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# ALGEBRA IN A GROTHENDIECK TOPOS: INJECTIVITY IN QUASI-EQUATIONAL CLASSES

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This paper gives a study of injectivity and some related notions in quasi-equational classes of algebras in an arbitrary Grothendieck topos **E**. The main purpose is to describe the relationship between the class mod $\Sigma$ of models of a set $\Sigma$ of quasi-equations in the category of sets Ens and the corresponding class mod($\Sigma$, **E**) of models of $\Sigma$ in **E** with respect to residual smallness, boundedness of essential extensions and injectivity. The basic nature of our results is that, for any given $\Sigma$, whatever holds in Ens, concerning these notions, also holds in **E**. In particular, this substantially improves the earlier results of Howlett [5] regarding the existence of enough injectives in mod($\Sigma$, **E**).

## 1. Preliminaries

**1.1.** *Algebras in a category.* Let **E** be a finitely complete category (in particular, it has a terminal object **1**). Given a family $\tau=(n_\lambda)_{\lambda\in\Omega}$ of finite cardinal numbers $n_\lambda$, indexed by a set $\Omega$, recall that *an algebra in* **E** is an entity $A=(|A|,(\lambda_A)_{\lambda\in\Omega})$, where $|A|$ is an object of **E**, called the underlying object of the algebra $A$ and, for each $\lambda\in\Omega$, the $\lambda$-th operation $\lambda_A:|A|^{n_\lambda}\to|A|$ of $A$ is a morphism in **E**, $n_\lambda$ being the *arity* of $\lambda_A$; the family $\tau=(n_\lambda)_{\lambda\in\Omega}$ is called the type of $A$. A *homomorphism* $h:A\to B$ from an algebra $A=(|A|,(\lambda_A)_{\lambda\in\Omega})$ to an algebra $B=(|B|,(\lambda_B)_{\lambda\in\Omega})$ is a morphism $|h|:|A|\to|B|$ such that the following diagram commutes, for each $\lambda\in\Omega$:

$$\begin{array}{ccc} |A|^{n_\lambda} & \xrightarrow{|h|^{n_\lambda}} & |B|^{n_\lambda} \\ {\scriptstyle\lambda_A}\downarrow & & \downarrow{\scriptstyle\lambda_B} \\ |A| & \xrightarrow{|h|} & |B| \end{array}$$

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The collection of all algebras (of the type $\tau$) in $\mathbf{E}$ and homomorphisms between them forms a category denoted by $\mathrm{Alg}(\tau)\mathbf{E}$ (or by $\mathrm{Alg}(\tau)$ if $\mathbf{E}=\mathrm{Ens}$).

For $A\in\mathrm{Alg}(\tau)\mathbf{E}$ and any natural number $n$, $\mathbf{E}(|A|^n,|A|)$ can easily be made into an algebra of the type $\tau$ in Ens, by defining the $\lambda$-th operation as

$$\lambda(\phi_1,\ldots,\phi_{n_\lambda})=\lambda_A\prod_{i=1}^{n_\lambda}\phi_i$$

for any $\phi_i:|A|^n\to|A|$ $(i=1,\ldots,n_\lambda)$, where $\prod_{i=1}^{n_\lambda}\phi_i$ is the morphism $|A|^n\to|A|^{n_\lambda}$ determined by the $\phi_i$. Let $F$ be the absolutely free algebra of the type $\tau$ on a set $X=\{x_1,\ldots,x_n\}$ of $n$ elements. Extend the map $x_i\rightsquigarrow\mathrm{pr}_i$ ($\mathrm{pr}_i:|A|^n\to|A|$ the projections) from $X$ to $\mathbf{E}(|A|^n,|A|)$ freely to a homomorphism $\phi:F\to\mathbf{E}(|A|^n,|A|)$ and denote $\phi(P)$ by $P_A$ for any $P\in F$. For a quasi-equation

$$\sigma:=\bigwedge_{i=1}^{k}(p_i=q_i)\to(p=q),$$

we say that $A$ satisfies $\sigma$, written as $A\models\sigma$, iff the pullback

$$\begin{array}{ccccc} & \nearrow & \mathrm{Eq}_1(p_{1A},q_{1A}) & \searrow & & & \\ & & \vdots & & & & \\ P & \longrightarrow & \mathrm{Eq}_i(p_{iA},q_{iA}) & \rightarrowtail & |A|^n & \overset{p_{iA}}{\underset{q_{iA}}{\rightrightarrows}} & |A| \\ & & \vdots & & & & \\ & \searrow & \mathrm{Eq}_k(p_{kA},q_{kA}) & \nearrow & & & \end{array}$$

of the equalizers

$$\mathrm{Eq}_i(p_{iA},q_{iA})\to|A|^n\overset{p_{iA}}{\underset{q_{iA}}{\rightrightarrows}}|A|\quad(i=1,\ldots,k)$$

factors through the equalizer

$$\mathrm{Eq}(p_A,q_A)\to|A|^n\overset{p_A}{\underset{q_A}{\rightrightarrows}}|A|.$$

In particular, $A$ satisfies an equation $(p=q)$ iff $p_A=q_A$. The full subcategory of $\mathrm{Alg}(\tau)\mathbf{E}$ given by the class of all algebras in $\mathbf{E}$ satisfying $\Sigma$ will be denoted by $\mathrm{mod}(\Sigma,\mathbf{E})$ (or $\mathrm{mod}\,\Sigma$ if $\mathbf{E}=\mathrm{Ens}$) and is called a quasi-equational class (or an equational class if $\Sigma$ is a set of equations). We note that, if $\mathbf{E}$ is a topos, then the above definition of $A\models\sigma$ coincides with the usual notion of satisfaction as defined for an arbitrary first order sentence in a topos. From now on $\Sigma$ always denotes a set of quasi-equations.

Let $k:\mathbf{E}\to\mathbf{F}$ be a functor, preserving finite limits; then $k$ induces another functor $f:A\to B$ in $\mathrm{Alg}(\tau)\mathbf{E}$. Since $k$ preserves finite limits, it preserves pullback and equalizer diagrams; and hence if $\sigma$ is a quasi-equation and $A\models\sigma$, for $A\in\mathrm{Alg}(\tau)\mathbf{E}$, then $\bar{k}A\models\sigma$. We thus get a functor

$$\bar{k}\,|\,\mathrm{mod}(\Sigma,\mathbf{E}):\mathrm{mod}(\Sigma,\mathbf{E})\to\mathrm{mod}(\Sigma,\mathbf{F})$$

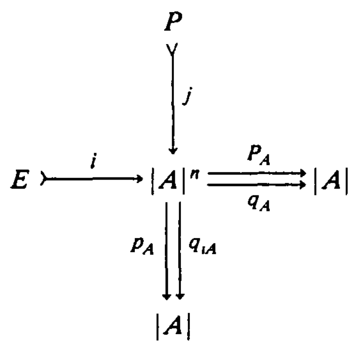
for any given set $\Sigma$ of quasi-equations.

**1.2. Lemma.** *Let* $\mathbf{E}$ *have a set* $\Phi$ *of generators. Then, for any* $A \in \mathrm{Alg}(\tau)\mathbf{E}$ *and any set* $\Sigma$ *of quasi-equations,* $A \in \mathrm{mod}(\Sigma, \mathbf{E})$ *iff* $\bar{h}_G(A) \in \mathrm{mod}\,\Sigma$ *for each* $G \in \Phi$, *where* $h_G = \mathbf{E}(G, -)$.

**Proof.** That $A \in \mathrm{mod}(\Sigma, \mathbf{E})$ implies $\bar{h}_G(A) \in \mathrm{mod}\,\Sigma$ is clear, by what has been discussed above. Conversely, let

$$\sigma := \bigwedge_{i=1}^{k} (p_1 = q_i) \to (p = q)$$

be any quasi-equation and $A \in \mathrm{Alg}(\tau)\mathbf{E}$ such that $\bar{h}_G(A) \models \sigma$ for all $G \in \Phi$. Consider the diagram



with $P$ the pullback of the equalizers of the pairs $(p_{iA}, q_{iA})$. By the hypothesis on $A, h_G(p_A j) = h_G(q_A j)$ for all $G \in \Phi$ which this implies that $p_A j = q_A j$, the latter because the set $\Phi$ of generators is collectively faithful, and hence $j$ factors through $i$, by definition of equalizers; thus $A \models \sigma$.

**1.3.** *Algebras in a Grothendieck topos.* Let $\mathbf{C}$ be a small category. The category $\hat{\mathbf{C}} = \mathrm{Ens}^{\mathbf{C}^*}$ ($\mathbf{C}^*$ the dual category of $\mathbf{C}$) is called the category of presheaves on $\mathbf{C}$. Recall that, up to equivalence, a category $\mathbf{E}$ is a *Grothendieck topos* iff it is a full subcategory of some $\hat{\mathbf{C}}$ for which there is a reflection functor $R : \hat{\mathbf{C}} \to \mathbf{E}$ (i.e. a left adjoint to the inclusion function $\mathbf{E} \xrightarrow{i} \hat{\mathbf{C}}$) which is also left exact (i.e. preserves finite limits) (Johnstone [6], p. 12 and p. 105). Let

$$\mathbf{E} \underset{R}{\overset{i}{\rightleftarrows}} \hat{\mathbf{C}}$$

be a Grothendieck topos. The discussion in the last sections, in particular, shows that the category $\mathrm{Alg}(\tau)\hat{\mathbf{C}}$ is isomorphic to the category of all $\mathrm{Alg}(\tau)$-valued presheaves on $\mathbf{C}$; and since $R$ preserves finite limits, it can be lifted to $\bar{R} : \mathrm{mod}(\Sigma, \hat{\mathbf{C}}) \to \mathrm{mod}(\Sigma, \mathbf{E})$ (we denote $\bar{R}$ by the same letter $R$). Moreover, since $\mathbf{E}$ has a set of generators, namely $\{R(h_U) : U \in \mathbf{C}\}$, where $h_U = (-, U)$ is the presheaf represented by $U$, Lemma 1.3 implies that $A \in \mathrm{mod}(\Sigma, \mathbf{E})$ iff $AU \in \mathrm{mod}\,\Sigma$ for all $U \in \mathbf{C}$.

A natural question to ask would be, what is the relationship between the behaviour of a certain classical algebraic notion in mod $\Sigma$ and in mod($\Sigma$, **E**). In this paper, we primarily consider the notion of injectivity, which has been extensively investigated for the case of equational classes of algebras in Ens (e.g. [1], [7]), and show that the properties of mod $\Sigma$, regarding this notion, survive the passage to mod($\Sigma$, **E**), for a set $\Sigma$ of quasi-equations and an *arbitrary* Grothendieck topos **E**. For example, we prove that mod($\Sigma$, **E**) has enough injectives iff it is residually small and pushouts transfer monomorphisms, which is a counterpart of a result for equational classes of algebras in Ens; further, we show that mod($\Sigma$, **E**) has enough injectives iff mod $\Sigma$ has enough injectives.

## 2. The adjointness of $\hat{\mathbf{C}}$

**2.1.** Here, we construct a pair of adjoint functors

$$\hat{\mathbf{C}} \underset{H}{\overset{G}{\rightleftarrows}} \mathrm{Ens}^{|\mathbf{C}|}$$

with $G$ a left adjoint to $H$, where $|\mathbf{C}|$ denotes the set of object of **C**.

Define $G$ by $GP=(PU)_{U\in\mathbf{C}}$, for $P\in\hat{\mathbf{C}}$, and for any map $f:P\to Q$ in $\hat{\mathbf{C}}$, $Gf=(f_U)_{U\in\mathbf{C}}$. That $G$ is a functor is easily checked.

Define $H$ by: for any $B=(B_V)_{V\in\mathbf{C}}$ in $\mathrm{Ens}^{|\mathbf{C}|}$, $HB=\bar{B}$ is the presheaf given as follows:

$$\bar{B}U=\prod_{V\in\mathbf{C}} B_V^{(V,U)}$$

for each $V\in\mathbf{C}$ and to define $\bar{B}s:\bar{B}W\to\bar{B}U$, for each $s:U\to W$, notice that $s$ induces a natural transformation $s^*:h_U\to h_W$ with components $s_V^*$ given by $s_V^*(t)=st$ ($V\in\mathbf{C}$), for $t:V\to U$; then $s^*$ induces a map $s_V^B:B_V^{(V,W)}\to B_V^{(V,U)}$ (composition from the left). Hence, define $\bar{B}s=\prod_{V\in\mathbf{C}} s_V^B$. One easily checks that $\bar{B}$ is a presheaf. To complete the definition of the functor $H$, let $f=(f_V)_{V\in\mathbf{C}}$ be any map from $B=(B_V)_{V\in\mathbf{C}}$ to $C=(C_V)_{V\in\mathbf{C}}$. For any pair $V$ and $U$ in **C**, define $f_V^U:B_V^{(V,U)}\to C_V^{(V,U)}$ by $f_V^U(\alpha)=f_V\alpha$ for any $\alpha:(V,U)\to B_V$. Define $\bar{f}=Hf$ by $\bar{f}_U=\prod_V f_V^U$, ($U\in\mathbf{C}$). That $\bar{f}$ is a natural transformation and $H$ is actually a functor is easily checked.

Next, we define two natural transformations $\eta:1\to HG$ and $\varepsilon:GH\to 1$. To define $\eta$, let $P\in\hat{\mathbf{C}}$; combining the maps $PU\to PV^{(V,U)}$ given by $a\rightsquigarrow a_V$ with $a_V(s)=(Ps)a$, for $a\in PU$ and $s:V\to U$, we get a map

$$\eta_{PU}:PU\to\prod_{V\in\mathbf{C}} PV^{(V,U)}=(HGP)\quad(U\in\mathbf{C}).$$

Define $\eta_P:P\to HGP$ by $(\eta_P)_U=\eta_{PU}$. It is not hard to check that each $\eta_P$ and then $\eta$ are natural transformations. Note that, for $V=U$ and $s=1_U$, $\bar{a}(s)=a$ and hence $\eta_P$ is a monomorphism for each $P\in\hat{\mathbf{C}}$. We define $\varepsilon$ such that, for each $B=(B_V)_{V\in\mathbf{C}}$, $\varepsilon_B$ has the composite

$$e\,\mathrm{pr}_u:\prod_V B_V^{(V,U)}\to B_U^{(U,U)}\to B_U$$

as its $U$-th component ($U \in \mathbf{C}$) and $e(\alpha) = \alpha(1_U)$ for $\alpha : (U, U) \to B_U$.

**2.2. Proposition.** *$G$ is a left adjoint to $H$, with $\eta$ and $\varepsilon$ as front and back adjunctions.*

**Proof.** It only remains to show that

$$1_G = G \xrightarrow{G * \eta} GHG \xrightarrow{\varepsilon * G} G$$

and

$$1_H = H \xrightarrow{\eta * H} HGH \xrightarrow{H * \varepsilon} H,$$

that is to show, for any $A \in \hat{\mathbf{C}}$ and $B = (B_V)_{V \in \mathbf{C}}$, $\varepsilon_{GA} G\eta_A = 1_{GA}$ and $H\varepsilon_B \eta_{HB} = 1_{HB}$. Let $U \in \mathbf{C}$ and $a \in AU$, we have $(GA)U \to (GHGA)U \to (GA)U$ given by $a \rightsquigarrow (\bar{a}_V)_{V \in \mathbf{C}} \rightsquigarrow e(\bar{a}_V)$, but $e(\bar{a}_U) = \bar{a}_U(1_U) = A(1_U)(a) = a$, and hence $\varepsilon_{GA} G\eta_A = 1_{GA}$. Similarly, one can show $H\varepsilon_B \eta_{HB} = 1_{HB}$.

**2.3. Remark.** By the definition of $G$ and since $H$ is a right adjoint, they both preserve finite limits, and hence can be lifted to

$$\operatorname{mod}(\Sigma, \hat{\mathbf{C}}) \underset{\bar{H}}{\overset{\bar{G}}{\rightleftarrows}} \operatorname{mod} \Sigma^{|\mathbf{C}|}$$

with $\bar{G}$ a left adjoint to $\bar{H}$.

**2.4. Remark.** The particular case of the above construction for a monoid $\mathbf{M}$ as the category $\mathbf{C}$, and hence the category of $\mathbf{M}$-sets as $\hat{\mathbf{C}}$, is due to Berthiaume [4].

## 3. Residual smallness

**3.1. Definition.** A monomorphism $h : A \to B$ in a category $\mathbf{K}$ is called *essential* iff, for any $g : B \to C$ in $\mathbf{K}$, whenever $gh$ is a monomorphism, then so is $g$.

Let $\mathbf{E}$ be a Grothendieck topos and $\Sigma$ a set of quasi-equations. Note that, as in the case of Ens, $\operatorname{mod}(\Sigma, \mathbf{E})$ is closed under direct limits in $\operatorname{Alg}(\tau)\mathbf{E}$ because any colimit in $\mathbf{E}$ is formed by first forming it in $\hat{\mathbf{C}}$ and then reflecting it to $\mathbf{E}$ where the latter preserves the conditions for $A \models \Sigma$.

**3.2. Lemma.** *In* $\operatorname{mod}(\Sigma, \mathbf{E})$,

(i) *any composite of essential monomorphisms is an essential monomorphism, and*

(ii) *any direct limit of essential monomorphisms is an essential monomorphism.*

**Proof.** (i) is trivial. To prove (ii), let $f : A \to \varinjlim B_\alpha$ be a direct limit in $\operatorname{mod}(\Sigma, \mathbf{E})$ of essential monomorphisms $f_\alpha: A \to B_\alpha$ and diagram maps $g_{\alpha\beta}: B_\alpha \to B_\beta$ ($\beta \geq \alpha$). Since each $f_\alpha$ is a monomorphism, $f$ is a monomorphism, because this is true in Ens and hence in $\hat{\mathbf{C}}$, and the reflection to $\mathbf{E}$ preserves monomorphisms by exactness. To show that $f$ is essential, let $hf: A \to D$ be a monomorphism, with $D$ in mod($\Sigma$, E). Then essentialness of $f_\alpha$ implies that all $hg_\alpha$ are monomorphisms, and hence $h$ is a monomorphism; the latter again because of the way direct limits are formed in mod($\Sigma$, E). This proves that $f$ is an essential monomorphism.

**3.3.** The following is a counterpart of (E3) in [1].

**Lemma.** *In* mod($\Sigma$, E), *for any monomorphism $h: A \to B$ there exists a homomorphism $g: B \to C$ with $gh$ an essential monomorphism.*

**Proof.** Take all the congruences $\Theta$ on $B$ such that $B/\Theta \in$ mod($\Sigma$, E) and $A \xrightarrow{vh} B/\Theta$ ($v: B \to B/\Theta$ the quotient map) is a monomorphism. Then by the observation in 3.1 and the exactness argument in the proof of 3.2, any join of a chain of such congruences is again such a congruence; and hence, there exists a maximal such congruence, say $\Theta_0$. Maximality of $\Theta_0$ then implies that $A \to B/\Theta_0$ is essential.

**3.4. Corollary.** *In* mod($\Sigma$, E), *an algebra A is an absolute retract iff it has no proper essential extension.*

**Proof.** ($\Rightarrow$) If $f: A \to B$ is an essential monomorphism and $h: B \to A$ is a retraction, then, by essentialness of $f$, $h$ is a monomorphism, and hence $A \cong B$.

($\Leftarrow$) Given any monomorphism $f: A \to B$, continue it to an essential monomorphism $A \xrightarrow{gf} C$, by the last lemma. By hypothesis on $A$, $gf$ is an isomorphism and then $(gf)^{-1}g$ is the desired retraction.

**3.5. Definition.** A category **K** is called *residually small* iff, it has a set of cogenerators.

**3.6. Definition.** A category **K** is called *essentially bounded* iff, for each $A \in \mathbf{K}$ there exists, up to isomorphism, only a set of essential extensions in **K**.

**3.7. Lemma.** *For any well powered category with products and a set $\Phi$ of generators, residual smallness implies essential boundedness.*

**Proof.** Let $h: A \to B$ be any essential monomorphism, and embed $B \xrightarrow{e} \prod_{\alpha \in I} C_\alpha$, for $C_\alpha$ from a suitable set of cogenerators. Then, for any $G \in \Phi$ and a pair of distinct maps $G \overset{s}{\underset{t}{\rightrightarrows}} A$, we have $ehs \neq eht$, and hence $p_\alpha ehs \neq p_\alpha eht$, for some projection $p_\alpha: \prod C_\alpha \to C_\alpha$. Pick $\alpha_{st}$ as one such, then $A \to \prod_{\beta \in J} C_\beta$ is a monomorphism, where $J = \{\alpha_{st}: s \neq t: G \to A\}$ and Card $J \leq$ Card $\bigcup_{G \in \Phi} (G, A)^2$. Essentialness of $h$ implies that $B \to \prod_J C_\beta$ is a monomorphism, and since there exists only a set of products $\prod_J C_\beta$, we are done.

**3.8. Lemma.** *For* mod($\Sigma$, E), *essential boundedness implies residual smallness.*

**Proof.** For any $A \in$ mod($\Sigma$, E), take all $B_\alpha \leq A$, generated by

$$\alpha : U \amalg U \to |A| \quad (U \in \mathbf{C})$$

and then continue them to essential extensions

$$B_\alpha \xrightarrow{i_\alpha} A \xrightarrow{f_\alpha} C_\alpha,$$

by Lemma 3.3. The homomorphism $\prod f_\alpha : A \to \prod C_\alpha$ is a monomorphism, for otherwise there exists some $\alpha$ with $B_\alpha \to A \to \prod C_\alpha$ not a monomorphism which is a contradiction to the fact that all $B_\alpha \to C_\alpha$ are monomorphisms. Since there exists, up to isomorphisms, only a set of $B$ generated by some $U \amalg U \to |B|$, and only a set of essential extensions of those $B$, any set representing, up to isomorphism, all essential extensions of such $B$ is a cogenerating set; hence, we are done.

**3.9. Corollary.** *For* mod($\Sigma$, E), *essential boundedness is equivalent to residual smallness.*

**Proof.** One way this is true by the last lemma, and since mod($\Sigma$, E) has a set of generators, namely the mod($\Sigma$, E)-free algebras on the reflection of representable presheaves $h_U$ ($U \in \mathbf{C}$), Lemma 3.7 implies the converse.

**3.10. Proposition.** mod($\Sigma$, E) *is residually small iff* mod $\Sigma$ *is residually small.*

**Proof.** ($\Rightarrow$) Consider the following pair of adjoint functors:

$$\text{mod}(\Sigma, \mathbf{E}) \underset{\Delta}{\overset{\Gamma}{\rightleftarrows}} \text{mod}\, \Sigma$$

where $\Gamma = (1, -)$ and $\Delta$ left exact, left adjoint to $\Gamma$; in fact $\Delta$ is the composite

$$\text{mod}\, \Sigma \xrightarrow{\Delta_0} \text{mod}\, \Sigma^{|\mathbf{C}|} \xrightarrow{|\bar{H}|} \text{mod}(\Sigma, \hat{\mathbf{C}}) \xrightarrow{R} \text{mod}(\Sigma, \mathbf{E})$$

where $\Delta_0$ takes mod $\Sigma$ to the constant families induced by $|\mathbf{C}|$ and $\bar{H}$ as in Section 2. One can then easily check that the functor $\Gamma$ transfers the set of cogenerators, and hence mod $\Sigma$ is residually small.

($\Leftarrow$) Let mod $\Sigma$ be residually small. Then so is any mod $\Sigma^I$ = mod($\Sigma$, Ens$^I$). The functor $\bar{H}$ transfers the set of cogenerators of mod($\Sigma$, Ens$^{|\mathbf{C}|}$) to a set of cogenerators in mod($\Sigma$, $\hat{\mathbf{C}}$), and hence the latter is essentially bounded, by Lemma 3.7. Since monomorphisms in mod($\Sigma$, E) are also monomorphisms in mod($\Sigma$, $\hat{\mathbf{C}}$) and the reflection functor preserves monomorphisms, essential monomorphisms in mod($\Sigma$, E) are also essential in mod($\Sigma$, $\hat{\mathbf{C}}$), and hence mod($\Sigma$, E) is also essentially bounded. Then mod($\Sigma$, E) is residually small, by Lemma 3.8.

## 4. Injective algebras in mod($\Sigma$, E)

**4.1. Definition.** In any category **K**, *pushouts transfer monomorphisms* iff, for any pushout diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle v} \\ C & \xrightarrow{g} & D \end{array}$$

whenever $f$ is a monomorphism, then $g$ is also a monomorphism. If $K$ has pushouts, one can easily check that this is equivalent to saying that, any diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle u}\downarrow & & \\ C & & \end{array}$$

with $f$ a monomorphism can be completed to a *commutative* diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle v} \\ C & \xrightarrow{g} & D \end{array}$$

with $g$ a monomorphism. This is the condition (E4) in [1].

**4.2. Proposition.** *Pushouts transfer monomorphisms in* mod($\Sigma$, E) *iff they do in* mod $\Sigma$.

**Proof.** ($\Rightarrow$) Using the pair of adjoint functors $\Delta \dashv \Gamma$ given in the proof of Proposition 3.10 and the fact that $\Delta$ is faithful, one can easily check this.

($\Leftarrow$) If the diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle v} \\ C & \xrightarrow{g} & D \end{array}$$

is a pushout in mod($\Sigma$, E) with $f$ a monomorphism, then, by the construction of pushouts in mod($\Sigma$, E), $D$ is the reflection of some $P \in$ mod($\Sigma$, $\hat{\mathbf{C}}$) with

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle v} \\ C & \xrightarrow{\bar g} & P \end{array}$$

a pushout in mod($\Sigma, \hat{\mathbf{C}}$), $R\bar g = g$, and $f$ a monomorphism. Now, for each $U \in \mathbf{C}$,

$$\begin{array}{ccc} AU & \xrightarrow{f_U} & BU \\ \downarrow & & \downarrow \\ CU & \xrightarrow{\bar g_U} & PU \end{array}$$

is a pushout in mod $\Sigma$ with $f_U$ a monomorphism. Then, by the hypothesis on mod $\Sigma$, $\bar g_U$ (for all $U$), and hence $\bar g$ are monomorphisms. Since $R\bar g = g$ and $R$ preserves monomorphisms, $g$ is a monomorphism.

**4.3. Lemma.** *The category* mod($\Sigma$, **E**) *has enough injectives iff, it is residually small and pushouts transfer monomorphisms.*

**Proof.** ($\Rightarrow$) To show that mod($\Sigma$, **E**) is residually small is to show it is essentially bounded, by Corollary 3.9. All the essential extensions of $A \in$ mod($\Sigma$, **E**) can be embedded in any injective extension of $A$, hence there exists a set of essential extensions for $A$. To prove the second part, let $A \xrightarrow{f} B$ be a monomorphism and $A \xrightarrow{g} C$ any homomorphism. Embedding $C$ into an injective $E$ we get the following commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle h} \\ C & \xrightarrow{j} & E \end{array}$$

where $h$ is obtained because $f$ is a monomorphism and $E$ is injective. This, by an earlier remark, proves that pushouts transfer monomorphisms.

($\Leftarrow$) For $A \in$ mod($\Sigma$, **E**), take a maximal extension $f : A \to E$ of $A$, which exists by Lemma 3.2 (ii). We claim that $E$ is injective. To prove this, let $g : B \to C$ be any monomorphism and $h : B \to E$ any homomorphism. Form the following pushout:

$$\begin{array}{ccc} B & \xrightarrow{g} & C \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle v} \\ E & \xrightarrow{u} & P \end{array}$$

By hypothesis $u$ is a monomorphism, and hence retractable by Corollary 3.4. This proves that $E$ is injective.

**4.4. Proposition.** *The category* mod($\Sigma$, E) *has enough injectives iff* mod $\Sigma$ *has enough injectives.*

**Proof.** This follows from Propositions 3.10 and 4.2 and Lemma 4.3.

This result substantially improves a similar result by Howlett [5]. Here, we deal with quasi-equational classes of algebras rather than equational classes as [5] does, but more importantly, our proof does not use the points of the topos whereas [5] only proves this result for a Grothendieck topos with enough points. Moreover, Proposition 3.10 provides a positive answer to Howlett's question [5, p. 108] whether essential boundedness of mod $\Sigma$ directly implies that of mod($\Sigma$, E).

## 5. Behaviour of injectivity in mod($\Sigma$, E)

**5.1.** Banaschewski in [1] calls the notion of injectivity in a category **K** *properly behaved* iff the following three propositions hold, which describe the relationship between essential boundedness, residual smallness and the existence of injective hulls in **K**. Actually [1] deals with injectivity with respect to a more general type of morphism, but of course, here we only consider injectivity with respect to all monomorphisms.

(I) For any $A \in \mathbf{K}$ the following conditions are equivalent:
 (I1) $A$ is injective.
 (I2) $A$ is an absolute retract.
 (I3) $A$ has no proper essential extensions.

(E) Every $A \in \mathbf{K}$ has an injective hull, unique up to isomorphisms.

(H) For any monomorphism $f: A \to B$, the following conditions are equivalent:
 (H1) $f: A \to B$ is an injective hull of $A$.
 (H2) $f: A \to B$ is a maximal essential extension.
 (H3) $f: A \to B$ is a minimal injective extension.

**5.2.** [1] also gives sufficient conditions for the proper behaviour of injectivity in **K** as follows:

(E3) For any monomorphism $f: A \to B$, there exists a homomorphism $g: B \to C$ with $gf$ an essential monomorphism.

(E4) Any diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle g}\downarrow & & \\ C & & \end{array}$$

with $f$ a monomorphism can be completed to a commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle v} \\ C & \xrightarrow{u} & D \end{array}$$

such that $u$ is a monomorphism.

(E5) Any direct limit of monomorphisms is a monomorphism.

(E6) The category **K** is essentially bounded.

For $\mathbf{K} = \mathrm{mod}(\Sigma, \mathbf{E})$, we now have the following counterpart of Proposition 5 in [1] for equational classes of algebras in Ens.

**5.3. Proposition.** *For* $\mathrm{mod}(\Sigma, \mathbf{E})$, *the following are equivalent:*

(i) *Injectivity is properly behaved.*

(ii) $\mathrm{Mod}(\Sigma, \mathbf{E})$ *has enough injectives.*

(iii) $\mathrm{Mod}(\Sigma, \mathbf{E})$ *is residually small and pushouts transfer monomorphisms.*

(iv) (E4) *and* (E6) *are satisfied.*

**Proof.** (i) ⇒ (ii): By (E) in the definition of proper behaviour of injectivity.

(ii) ⇒ (iii): By Lemma 4.3.

(iii) ⇒ (iv): (E4) is trivial, by completing any diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle g}\downarrow & & \\ C & & \end{array}$$

to a pushout, and (E6) holds by Corollary 3.9.

(iv) → (i): It remains to show (E3) and (E5). But Lemma 3.3 proves (E3), and (E5) is discussed in the proof of Lemma 3.2 (ii).

In particular, one has, by Proposition 4.4:

*Injectivity is properly behaved in* $\mathrm{mod}(\Sigma, \mathbf{E})$ *iff it is properly behaved in* $\mathrm{mod}\,\Sigma$.

We conclude with a couple of comments on injectivity in mod($\Sigma$, **C**).

Since each of the functors

$$\text{mod}(\Sigma, \mathbf{E}) \xrightarrow{(V,-)} \text{mod}\,\Sigma \quad (V \in \mathbf{C})$$

has a left adjoint preserving monomorphisms, and by the well known fact that such a functor preserves injectives, if $A \in \text{mod}(\Sigma, \mathbf{E})$ is injective, then so is each $AV$ ($V \in \mathbf{C}$). However, the converse of this is not true; for counter examples, in the case of abelian groups, the reader is referred to [2].

For certain $\Sigma$, one has characterizations of the injective $A \in \text{mod}\,\Sigma$ by properties of $A$ in terms of its elements of its elements and subsets, for example: divisibility for abelian groups, completeness for Boolean algebras, and completeness and Booleanness for distributive lattices. An obvious question to ask is to what extent, that is for what **E**, such characterizations remain valid in mod($\Sigma$, **E**). The only case where anything is known about this is that of abelian groups: divisibility = injectivity for abelian groups in the category Sh **L** of sheaves on a locale **L** iff **L** is Boolean [3].

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