Journal of Pure and Applied Algebra 24 (1982) 227–233
North-Holland Publishing Company

A HOMOLOGICAL APPROACH TO REPRESENTATIONS OF ALGEBRAS I: THE WILD CASE

Dagmar BAER and Helmut LENZING
Fachbereich Mathematik, Universität-Gesamthochschule Paderborn
D-4790 Paderborn, W.-Germany

Communicated by H. Bass
Received 8 April 1981
Revised 12 August 1981

We prove that the pure global dimension of a polynomial ring over an integral domain k in a finite or countable number $n\geq 2$ of commuting (non-commuting, resp.) variables is $t+1$, provided $|k|=\aleph_t$. As an application, we determine the pure global dimension of wild algebras of quiver type, also (in case k is an algebraically closed field) of the wild local and wild commutative algebras of finite k-dimension.

Keywords: Pure global dimension, algebraic compactness, polynomial rings, wild algebras.

1. Introduction

According to Auslander [1] and Tachikawa [23], a finite dimensional k-algebra R (k a commutative field) has pure global dimension zero, i.e. every R-module is a direct sum of finitely generated ones, if and only if R is of finite representation type, i.e. R has only finitely many isomorphism classes of indecomposable R-modules of finite length. This suggests to consider the pure global dimension of R (p.gl.dim R) as a natural measure for the complexity of the representation theory of R. However, a theorem of Gruson and Jensen [9, see also 10, 14] states that p.gl.dim $R\leq t+1$, provided R has cardinality $\aleph_t$. In particular, in case k is countable, any finite dimensional k-algebra R, which is not of finite representation type, has pure global dimension one.

Therefore, pure global dimension will not reflect the representation type of R in the countable case. Obviously, this result is responsible for the fact, that nothing is known on the pure global dimension of finite dimensional algebras, apart from the theorems of Auslander, Gruson–Jensen, cited above, and a result of Okoh [16], stating that the Kronecker algebra, a noncommutative k-algebra of dimension 4, which is the path algebra of the quiver $\cdot\rightrightarrows\cdot$, has pure global dimension ≥ 2.

The aim of this paper is to prove that in contrast, pure global dimension reflects

0022-4049/82/0000–0000/$02.75

the representation theoretic behaviour of algebras quite well. It is only necessary to enlarge the base field: in fact, we prove that the wild algebras R of quiver type [4, 8, 15], also in case k is algebraically closed, the wild local, and the wild commutative algebras of finite k-dimension [20] will all attain the maximal possible value of the pure global dimension. Namely, p.gl.dim $R=t+1$, provided $|k|=\aleph_t$ (Theorem 4.1). Further, the same result holds true for polynomial rings $k[X_1,\dots,X_n]$ ($k\langle X_1,\dots,X_n\rangle$ resp.) in a finite or countable number of commuting (resp. non-commuting) variables, if $n\geq 2$ (Theorem 3.2). This extends results of Kielpinski and Simson [14]. Of course, p.gl.dim $k[X]=1$, by a well-known theorem of Kulikov [13]. However, as is shown in [5], even a rather weak form of the Kulikov-property forces a finite dimensional algebra to be of finite representation type.

Consequently, wild algebras also behave wild from the homological point of view. In contrast, as we will show in a forthcoming paper [2], the pure global dimension of a tame, hereditary algebra [6], in particular of an algebra of a tame quiver [15], is always ≤ 2, more precisely is one in the countable and two in the uncountable case.

Our determination of the pure global dimension of wild algebras depends vitally on the determination of the projective dimension of the field of rational functions $k(X_1,\dots,X_n)$, considered as a module over the polynomial ring $k[X_1,\dots,X_n]$ by Osofsky [17, or 18, 19], see also [12]. We further use some embedding functors, well-known in representation theory, due mainly to Brenner [4], Gabriel [7, 8], Nazarova [15] and Ringel [20]. The final step in our argument is a criterion for functors which preserve pure injective dimension (Proposition 2.2).

Quite obvious, the statement p.gl.dim $R=t+1$, provided R is wild and $|R|=\aleph_t$, is homological in nature and equivalent to a certain form of the continuum hypothesis. Since the aspect is already developed in [17, 18, 19], we will not discuss it here.

2. A change of rings theorem for pure global dimension

In this section k is a commutative ring and R is a k-algebra, associative with 1. All notions on rings and modules are defined with respect to the left side. Therefore, R-Mod is the category of all left R-modules, and pure global dimension (p.gl.dim R) denotes the supremum of the pure-projective dimensions (p.proj.dim. M), equivalently of the pure-injective dimensions (p.inj.dim. M) of all left R-modules M.

Of course, pure-projective and pure-injective dimension are defined in the sense of relative homological algebra via pure-projective (pure-injective, resp.) resolutions, which are pure-exact by definition. We further recall a useful *splitting criterion for pure-injectivity* (=algebraic compactness) which belongs to the folklore of the subject (compare [22]): If M is an R-module, there is an obvious notion of dual $M^+=\mathrm{Hom}_k(M,E)$, where E denotes an injective cogenerator in the category of k-modules. Now,

2.1. Lemma. *An R-module M is pure-injective if and only if the canonical embedding* $M \to M^{++}$ *splits.*

The following proposition establishes the prototype of argument for functors preserving the pure-injective dimension. In fact, for the applications we have in mind, we shall need a several-object-version of Proposition 2.2, which however we will not state explicitly.

2.2. Proposition. *Suppose* R_1, R_2 *are k-algebras and* $T: R_1\text{-Mod} \to R_2\text{-Mod}$ *is a functor, which commutes with the forgetful functors from* R_i*-modules to k-modules* $(i=1,2)$ *and further satisfies the following condition:*

(*) *If a monomorphism Tf splits in* R_2-Mod, *then f splits in* R_1-Mod.

Then T preserves pure-injective dimension. In particular, $\text{p.gl.dim}\, R_1 \leq \text{p.gl.dim}\, R_2$.

Proof. Since T commutes with direct limits, T preserves pure-exactness. It is therefore sufficient to prove that T preserves and reflects pure-injectivity. This is obvious from Lemma 2.1, since $T(M \to M^{++})$ is just the canonical embedding $TM \to (TM)^{++}$.

The following immediate consequence appears already in [14].

2.3. Corollary. (i) *If I is a two-sided ideal of R, then* $\text{p.gl.dim}\, R/I \leq \text{p.gl.dim}\, R$.

(ii) *If S denotes a multiplicatively closed subset of R, then* $\text{p.gl.dim}\, S^{-1}R \leq \text{p.gl.dim}\, R$.

We recall that a left noetherian, left self-injective ring is artinian and self-injective on both sides.

2.4. Corollary. *Suppose R is self-injective, artinian. Then* $\text{p.gl.dim}\, R = \text{p.gl.dim}\, R/\text{Soc}\, R$.

Proof. Any R-module has the form $M = Q \oplus N$, where Q is injective and N has no nonzero injective submodule. N therefore satisfies $S \cdot N = 0$, where $S = \text{Soc}\, R$ denotes the socle of R. Now, $\text{p.inj.dim}_R\, M = \text{p.inj.dim}_R\, N = \text{p.inj.dim}_{R/S}\, N$ by Proposition 2.2.

3. The pure global dimension of polynomial rings

Let $P_n = k[X_1, \ldots, X_n]$ and $S_n = k\langle X_1, \ldots, X_n \rangle$ denote the ring of polynomials over k in n commuting (resp. non-commuting) variables. Then:

3.1. Lemma. *For an arbitrary ring k, the pure global dimensions of* P_n, S_n, P_2 *and* S_2 *coincide for each integer* $n \geq 2$.

Proof. We first observe that an S_n-module (P_n-module, resp.) is just a k-module V together with n k-linear maps (pairwise commuting k-linear maps, resp.) $f_i: V \to V$. Now the functor

$$T: \quad S_n\text{-Mod} \to S_2\text{-Mod}, \qquad (V;\, f_1, \ldots, f_n) \mapsto (V^{n+2};\, F_1, F_2),$$

where

$$F_1 = \begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ & & & & 0 \end{bmatrix} \quad \text{and} \quad F_2 = \begin{bmatrix} 0 & & & & & \\ 1 & 0 & & & & \\ f_1 & 1 & 0 & & & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & f_n & 1 & 0 \end{bmatrix}$$

is a full embedding [4, 8]. Therefore, an easy variant of Proposition 2.2 proves in combination with Corollary 2.3 that

$$\text{p.gl.dim}\, S_2 \geq \text{p.gl.dim}\, S_n \geq \text{p.gl.dim}\, P_n \geq \text{p.gl.dim}\, P_2.$$

It remains therefore to prove that p.gl.dim $P_2 \geq$ p.gl.dim S_2. This is done by the following embedding:

$$T: \quad S_2\text{-Mod} \to P_2\text{-Mod}, \qquad (V;\, f_1, f_2) \mapsto (V^4;\, F_1, F_2),$$

where

$$F_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad F_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & f_2 & 0 & 0 \\ f_1 & 0 & f_2 & 0 \end{bmatrix}$$

T satisfies condition (∗) of Proposition 2.2. Again, an easy variant of Proposition 2.2 applies and proves the claim.

We are now in a position to determine the pure global dimension of polynomial rings even for a small number of variables, thereby extending results of Kielpinski and Simson [14]. We also refer to [14] for complete information on the pure global dimension of polynomial rings in an infinite number of variables.

3.2. Theorem. *If R is a polynomial ring over a commutative integral domain in a finite or countable number $n \geq 2$ of commuting (resp. non-commuting) variables. Then*

$$\text{p.gl.dim.}\, R = t + 1,$$

provided $|R| = \max(\aleph_0, |k|) = \aleph_t$.

(Here, we use the convention that $t + 1 = \infty$ for an infinite ordinal t.)

Proof. By localization (see Corollary 2.3) we may assume that k is a commutative field. Of course, p.gl.dim $R \le t+1$ by the theorem of Gruson and Jensen [9, 10, 14]. On the other hand, by Lemma 3.1 we may assume that $R = k[X_1, \ldots, X_n]$ for an arbitrary integer $n \ge 2$. Now, if Q denotes the quotient field of R, Q is flat as an R-module. Therefore,

$$\text{p.gl.dim}\, R \ge \text{p.proj.dim}_R\, Q = \text{proj.dim}_R\, Q = \min(n, t+1)$$

by a famous result of Osofsky [17, 18, 19]. Taking n large we therefore get p.gl.dim $R \ge t+1$.

4. The pure global dimension of wild quivers

For the notion of a representation (resp. algebra) of a quiver $\mathbb{K}$ we refer to [7, 8]. We recall that the k-linear representations of $\mathbb{K}$ are just the modules over the quiver algebra $k[\mathbb{K}]$ of $\mathbb{K}$, which is a finite dimensional k-algebra if $\mathbb{K}$ has no oriented cycles. We further recall [4, 8, 15] that a quiver $\mathbb{K}$ is of wild representation type, provided $\mathbb{K}$ contains a connected subquiver $\mathbb{L}$ consisting of some of the euclidian quivers $\tilde{\mathbb{A}}_n$ $(n \ge 0)$, $\tilde{\mathbb{D}}_n$ $(n \ge 4)$, $\tilde{\mathbb{E}}_6, \tilde{\mathbb{E}}_7, \tilde{\mathbb{E}}_8$ enlarged by some additional arrow. Now, the list of Brenner [4] defines a full embedding of $k\langle X_1, X_2\rangle$-Mod into $k[\mathbb{K}]$-Mod for any wild quiver $\mathbb{K}$, provided $\mathbb{K}$ has standard orientation. In each case, by a variant of Proposition 2.2 we conclude that p.gl.dim $k[\mathbb{K}] = t+1$, if $|k| = \aleph_t$ and $\mathbb{K}$ is a wild quiver with standard orientation. This proves nearly the first part of the following theorem.

4.1. Theorem. *Suppose R is either*

(i) *the algebra $k[\mathbb{K}]$ of a wild quiver $\mathbb{K}$ with respect to a field k, or*

(ii) *in case k is an algebraically closed field, a wild local (or commutative) algebra of finite k-dimension.*

Then p.gl.dim $R = t+1$, *provided* $\max(|k|, \aleph_0) = \aleph_t$.

Proof. We first prove (i) ⇒ (ii). It follows from the work of Ringel [20, 21] that a finite dimensional, local algebra R over an algebraically closed field is wild if and only if R has a quotient of the form

$$R_1 = k\langle X, Y, Z\rangle/(X, Y, Z)^2, \qquad R_2 = k\langle X, Y\rangle/(X^2, XY, Y^2X, Y^3),$$

$$R_3 = R_2^{\text{op}}, \qquad R_4 = k\langle X, Y\rangle/(X^2, XY - \lambda YX, Y^2X, Y^3) \quad \text{with } \lambda \ne 0, \text{ or}$$

$$R_5 = k\langle X, Y\rangle/(X^2 - Y^2, YX).$$

Now, in [20] there is a list of embeddings from the category of representations of some wild quiver $\mathbb{K}_i$ into R_i-Mod, to which the argument of Proposition 2.2 applies. This proves the claim in the case of local algebras. If R is commutative, R is a finite product of local algebras, which concludes the proof of (ii).

In view of the remarks above the proof of (i) rests on the following lemma which is proved with the aid of the reflection functors introduced by Bernstein, Gelfand and Ponomarev in [3].

4.2. Lemma. *Suppose p is a source of the quiver* $\mathbb{K}$ *(i.e. all arrows connected with p are starting at p). If* $\bar{\mathbb{K}}$ *denotes the quiver constructed from* $\mathbb{K}$ *by converting the orientation of all arrows starting at p, then*

$$\text{p.gl.dim}\, k[\mathbb{K}] = \text{p.gl.dim}\, k[\bar{\mathbb{K}}].$$

Proof. We consider the reflection functors [3]

$$F^+:\quad k[\mathbb{K}]\text{-Mod} \to k[\bar{\mathbb{K}}]\text{-Mod} \quad\text{and}\quad F^-:\quad k[\bar{\mathbb{K}}]\text{-Mod} \to k[\mathbb{K}]\text{-Mod}$$

with respect to p. There are full subcategories $\mathscr{U}$ and $\bar{\mathscr{U}}$ of $k[\mathbb{K}]$-Mod ($k[\bar{\mathbb{K}}]$-Mod, resp.) which are closed against the formation of direct limits and satisfy the following conditions:

(i) $F^+ : \mathscr{U} \to \bar{\mathscr{U}}$ and $F^- : \bar{\mathscr{U}} \to \mathscr{U}$ are equivalences, inverse to each other.

(ii) Every M of $k[\mathbb{K}]$-Mod ($\bar{M}$ of $k[\bar{\mathbb{K}}]$-Mod, resp.) has the form $M = Q \oplus U$ ($\bar{M} = \bar{Q} \oplus \bar{U}$, resp.), where U ($\bar{U}$, resp.) is in $\mathscr{U}$ ($\bar{\mathscr{U}}$, resp.) and Q ($\bar{Q}$, resp.) is concentrated at p and is the maximal direct summand with this property.

In particular, Q and $\bar{Q}$ are both pure-projective and pure-injective. As further consequences of (i), (ii) we note:

(iii) If M is the pure-injective hull [22] (a pure quotient, resp.) of $V \in \mathscr{U}$, then also $M \in \mathscr{U}$. The same statement holds true for $\bar{\mathscr{U}}$.

(iv) $F^+ : \mathscr{U} \to \bar{\mathscr{U}}$ preserves pure injective dimension.

Concerning (iii) we argue as follows: If $M = Q \oplus U$ (in the notation of (ii)) is the pure-injective hull of V then $Q \cap V = 0$, hence $Q = 0$. If $M = Q \oplus U$ is a pure quotient of V, Q may be lifted to V by pure-projectivity. This proves $Q = 0$.

Concerning (iv) we observe that according to (iii) any U in $\mathscr{U}$ has a pure-injective resolution also lying in $\mathscr{U}$. Since F^+, restricted to $\mathscr{U}$, commutes with direct limits, it also preserves pure-exactness. It therefore remains to prove that $F^+ : \mathscr{U} \to \bar{\mathscr{U}}$ preserves and reflects pure-injectivity, which again is a direct consequence of (iii).

Now, the combination of (ii) and (iv) concludes the proof of Lemma 4.2.

4.3. Remark. As is obvious from Theorems 3.2 and 4.1, there is a clear distinction in the behaviour of global dimension and pure global dimension for k-algebras. As is shown in [11] the global dimension of finite-dimensional algebras is usually preserved under extensions K/k of the base field k. To be precise, K/k has to be separable in the sense of McLane.

In contrast, in order to see the effects of pure global dimension it is usually necessary to enlarge the base field k.

References

[1] M. Auslander, Representation theory of Artin Algebras II, Comm. Algebra 1 (1974) 269–310.
[2] D. Baer, H. Brune and H. Lenzing, A homological approach to representations of algebras II: Tame hereditary algebras, J. Pure Appl. Algebra, to appear.
[3] I.N. Bernstein, I.M. Gelfand and V.A. Ponomarev, Coxeter functors and Gabriel's theorem, Russian Math. Surveys 28 (1973) 17–32.
[4] S. Brenner, Decomposition properties of some small diagrams of modules, Ist. Naz. Alta Mat. Symposia Math. 13 (1974) 127–141.
[5] H. Brune, On finite representation type and a theorem of Kulikov, in: Representation Theory II, Proc. 2nd ICRA, Ottawa, 1979, Lecture Notes in Mathematics No. 832 (Springer, Berlin–New York, 1980).
[6] V. Dlab and C.M. Ringel, The representations of tame hereditary algebras, in: R. Gordon, Ed., Representation Theory of Algebras (Marcel Dekker, New York, 1978).
[7] P. Gabriel, Unzerlegbare Darstellungen I, Manuscripta Math. 6 (1972) 71–103.
[8] P. Gabriel, Représentations indécomposables, in: Séminaire Bourbaki (1973–74), Lecture Notes in Mathematics No. 431 (Springer, Berlin–New York, 1975).
[9] L. Gruson and C.U. Jensen, Modules algébriquement compacts et foncteurs $\varprojlim^{(i)}$, C.R. Acad. Sci. Paris 276 (1973) 1651–1653.
[10] L. Gruson and C.U. Jensen, Dimensions cohomologiques reliées aux foncteurs $\varprojlim^{(i)}$, in: Séminaire d'Algèbre Paul Dubreil et Marie-Paule Malliavin, Lecture Notes in Mathematics (Springer, Berlin–New York) to appear.
[11] C.U. Jensen and H. Lenzing, Homological dimension and representation type of algebras under base field extension, to appear.
[12] I. Kaplansky, The homological dimension of a quotient field, Nagoya Math. J. 27 (1966) 139–142.
[13] I. Kaplansky, Infinite Abelian Groups (The University of Michigan Press, Ann Arbour, 1971).
[14] R. Kielpinski and D. Simson, On pure homological dimension, Bull. Acad. Pol. Sci. 23 (1975) 1–6.
[15] L.A. Nazarova, Representations of quivers of infinite type, Math. USSR Izvestija 7 (1973) 749–792.
[16] F. Okoh, Hereditary algebras that are not pure hereditary, in: Representation Theory II, Proc. 2nd ICRA, Ottawa, 1979, Lecture Notes in Mathematics No. 832 (Springer, Berlin–New York, 1980).
[17] B.L. Osofsky, Homological dimension and the continuum hypothesis, Trans. Amer. Math. Soc. 132 (1968) 217–230.
[18] B.L. Osofsky, Homological dimensions of modules, Regional Conference Series in Mathematics 12 (Amer. Math. Soc., Providence, RI, 1973).
[19] B.L. Osofsky, The subscript of $\aleph_n$, projective dimension and the vanishing of $\varprojlim^{(n)}$, Bull. Amer. Math. Soc. 80 (1974) 8–26.
[20] C.M. Ringel, The representation type of local algebras, in: Representations of Algebras, Proc. ICRA, Ottawa, 1974, Lecture Notes in Mathematics No. 488 (Springer, Berlin–New York, 1975).
[21] C.M. Ringel, Tame group algebras, in: Darstellungstheorie endlich dimensionaler Algebren, Oberwolfach (1977).
[22] R.B. Warfield, Purity and algebraic compactness for modules, Pac. J. Math. 28 (1969) 699–719.
[23] H. Tachikawa, Quasi-Frobenius Rings and Generalizations, Lecture Notes in Mathematics No. 351 (Springer, Berlin–New York, 1973).