\ F|_{V_g}$ in Zariski topology and $b_i(Y)x_i - a_i(Y)$ is 0 on $\pi^{-1}(U) \bigcap graph\ F|_{V_g}$. Thus, $b_i(Y)x_i - a_i(Y) \in I(graph\ F|_{V_g})$ since V and $graph\ F|_{V_g}$ both are irreducible. $\square$

3. Ordering for Gröbner basis

In order to deal with our problems using Gröbner basis, we must first impose the admissible ordering (Lexicographic order) on N^{m+n+1} used, where N is the set of integers $>$ or equal 0.

Let $k[x, x_1, , \dots, x_m, y_1, \dots, y_n]$ and $k[x_1, \dots, x_m, y_1, \dots, y_n]$ be two polynomial rings. We will use the following lexicographic order (still denoted by $>$) so that

$$x > x_1 > \cdots > x_m > y_1 > \cdots > y_n,$$

and

$$x_1 > \cdots > x_m > y_1 > \cdots > y_n.$$

The definition of lexicographic order can be found in detail in [2].

Remark 3.1. If β is a Gröbner basis (respectively, the reduced Gröbner basis) of an ideal $I \subset k[x, x_1, \dots, x_m, y_1, \dots, y_n]$ with respect to the above lexicographic order, then

$$\beta \bigcap k[x_1, \dots, x_m, y_1, \dots, y_n],$$

(respectively, $\beta \bigcap k[y_1, \dots, y_n]$), is a Gröbner basis (respectively, the reduced Gröbner basis) of $I \bigcap k[x_1, \dots, x_m, y_1, \dots, y_n]$ (respectively, $I \bigcap k[y_1, \dots, y_n]$), for the induced ordering of N^{m+n} (respectively, N^n) (see [1, 2]).

4. Computing the image of a rational map

In this section we describe the image of an affine variety in a rational map, in terms of a Gröbner basis of an ideal with given generators. Because there exist algorithms for computing Gröbner bases from a finite set of generators, the following theorem yields effective algorithms for computing the ideal of the closure of the image.

Theorem 4.1. *Let β be a Gröbner basis (respectively, the reduced Gröbner basis) of the ideal*

$$J = (h_1(X), \dots, h_s(X), g_1(X)y_1 - f_1(X), \dots, g_n(X)y_n - f_n(X), 1 - g(X)x)$$

with respect to the lexicographic order described in section 3. Then $\beta \cap k[y_1, \dots, y_n]$ is a Gröbner basis (respectively, the reduced Gröbner basis) of the ideal $I(\overline{F(V_g)}) \subset k[y_1, \dots, y_n]$ for the induced ordering of N^n. In particular, $\overline{F(V_g)} = k^n$ iff $\beta \cap k[y_1, \dots, y_n] = \emptyset$.

To prove Theorem 4.1, note that

$$\begin{aligned} I(\overline{F(V_g)}) &= I(graph\, F|_{V_g}) \cap k[y_1, \dots, y_n] \\ &= J \cap k[Y] \end{aligned}$$

and apply Lemmas 2.1, 2.2 and Remark 3.2.

5. Criterion for birational isomorphisms

In this section, we present criteria for a given rational map defined by a rational parametrization to be birational and a method of computing the local inverse. Our main result is the following theorem, which yields an effective algorithm.

Theorem 5.1. *Let β be the reduced Gröbner basis of the ideal*

$$\begin{aligned} J &= (h_1(X), \dots, h_s(X), g_1(X)y_1 - f_1(X), \dots, g_n(X)y_n - f_n(X), 1 - g(X)x) \\ &\subset k[x, x_1, \dots, x_m, y_1, \dots, y_n], \end{aligned}$$

with respect to the lexicographic order described in Section 3. Then the following statements are equivalent:

(1) *$F : V \cdots \to \overline{F(V_g)}$ is birational.*

(2) *F has a local left inverse defined by a rational parametrization.*

(3) *J contains polynomials of the type $b_i(Y)x_i - a_i(Y)$, $i = 1, \dots, m$, where for some $b_i(Y)$, $a_i(Y) \in k[Y]$, and $b_i(Y)$ are non-zero on $\overline{F(V_g)}$ for $i = 1, \dots, m$.*

(4) *β contains the following two sets β_{inverse} and β_{image}, here*

$$\beta_{\text{inverse}} = \{b_{m,0}(Y)x_m - b_{m,1}(Y), b_{m-1,0}(Y)x_{m-1} - b_{m-1,1}(Y, x_m), \dots,$$

$$b_{1,0}(Y)x_1 - b_{1,1}(Y, x_m, x_{m-1}, \dots, x_2)\}$$

and $\beta_{\text{image}} = \emptyset$ or $\beta_{\text{image}} = \{Q_1(Y), \dots, Q_m(Y)\}$ for some $Q_1(Y), \dots, Q_m(Y) \in k[Y]$, where

$$b_{i,0}(Y) \in k[Y], b_{i,1}(Y, x_m, x_{m-1}, \dots, x_{i+1}) \in k[Y, x_m, x_{m-1}, \dots, x_{i+1}], \textit{and } b_{i,0}(Y)$$

does not belong to J, for all i.

Furthermore, if the above four conditions are satisfied, then β_{image} is the reduced Gröbner basis of $I(\overline{F(V_g)})$ for the induced ordering and the local inverse of F is given by $x_i = d_i(Y)/c_i(Y)$, obtained by solving the system of equations defined by the polynomials in β_{inverse} successively (first solve for x_m, then substitute x_m and solve for x_{m-1}, etc.).

Proof of Theorem 5.1. (1) ⇒ (2) is trivial. To prove (2) ⇒ (3), note that if $a_i(Y)/b_i(Y)$ are coordinates of the local left inverse, then

$$d(X)\left[b_i\left(\frac{f_1(X)}{g_1(X)}, \dots, \frac{f_n(X)}{g_n(X)}\right)x_i - a_i\left(\frac{f_1(X)}{g_1(X)}, \dots, \frac{f_n(X)}{g_n(X)}\right)\right]$$

belongs to $I(V)$, where $d(X) \in k[X]$ such that

$$d(X)\left[b_i\left(\frac{f_1(X)}{g_1(X)}, \dots, \frac{f_n(X)}{g_n(X)}\right)x_i - a_i\left(\frac{f_1(X)}{g_1(X)}, \dots, \frac{f_n(X)}{g_n(X)}\right)\right]$$

in $k[X]$. Hence, $b_i(Y)x_i - a_i(Y) \in I(\textit{graph } F|_{V_g})$.

(3) ⇒ (4): By hypothesis (3) we have

(∗) $b_m(Y)x_m - a_m(Y) \in J$ and $b_m(Y)$ not in J (since $b_m(Y)$ does not vanish on $\overline{F(V_g)}$).

On the other hand, our chosen lexicographic order is

$$x > x_1 > \cdots > x_m > y_1 > \cdots > y_n$$

and β is the reduced Gröbner basis of J w.r.t. $>$.

We deduce that there exists an element

$b_{m,0}(Y)x_m - b_{m,1}(Y)$ in β with $b_{m,0}(Y), b_{m,1}(Y) \in k[Y]$, and $b_{m,0}(Y)$ not in J as follows:

From (∗) and the definition of Gröbner basis there exist a term s and an element b in β such that

$\mathrm{Lt}(b_m(Y)x_m) = s\mathrm{Lt}(b)$ (Lt means leading term). So, $s\mathrm{Lt}(b) = Y^p x_m$ for some p. If x_m does not divide s then $\mathrm{Lt}(b) = Y^q x_m$ for some q. So $b = b_{m,0}(Y)x_m - b_{m,1}(Y)$. Furthermore, $b_{m,0}(Y)$ is not in J, for if it is then b can be reduced, contradiction with β being reduced Gröbner basis of J. Finally, assume that x_m divides s, say $s = x_m Y^q$ for some q. Then $\mathrm{Lt}(b_m(Y)) = Y^q\mathrm{Lt}(b)$. In particular, b in $k[Y]$ and for suitable c in k, $\mathrm{Lt}(b_m(Y) - cY^q b) < \mathrm{Lt}(b_m(Y))$. Now, consider

$(b_m(Y) - cY^q b)x_m - a_m(Y) = (b_m(Y)x_m - a_m(Y)) - cx_m Y^q b$ in J, since $b \in \beta$ in J.

So, we get a new element $b'_m(Y)x_m - a_m(Y)$ in J with $\mathrm{Lt}(b'_m(Y)) < \mathrm{Lt}(b_m(Y))$. Now repeat this process. It will stop after a finite number of steps with an element $b''_m(Y)x_m - a_m(Y)$ say, such that, with the notation above, x_m does not divide s (since b_m is not in J). Then we can conclude as above.

Similarly, we can verify that β must also contain the rest types of polynomials in β_{inverse} by the definition of the reduced Gröbner basis and the Lexicographic order.

Hence, only (4) ⇒ (1) and the concluding remarks remain to be proved.

The fact β_{image} is the reduced Gröbner basis of $I(\overline{F(V_g)})$ follows directly from Theorem 4.1.

Let $c_i(Y)x_i - d_i(Y)$, $i = 1, \ldots, m$ be obtained by Gauss elimination applied to β_{inverse}, and let $G: k^n \to k^m$ be the rational map determined by coordinates $\{x_i = d_i(Y)/c_i(Y)\}$. Set $b(Y) = \prod_{i=1}^m c_i(Y)$. Then $V_b = F(V_g)\setminus V(b)$ is an open subset of $\overline{F(V_g)}$.

We first show that $G(V_b) \subset V_g$. Indeed, if $y \in V_b$, then $Q_i(y) = 0$, $i = 1, \ldots, r$, hence $(G(y), y) \in V(\beta \bigcap k[X, Y]) = \mathit{graph}\ F|_{V_g}$ from Lemma 2.2 and so $G(y) \in V_g$.

Now we show that G is indeed the local inverse of F. Take $d(X) = g(X)^r$, where r is sufficiently large such that

$$d(X)\left[c_i\left(\frac{f_1(X)}{g_1(X)}, \ldots, \frac{f_n(X)}{g_n(X)}\right)x_i - d_i\left(\frac{f_1(X)}{g_1(X)}, \ldots, \frac{f_n(X)}{g_n(X)}\right)\right]$$

belongs to $k[X]$. Then, using that $c_i(Y)x_i - d_i(Y)$ belongs to $I(\mathit{graph}\ F|_{V_g})$, it follows from Lemma 2.3 that

$$d(X)\left[c_i\left(\frac{f_1(X)}{g_1(X)}, \ldots, \frac{f_n(X)}{g_n(X)}\right)x_i - d_i\left(\frac{f_1(X)}{g_1(X)}, \ldots, \frac{f_n(X)}{g_n(X)}\right)\right]$$

belongs to $I(V)$, which implies that G is the local left inverse of F.

It remains to prove that it is the right inverse. Consider the projection

$$\pi_n : k^{m+n} \to k^n$$

$$(x_1, \ldots, x_m, y_1, \ldots, y_n) \mapsto (y_1, \ldots, y_n).$$

Set

$$V' = \pi_n^{-1}(V_b) \bigcap (\mathit{graph}\ F|_{V_g}).$$

Then V' is an open subset of $graph\ F|_{V_g}$. It is obvious that $g_j(X)y_j - f_j(X) \in I(V')$. But $I(V') = I(graph\ F|_{V_g})$. Hence, by $g_j(X)y_j - f_j(X) \in I(graph\ F|_{V_g})$ and Lemma 2.3, we obtain that

$$c(Y)\left[g_j\left(\frac{d_1(Y)}{c_1(Y)}, \ldots, \frac{d_m(Y)}{c_m(Y)}\right)y_j - f_j\left(\frac{d_1(Y)}{c_1(Y)}, \ldots, \frac{d_m(Y)}{c_m(Y)}\right)\right]$$

in $I(\bar{V}_b) = I(\overline{F(V_g)})$ as desired, where $c(Y) = b(Y)^t$ where t is sufficiently large such that

$$c(Y)\left[g_j\left(\frac{d_1(Y)}{c_1(Y)}, \ldots, \frac{d_m(Y)}{c_m(Y)}\right)y_j - f_j\left(\frac{d_1(Y)}{c_1(Y)}, \ldots, \frac{d_m(Y)}{c_m(Y)}\right)\right]$$

belongs to $k[Y]$. The rest of the assertion in the theorem follows easily. □

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