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A method for the construction of complete congruences on lattices of pseudovarieties

K. Auinger*

Institut für Mathematik, Universität Wien, Strudlhofgasse 4, A-1090 Wien, Austria

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Abstract

We present a method for the construction of complete congruences on lattices of pseudovarieties, thereby generalizing and unifying the concept of radical congruence system and other constructions invented by Petrich and Reilly. For similarly defined classes such as varieties, generalized varieties, quasivarieties, existence varieties, etc., similar methods can be obtained. © 1998 Elsevier Science B.V.

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1. Introduction

The present paper is an extension of [6] where the authors started the investigation of certain complete congruences and operators on $\mathscr{L}(\mathbf{S})$, the lattice of all pseudovarieties of finite semigroups. Recall that – in general – a pseudovariety $\mathbf{V}$ is a class of finite algebras (of any type of algebra) which is closed under taking finitary direct products, subalgebras and morphic images. Of particular interest is the pseudovariety $\mathbf{S}$ of all finite semigroups due to its strong relationship to language theory (see, for example, the books of Almeida [1], Eilenberg [9], Howie [12], and Pin [22], and the references given there). Similarly to many lattices of varieties it is more or less hopeless to obtain a complete (good) description of the lattice $\mathscr{L}(\mathbf{S})$. However, the study of complete congruences and operators on lattices of varieties has turned out to be very fruitful for the investigation of the local as well as the global behaviour of such lattices (see, for instance, Polák's work on the lattice of completely regular semigroup varieties

* E-mail: karla@banach.mat.univie.ac.at.

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[24–26]). We intend to carry over some of these advantages to lattices of pseudovarieties, in particular to the lattice $\mathscr{L}(\mathbf{S})$.

The purpose of this paper is to introduce a general method for finding complete congruences on lattices of pseudovarieties (the results in part remain true for lattices of varieties as well (see [3])). This method generalizes and unifies certain concepts found earlier by Petrich and Reilly [20, 21]. The crucial notion will be that of a *divisor system* $\mathscr{D}$ for a pseudovariety $\mathbf{V}$ (where $\mathbf{V}$ is a pseudovariety of any type). Recall that by a divisor of an algebra A is meant any morphic image of some substructure of A. Such a divisor system $\mathscr{D}$ selects for each algebra $A \in \mathbf{V}$ a set of divisors $\mathscr{D}(A)$ of A such that the following conditions are satisfied:

(D1) if $A \in \mathscr{D}(B)$ for some $B \in \mathbf{V}$ then $A \in \mathscr{D}(A)$;

(D2) each member of $\mathscr{D}(A \times B)$ divides a direct product $A' \times B'$ for some $A' \in \mathscr{D}(A)$ and $B' \in \mathscr{D}(B)$;

(D3) if A divides B then each member of $\mathscr{D}(A)$ divides a member of $\mathscr{D}(B)$.

It turns out that for every divisor system $\mathscr{D}$ of $\mathbf{V}$, the binary relation on the lattice of subpseudovarieties $\mathscr{L}(\mathbf{V})$ of $\mathbf{V}$, defined by

$$\mathbf{A} D \mathbf{B} \Leftrightarrow \mathscr{D}(\mathbf{A}) = \mathscr{D}(\mathbf{B}),$$

is a complete congruence (where for any class $\mathbf{C}$, $\mathscr{D}(\mathbf{C})$ stands for $\bigcup_{C \in \mathbf{C}} \mathscr{D}(C)$). Moreover, the congruence classes $\mathbf{A}D$ with respect to this congruence (these classes are of course bounded intervals) can be described very neatly. Furthermore, having two divisor systems $\mathscr{D}$ and $\mathscr{D}'$, say, with associated congruences D and D', then the join $D \sqcup D'$ within the lattice of all complete congruences on $\mathscr{L}(\mathbf{V})$ can be described as

$$\mathbf{A}\; D \sqcup D'\; \mathbf{B} \Leftrightarrow \mathscr{D}(\mathbf{A}) \cap \mathscr{D}'(\mathbf{A}) = \mathscr{D}(\mathbf{B}) \cap \mathscr{D}'(\mathbf{B}).$$

We shall also give a necessary and sufficient condition in order that $D \sqcup D'$ coincides with $D \vee D'$ where the latter denotes the join of D and D' within the lattice of all congruences on $\mathscr{L}(\mathbf{S})$. The results mentioned so far will be obtained in Section 3.

The pseudovariety $\mathbf{S}$ of all semigroups turns out to be quite well equipped as far as divisor systems are concerned. All congruences obtained in [6] by means of "radical congruence systems" can be easily seen to be obtained from the present more general concept of divisor system. However, in addition to these, we will find a number of divisor systems for $\mathbf{S}$ which will in turn give us many more congruences on the lattice of pseudovarieties of semigroups. Among these will be the following:

- the set of all local submonoids of S;
- the (set of the) idempotent generated subsemigroup of S;
- the set of all simple groups dividing S;
- the set of all regular principal factors of S.

These and other applications of the results of Section 3 to the lattice of pseudovarieties of semigroups will be obtained in Section 4.

2. Preliminaries

For undefined notions and unproved results of semigroup theory the reader is referred to the books of Clifford and Preston [7] or Howie [11]; background knowledge in pseudovarieties can be found in [1, 9, 12] or [22]; as a reference in universal algebra the reader may consult [10]. On any set, the identical and universal relations will be denoted by ε and ω, respectively. For any semigroup S, $E(S)$, or E if the context is clear, denotes the set of idempotents of S. The *core* of S, that is, the subsemigroup of S generated by its idempotents, will be denoted by $\langle E(S)\rangle$. More generally, if A is a subset of the semigroup S then $\langle A\rangle$ denotes the semigroup generated by A (within S). An element $x\in S$ is regular if $x = x\,y\,x$ for some $y\in S$. By $\mathrm{Reg}(S)$ we denote the set of all regular elements of S. By $\mathrm{Gr}(S)$ we mean the set of elements in subgroups of S.

We say that a semigroup S *divides* a semigroup S', symbolically $S \prec S'$, if S is a morphic image of a subsemigroup of S'. A *pseudovariety* $\mathbf{V}$ of semigroups is a class of finite semigroups which is closed under the taking of finitary direct products, subsemigroups and morphic images. Equivalently, $\mathbf{V}$ is closed under finitary direct products and division. By $\mathbf{S}$ we denote the pseudovariety of all finite semigroups, by $\mathscr{L}(\mathbf{S})$ the lattice of all pseudovarieties of semigroups. For any class $\mathbf{V}\subseteq\mathbf{S}$ denote by $\operatorname{pvar}\mathbf{V}$ the pseudovariety generated by $\mathbf{V}$. We shall not give the notion of pseudoidentity, pseudoword (implicit operation) etc. here; the interested reader is referred to the book of Almeida [1, Chapter 3]. We list some pseudovarieties which will be used later:

$E\mathbf{I} = [\![x^\omega = y^\omega]\!]$	idempotents are trivial (= one element),
$E\mathbf{LZ} = [\![x^\omega y^\omega = x^\omega]\!]$	idempotents are a left zero semigroup,
$E\mathbf{RZ} = [\![x^\omega y^\omega = y^\omega]\!]$	idempotents are a right zero semigroup,
$L\mathbf{G} = [\![(x^\omega y x^\omega)^\omega = x^\omega]\!]$	local groups,
$\mathbf{N} = [\![x^\omega = 0]\!]$	nilpotent semigroups,
$\mathbf{K} = [\![x^\omega y = x^\omega]\!]$	idempotents are an ideal and a left zero semigroup,
$\mathbf{D} = [\![y x^\omega = x^\omega]\!]$	idempotents are an ideal and a right zero semigroup,
$L\mathbf{I} = [\![x^\omega y x^\omega = x^\omega]\!]$	locally trivial semigroups,
$\mathbf{S}l = [\![x^2 = x,\ xy = yx]\!]$	semilattices,
$\mathbf{I} = [\![x = y]\!]$	trivial semigroups.

The next result will be of essential use in the paper. For a proof see [18, Lemma 4.13].

Lemma 2.1. *Let ρ be an equivalence relation on the complete lattice L such that each ρ-class $x\rho$ is a bounded interval $x\rho = [x_\rho, x^\rho]$. Then ρ is a complete congruence if and only if for all $x, y\in L$, $x \le y$ implies $x_\rho \le y_\rho$ and $x^\rho \le y^\rho$.*

Finally, for some applications, the following lemma due to Rhodes [29, Proposition 3.2] turns out to be useful.

Lemma 2.2. *Let $\phi : S_1 \to S_2$ be an epimorphism between finite semigroups S_1 and S_2. Let J_2 be a $\mathscr{J}$-class of S_2. Then $J_2\phi^{-1}$ is a union of $\mathscr{J}$-classes of S_1. If J_1 is* *a $\mathscr{J}$-class in $J_2\phi^{-1}$, minimal between these with respect to the usual ordering of $\mathscr{J}$-classes, then $J_1\phi = J_2$. The map $\phi|J_1$ extends to a surjective morphism among the corresponding principal factors $J_1^0 \to J_2^0$. Moreover, each $\mathscr{R}$ (respectively $\mathscr{L}$)-class of S in J_1 is mapped onto an $\mathscr{R}$ (respectively $\mathscr{L}$)-class in J_2 and each $\mathscr{R}$ (respectively $\mathscr{L}$)-class in J_2 is obtained in this way. Also J_1 is regular if and only if J_2 is regular. In this case, each maximal subgroup in J_1 is mapped onto a maximal subgroup in J_2, and each maximal subgroup in J_2 is obtained in this way.*

3. Congruence relations on $\mathscr{L}(\mathbf{S})$ via divisor systems

All results in this section hold for pseudovarieties $\mathbf{V}$ of any type of algebra. However, we formulate the results for the pseudovariety $\mathbf{S}$ of semigroups. Let $\mathscr{D}$ be an operator which assigns to each finite semigroup S a collection $\mathscr{D}(S)$ of divisors of S. For arbitrary classes $\mathbf{A}, \mathbf{B}$ of finite semigroups put $\mathbf{A} \times \mathbf{B} = \{A \times B \mid A \in \mathbf{A}, B \in \mathbf{B}\}$ and $\mathscr{D}(\mathbf{A}) = \bigcup_{A \in \mathbf{A}} \mathscr{D}(A)$. Furthermore, we say that $\mathbf{A}$ *divides* $\mathbf{B}$, denoted by $\mathbf{A} \prec \mathbf{B}$, if each member of $\mathbf{A}$ divides a member of $\mathbf{B}$. Now we give the central definition of the paper, in a concise form. The concepts and some of the proofs in this section are inspired by methods and results developed by Petrich and Reilly in [20, 21] and extended by Reilly and Zhang [27].

Definition 3.1. An operator $\mathscr{D}$ is a *divisor system* for the pseudovariety $\mathbf{S}$ if the conditions (D0–D3) are satisfied:

(D0) for each $S \in \mathbf{S}$, $\mathscr{D}(S)$ is a set of divisors of S;

(D1) if $S \in \mathscr{D}(S')$ for some S' then $S \in \mathscr{D}(S)$;

(D2) for any S, S', $\mathscr{D}(S \times S') \prec \mathscr{D}(S) \times \mathscr{D}(S')$;

(D3) if $S \prec S'$ then $\mathscr{D}(S) \prec \mathscr{D}(S')$.

Let $\mathbf{V}$ be any class of finite semigroups and $\mathscr{D}$ be any divisor system. We put $\mathbf{V}^{\mathscr{D}} = \{S \in \mathbf{S} \mid \mathscr{D}(S) \subseteq \mathbf{V}\}$. The *divisor class* associated with $\mathscr{D}$ is the class $\mathscr{D}(\mathbf{S}) = \bigcup_{S \in \mathbf{S}} \mathscr{D}(S)$. In view of (D1), we have that $\mathscr{D}(\mathbf{S}) = \{S \in \mathbf{S} \mid S \in \mathscr{D}(S)\}$.

Lemma 3.1. *Let $\mathscr{D}$ be a system satisfying* (D2) *and* (D3) *and let $\mathbf{V}$ be a pseudovariety; then $\mathbf{V}^{\mathscr{D}}$ is also a pseudovariety.*

Proof. Let $S, S' \in \mathbf{V}^{\mathscr{D}}$, that is $\mathscr{D}(S), \mathscr{D}(S') \subseteq \mathbf{V}$. Then also $\mathscr{D}(S) \times \mathscr{D}(S') \subseteq \mathbf{V}$. By (D2) $\mathscr{D}(S \times S') \prec \mathscr{D}(S) \times \mathscr{D}(S')$ and thus $\mathscr{D}(S \times S') \subseteq \mathbf{V}$. Hence $S \times S' \in \mathbf{V}^{\mathscr{D}}$. Let $S \prec S'$ and $S' \in \mathbf{V}^{\mathscr{D}}$. By (D3) we easily get $\mathscr{D}(S) \subseteq \mathbf{V}$, whence $S \in \mathbf{V}^{\mathscr{D}}$. □

We are able to prove one of the main results of this section.

Theorem 3.2. *Let $\mathscr{D}$ be a divisor system for the pseudovariety $\mathbf{S}$. Then the relation defined on $\mathscr{L}(\mathbf{S})$ by*

$$\mathbf{V} \, D \, \mathbf{W} \Leftrightarrow \mathbf{V} \cap \mathscr{D}(\mathbf{S}) = \mathbf{W} \cap \mathscr{D}(\mathbf{S}) \tag{1}$$

is a complete congruence on $\mathscr{L}(\mathbf{S})$. *The congruence classes with respect to this congruence are the intervals*

$$\mathbf{V}D = [\text{pvar}(\mathbf{V} \cap \mathscr{D}(\mathbf{S})), \mathbf{V}^{\mathscr{D}}]. \tag{2}$$

Proof. The proof is essentially by the same idea as the proof of Theorem 5.1 in [20]. First of all, the relation D defined by (1) is an equivalence relation all of whose equivalence classes are convex subsets of $\mathscr{L}(\mathbf{S})$. We first show that each equivalence class $\mathbf{V}D$ is given by (2). Let $\mathbf{V}, \mathbf{W} \in \mathscr{L}(\mathbf{S})$ with $\mathbf{V}\ D\ \mathbf{W}$, that is, $\mathbf{V} \cap \mathscr{D}(\mathbf{S}) = \mathbf{W} \cap \mathscr{D}(\mathbf{S})$. Then $\text{pvar}(\mathbf{V} \cap \mathscr{D}(\mathbf{S})) = \text{pvar}(\mathbf{W} \cap \mathscr{D}(\mathbf{S})) \subseteq \mathbf{W}$. Let $S \in \mathbf{W}$; then $\mathscr{D}(S) \subseteq \mathbf{W} \cap \mathscr{D}(\mathbf{S}) = \mathbf{V} \cap \mathscr{D}(\mathbf{S})$. So by definition we have $S \in \mathbf{V}^{\mathscr{D}}$. We have shown that $\mathbf{V}\ D \subseteq [\text{pvar}(\mathbf{V} \cap \mathscr{D}(\mathbf{S}), \mathbf{V}^{\mathscr{D}}]$. Next we show that $\text{pvar}(\mathbf{V} \cap \mathscr{D}(\mathbf{S}))\ D\ \mathbf{V}\ D\ \mathbf{V}^{\mathscr{D}}$. Clearly, we have that $\text{pvar}(\mathbf{V} \cap \mathscr{D}(\mathbf{S})) \cap \mathscr{D}(\mathbf{S}) \subseteq \mathbf{V} \cap \mathscr{D}(\mathbf{S}) \subseteq \text{pvar}(\mathbf{V} \cap \mathscr{D}(\mathbf{S}))$. Intersection with $\mathscr{D}(\mathbf{S})$ gives $\text{pvar}(\mathbf{V} \cap \mathscr{D}(\mathbf{S})) \cap \mathscr{D}(\mathbf{S}) = \mathbf{V} \cap \mathscr{D}(\mathbf{S})$, that is, $\text{pvar}(\mathbf{V} \cap \mathscr{D}(\mathbf{S}))\ D\ \mathbf{V}$. Since $\mathbf{V} \subseteq \mathbf{V}^{\mathscr{D}}$ we clearly have that $\mathbf{V} \cap \mathscr{D}(\mathbf{S}) \subseteq \mathbf{V}^{\mathscr{D}} \cap \mathscr{D}(\mathbf{S})$. To show the reverse inclusion, let $S \in \mathbf{V}^{\mathscr{D}} \cap \mathscr{D}(\mathbf{S})$. That is, $S \in \mathbf{V}^{\mathscr{D}}$ and there is $S' \in \mathbf{S}$ such that $S \in \mathscr{D}(S')$. By (D1), $S \in \mathscr{D}(S)$. But $S \in \mathbf{V}^{\mathscr{D}}$ says that $\mathscr{D}(S) \subseteq \mathbf{V}$, whence $S \in \mathscr{D}(S) \subseteq \mathbf{V}$. That is, $\mathbf{V}^{\mathscr{D}} \cap \mathscr{D}(\mathbf{S}) \subseteq \mathbf{V} \cap \mathscr{D}(\mathbf{S})$, so equality holds and therefore, $\mathbf{V}\ D\ \mathbf{V}^{\mathscr{D}}$. Since each D-class is a convex subset of $\mathscr{L}(\mathbf{S})$, $\mathbf{V}D$ is precisely the interval (2). Finally, if $\mathbf{V} \subseteq \mathbf{W}$ then clearly $\text{pvar}(\mathbf{V} \cap \mathscr{D}(\mathbf{S})) \subseteq \text{pvar}(\mathbf{W} \cap \mathscr{D}(\mathbf{S}))$ and $\mathbf{V}^{\mathscr{D}} \subseteq \mathbf{W}^{\mathscr{D}}$. By Lemma 2.1 we see that D is a complete congruence on $\mathscr{L}(S)$. □

For any pseudovariety $\mathbf{V}$, notice that $\mathbf{V} \cap \mathscr{D}(\mathbf{S}) = \mathscr{D}(\mathbf{V})$. So the definition of the congruence D in Theorem 3.2 could be formulated more compactly as

$$\mathbf{V}\,D\,\mathbf{W} \Leftrightarrow \mathscr{D}(\mathbf{V}) = \mathscr{D}(\mathbf{W}).$$

For several reasons we will, however, mostly use the formulation of Theorem 3.2. The last assertion in the above proof, namely that D is a complete congruence on $\mathscr{L}(\mathbf{S})$ is also a consequence of the next result. For any divisor system $\mathscr{D}$ put $\mathscr{L}_{\mathscr{D}}(\mathbf{S}) = \{\mathbf{V} \cap \mathscr{D}(\mathbf{S}) \mid \mathbf{V} \in \mathscr{L}(\mathbf{S})\}$. This latter set is partially ordered by inclusion, it is closed under taking arbitrary intersections and possesses the greatest element $\mathscr{D}(\mathbf{S})$. Hence $\mathscr{L}_{\mathscr{D}}(\mathbf{S})$ forms a complete lattice under inclusion.

Proposition 3.3. *The mapping* $\phi: \mathscr{L}(\mathbf{S}) \to \mathscr{L}_{\mathscr{D}}(\mathbf{S})$, $\mathbf{V} \mapsto \mathbf{V} \cap \mathscr{D}(\mathbf{S})$ *is a complete surjective lattice morphism.*

Proof. By the definition of $\mathscr{L}_{\mathscr{D}}(\mathbf{S})$, ϕ is surjective and ϕ clearly respects arbitrary intersections. So it remains to prove that ϕ respects arbitrary joins. Let $\{\mathbf{V}_i \mid i \in I\}$ be any set of pseudovarieties. Since ϕ is monotone for each particular $j \in I$, we have that $\mathbf{V}_j\phi \subseteq (\bigvee_{i\in I} \mathbf{V}_i)\phi$ which trivially implies that

$$\bigvee_{i\in I} (\mathbf{V}_i\phi) \subseteq \left(\bigvee_{i\in I} \mathbf{V}_i\right)\phi.$$

We therefore have to show the reverse inclusion

$$\left(\bigvee_{i\in I} \mathbf{V}_i\right)\phi \subseteq \bigvee_{i\in I} (\mathbf{V}_i\phi).$$

Recalling the definition of the join within $\mathscr{L}_{\mathscr{D}}(\mathbf{S})$ we notice that

$$\bigvee_{i\in I} (\mathbf{V}_i\phi) = \bigcap \left\{\mathbf{W}\phi \mid \mathbf{W} \in \mathscr{L}(\mathbf{S}) \text{ and } \left(\bigcup_{i\in I} \mathbf{V}_i\phi\right) \subseteq \mathbf{W}\phi\right\}.$$

So let $S \in (\bigvee \mathbf{V}_i)\phi = (\bigvee \mathbf{V}_i) \cap \mathscr{D}(\mathbf{S})$. Then $S \in \mathscr{D}(\mathbf{S})$ and $S \in \bigvee \mathbf{V}_i$. The first condition assures that $S \in \mathscr{D}(S)$. From the second condition we get that there are $i_1, \ldots, i_n \in I$ and $S_1 \in \mathbf{V}_{i_1}, \ldots, S_n \in \mathbf{V}_{i_n}$ such that $S \prec S_1 \times S_2 \times \cdots \times S_n$. From (D2) and (D3) we have $\mathscr{D}(S) \prec \mathscr{D}(S_1) \times \mathscr{D}(S_2) \times \cdots \times \mathscr{D}(S_n)$. Let $\mathbf{W}$ be any pseudovariety such that $\mathbf{V}_i \cap \mathscr{D}(\mathbf{S}) \subseteq \mathbf{W} \cap \mathscr{D}(\mathbf{S})$ for all $i \in I$. Then $\mathscr{D}(S_i) \subseteq \mathbf{W}$ and thus $\prod_{i=1}^n \mathscr{D}(S_i) \subseteq \mathbf{W}$. As $S \in \mathscr{D}(S) \prec \prod_{i=1}^n \mathscr{D}(S_i) \subseteq \mathbf{W}$ we get $S \in \mathbf{W}$ since $\mathbf{W}$ is closed under division. As already mentioned, $S \in \mathscr{D}(\mathbf{S})$. Therefore $S \in \mathbf{W} \cap \mathscr{D}(\mathbf{S})$. In turn, this fact holds for *each* $\mathbf{W}$ with $\bigcup_{i\in I}(\mathbf{V}_i \cap \mathscr{D}(\mathbf{S})) \subseteq \mathbf{W} \cap \mathscr{D}(\mathbf{S})$. Hence S is contained in the intersection of all such sets $\mathbf{W} \cap \mathscr{D}(\mathbf{S})$ which is precisely the join of all sets $\mathbf{V}_i \cap \mathscr{D}(\mathbf{S}) = \mathbf{V}_i\phi$ within the lattice $\mathscr{L}_{\mathscr{D}}(\mathbf{S})$. □

Again, if we write $\mathscr{D}(\mathbf{V})$ instead of $\mathbf{V} \cap \mathscr{D}(\mathbf{S})$ then Proposition 3.3 may be restated as: the mapping $\mathscr{D}: \mathscr{L}(\mathbf{S}) \to \mathscr{L}_{\mathscr{D}}(\mathbf{S})$, $\mathbf{V} \mapsto \mathscr{D}(\mathbf{V})$ is a complete surjective lattice morphism.

Remark. (1) Consider the axiom

(D2′) $\mathscr{D}(S \times S') \subseteq \mathrm{pvar}(\mathscr{D}(S) \cup \mathscr{D}(S'))$.

All results in this section will stay true if instead of (D2) the weaker axiom (D2′) is used to define a divisor system.

(2) Obvious modifications in the definition of divisor system lead to analogous concepts for varieties, existence varieties (see [3]]), generalized varieties (see [2; 1, Chapter 3]), quasivarieties and similarly defined classes. For instance, if we consider generalized varieties then (D2) has to be replaced by two axioms which reflect the fact that generalized varieties are closed under finitary direct products and arbitrary powers; for the case of quasivarieties, "divisors" have to replaced by "substructures", and (D2) has to be modified in order to reflect that quasivarieties are closed under arbitrary direct products and ultraproducts. The proofs in the present paper can be easily transformed to any of these cases, with the exceptions of the proofs of Theorems 3.4 and 3.5, which essentially require finiteness of the underlying structures. That is, these theorems remain true, for instance for e-pseudovarieties of regular semigroups (for a definition see [15]) but are no longer true for varieties. In contrast, Corollaries 3.6 and 3.7 do not require finiteness (see [3]).

In the following, we shall study the behaviour and interrelationship between two complete congruences D, B, say, given by two divisor systems $\mathscr{D}$ and $\mathscr{B}$. For these congruences we shall denote their join within the lattice of all congruences on $\mathscr{L}(\mathbf{S})$ by $D \vee B$, and their join within the lattice of all complete congruences on $\mathscr{L}(\mathbf{S})$ by $D \sqcup B$. Then $D \vee B$ is just the transitive closure of $D \cup B$ whereas $D \sqcup B$ is the intersection of all complete congruences on $\mathscr{L}(\mathbf{S})$ which contain $D \cup B$. In general, $D \vee B \subseteq D \sqcup B$, however, equality need not be true. It turns out that the join $D \sqcup B$ can be described conveniently by making use of composition of the corresponding divisor systems $\mathscr{D}$ and $\mathscr{B}$. For any $S \in \mathbf{S}$ we put $\mathscr{D}\mathscr{B}(S) = \bigcup_{T \in \mathscr{B}(S)} \mathscr{D}(T)$. Divisor systems may be viewed as relations $\mathbf{S} \to \mathbf{S}$. Then the above-defined composition $\mathscr{D}\mathscr{B}$ is associative and we may compose divisor systems arbitrarily without the necessity of writing brackets. In particular, for each $n \in \mathbb{N}$, $(\mathscr{D}\mathscr{B})^n(S)$ is well defined and so is $\mathbf{V}^{(\mathscr{D}\mathscr{B})^n}$ for each class $\mathbf{V} \subseteq \mathbf{S}$. If $\mathbf{V}$ is a pseudovariety then so is $\mathbf{V}^{(\mathscr{D}\mathscr{B})^n}$ for each $n \in \mathbb{N}$ since $\mathbf{V}^{(\mathscr{D}\mathscr{B})^n} = (\ldots((\mathbf{V}^{\mathscr{D}})^{\mathscr{B}})^{\mathscr{D}}\ldots)^{\mathscr{B}}$. Moreover, for all $n \in \mathbb{N}$ it is easy to see that $\mathbf{V}^{(\mathscr{D}\mathscr{B})^n} \subseteq \mathbf{V}^{(\mathscr{D}\mathscr{B})^{n+1}}$ provided $\mathbf{V}$ is a pseudovariety. Furthermore, put $\mathbf{V}^{(\mathscr{D}\mathscr{B})^\infty} = \bigvee_{n \geq 1} \mathbf{V}^{(\mathscr{D}\mathscr{B})^n}$, the pseudovariety join of the sequence $(\mathbf{V}^{(\mathscr{D}\mathscr{B})^n})_{n \in \mathbb{N}}$. Then $\mathbf{V}^{(\mathscr{D}\mathscr{B})^\infty} = \bigcup_{n \geq 1} \mathbf{V}^{(\mathscr{D}\mathscr{B})^n}$ since the set-theoretic union of an ascending chain of pseudovarieties is again a pseudovariety. In view of these notations we are able to formulate the next result.

Theorem 3.4. *Let $\mathscr{D}, \mathscr{B}$ be divisor systems for $\mathbf{S}$ with induced congruences D, B on $\mathscr{L}(\mathbf{S})$. Then the join $D \sqcup B$ within the lattice of all complete congruences on $\mathscr{L}(\mathbf{S})$ is given by*

$$\mathbf{V}\ D \sqcup B\ \mathbf{W} \Leftrightarrow \mathbf{V} \cap \mathscr{D}(\mathbf{S}) \cap \mathscr{B}(\mathbf{S}) = \mathbf{W} \cap \mathscr{D}(\mathbf{S}) \cap \mathscr{B}(\mathbf{S}). \qquad (*)$$

Moreover, the congruence classes with respect to $D \sqcup B$ are the intervals

$$\mathbf{V}(D \sqcup B) = [\mathrm{pvar}(\mathbf{V} \cap \mathscr{D}(\mathbf{S}) \cap \mathscr{B}(\mathbf{S})), \mathbf{V}^{(\mathscr{D}\mathscr{B})^\infty}]. \qquad (**)$$

Proof. (1) Let $\sim$ denote the equivalence relation on $\mathscr{L}(\mathbf{S})$ which is defined by the right-hand side of $(*)$. Similarly to the appropriate argument in the proof of Theorem 3.2 one easily shows that $\mathrm{pvar}(\mathbf{V} \cap \mathscr{D}(\mathbf{S}) \cap \mathscr{B}(\mathbf{S})) \sim \mathbf{V}$. We want to show that $\mathbf{V} \sim \mathbf{V}^{(\mathscr{D}\mathscr{B})^\infty}$. Since $\mathbf{V} \subseteq \mathbf{V}^{(\mathscr{D}\mathscr{B})^\infty}$ the inclusion $\mathbf{V} \cap \mathscr{D}(\mathbf{S}) \cap \mathscr{B}(\mathbf{S}) \subseteq \mathbf{V}^{(\mathscr{D}\mathscr{B})^\infty} \cap \mathscr{D}(\mathbf{S}) \cap \mathscr{B}(\mathbf{S})$ is trivial. So let $S \in \mathbf{V}^{(\mathscr{D}\mathscr{B})^\infty} \cap \mathscr{D}(\mathbf{S}) \cap \mathscr{B}(\mathbf{S})$. Then $S \in \mathbf{V}^{(\mathscr{D}\mathscr{B})^n}$ for some $n \in \mathbb{N}$ and $(\mathscr{D}\mathscr{B})^n(S) \subseteq \mathbf{V}$. But $S \in \mathscr{D}(\mathbf{S}) \cap \mathscr{B}(\mathbf{S})$ implies $S \in \mathscr{D}(S)$ as well as $S \in \mathscr{B}(S)$. Application of $\mathscr{D}$ to the latter relation yields $\mathscr{D}(S) \subseteq \mathscr{D}\mathscr{B}(S)$ and since $S \in \mathscr{D}(S)$ also $S \in \mathscr{D}\mathscr{B}(S)$. We may now iterate this process and alternatively apply $\mathscr{B}$, $\mathscr{D}$, $\mathscr{B}, \ldots$ until we eventually get $S \in (\mathscr{D}\mathscr{B})^n(S)$. We have thus shown that $\mathbf{V}^{(\mathscr{D}\mathscr{B})^n} \cap \mathscr{D}(\mathbf{S}) \cap \mathscr{B}(\mathbf{S}) \subseteq \mathbf{V} \cap \mathscr{D}(\mathbf{S}) \cap \mathscr{B}(\mathbf{S})$ for all $n \in \mathbb{N}$. Consequently, $\mathbf{V}^{(\mathscr{D}\mathscr{B})^\infty} \cap \mathscr{D}(\mathbf{S}) \cap \mathscr{B}(\mathbf{S}) \subseteq \mathbf{V} \cap \mathscr{D}(\mathbf{S}) \cap \mathscr{B}(\mathbf{S})$ and thus, since the reverse inclusion is trivial, $\mathbf{V} \sim \mathbf{V}^{(\mathscr{D}\mathscr{B})^\infty}$.

(2) Let $\mathbf{W}$, $\mathbf{V} \in \mathscr{L}(\mathbf{S})$ and suppose that $\mathbf{V} \sim \mathbf{W}$. Then clearly $\mathrm{pvar}(\mathbf{V} \cap \mathscr{D}(\mathbf{S}) \cap \mathscr{B}(\mathbf{S})) = \mathrm{pvar}(\mathbf{W} \cap \mathscr{D}(\mathbf{S}) \cap \mathscr{B}(\mathbf{S})) \subseteq \mathbf{W}$. We intend to show that $\mathbf{W} \subseteq \mathbf{V}^{(\mathscr{D}\mathscr{B})^\infty}$. We show by induction on the size $|S|$ of S that for each $S \in \mathbf{W}$ there is some $N \in \mathbb{N}$ such that $(\mathscr{D}\mathscr{B})^n(S) \subseteq \mathbf{V}$ for each $n \geq N$. If $|S| = 1$ then the assertion is trivial since each pseudovariety contains the trivial semigroup. So let $S \in \mathbf{W}$, $|S| > 1$ and suppose the assertion is true for each $T \in \mathbf{W}$ with $|T| < |S|$. Let us consider the set $\mathscr{D}\mathscr{B}(S)$.

Case 1: $S \in \mathscr{D}\mathscr{B}(S)$. In this case $S \in \mathscr{B}(S)$ and thus $S \in \mathscr{B}(\mathbf{S})$. For if $S \notin \mathscr{B}(S)$ then $\mathscr{B}(S)$ would consist entirely of proper divisors of S and so would $\mathscr{D}\mathscr{B}(S)$. Clearly, if $S \in \mathscr{D}\mathscr{B}(S)$ then $S \in \mathscr{D}(\mathbf{S})$ and thus $S \in \mathscr{D}(\mathbf{S}) \cap \mathscr{B}(\mathbf{S})$. But then $S \in \mathbf{W} \cap \mathscr{D}(\mathbf{S}) \cap \mathscr{B}(\mathbf{S})$ so that $S \in \mathbf{V}$. Hence $S \in \mathbf{V}^{(\mathscr{D}\mathscr{B})^n}$ for all $n \in \mathbb{N}$.

Case 2: $S \notin \mathscr{D}\mathscr{B}(S)$. Then $\mathscr{D}\mathscr{B}(S)$ consists entirely of proper divisors of S, say $\mathscr{D}\mathscr{B}(S) = \{S_1, \ldots, S_k\}$, and clearly we have $S_i \in \mathbf{W}$ and $|S_i| < |S|$ for each i. By the induction hypothesis there are $N_1, \ldots, N_k \in \mathbb{N}$ such that $(\mathscr{D}\mathscr{B})^n(S_i) \subseteq \mathbf{V}$ for each $n \geq N_i$ and for each i. Now put $N = \max\{N_1, \ldots, N_k\} + 1$. Then for each $n \geq N$

$$(\mathscr{D}\mathscr{B})^n(S) = \bigcup_{i=1}^{k} (\mathscr{D}\mathscr{B})^{n-1}(S_i)$$

and, by the definition of N, $(\mathscr{D}\mathscr{B})^{n-1}(S_i) \subseteq \mathbf{V}$ for each i. Consequently, $(\mathscr{D}\mathscr{B})^n(S) \subseteq \mathbf{V}$ for all $n \geq N$ and in particular $S \in \mathbf{V}^{(\mathscr{D}\mathscr{B})^N}$. It follows that $\mathbf{W} \subseteq \mathbf{V}^{(\mathscr{D}\mathscr{B})^\infty}$. Once more, since all equivalence classes with respect to $\sim$ are convex sets, we have shown that for each pseudovariety $\mathbf{V}$,

$$\mathbf{V}\sim\, = [\text{pvar}(\mathbf{V} \cap \mathscr{D}(\mathbf{S}) \cap \mathscr{B}(\mathbf{S})), \mathbf{V}^{(\mathscr{D}\mathscr{B})^\infty}]. \qquad (\dagger)$$

(3) By the same argument as in the proof of Theorem 3.2 (making use of Lemma 2.1) we get that $\sim$ is a complete congruence on $\mathscr{L}(\mathbf{S})$.

(4) Immediately from the definition of $\sim$ it follows that $D, B \subseteq \sim$. Namely, if $\mathbf{V} \cap \mathscr{D}(\mathbf{S}) = \mathbf{W} \cap \mathscr{D}(\mathbf{S})$ then also $\mathbf{V} \cap \mathscr{D}(\mathbf{S}) \cap \mathscr{B}(\mathbf{S}) = \mathbf{W} \cap \mathscr{D}(\mathbf{S}) \cap \mathscr{B}(\mathbf{S})$. That is, $D \subseteq \sim$, and by analogy, $B \subseteq \sim$. Hence $D \sqcup B \subseteq \sim$. Let us suppose that $\mathbf{V} \sim \mathbf{W}$. From (†) we have that $\mathbf{V}^{(\mathscr{D}\mathscr{B})^\infty} = \mathbf{W}^{(\mathscr{D}\mathscr{B})^\infty}$. On the other hand, as mentioned in the discussion before Theorem 3.4, for each pseudovariety $\mathbf{U}, \mathbf{U}^{\mathscr{D}\mathscr{B}} = (\mathbf{U}^{\mathscr{D}})^{\mathscr{B}}$. Therefore, for each $n \in \mathbb{N}$ we have: $\mathbf{V}^{(\mathscr{D}\mathscr{B})^n} \; D \; \mathbf{V}^{(\mathscr{D}\mathscr{B})^n\mathscr{D}} \; B \; \mathbf{V}^{(\mathscr{D}\mathscr{B})^{n+1}}$ so that $\mathbf{V} \; (DB)^n \; \mathbf{V}^{(\mathscr{D}\mathscr{B})^n}$ for all n. A fortiori, $\mathbf{V} \; D \sqcup B \; \mathbf{V}^{(\mathscr{D}\mathscr{B})^n}$ for all $n \in \mathbb{N}$. Forming infinite joins on both sides and taking into account that $D \sqcup B$ respects arbitrary joins we get

$$\mathbf{V} = \bigvee_{n \geq 1} \mathbf{V} \; D \sqcup B \; \bigvee_{n \geq 1} \mathbf{V}^{(\mathscr{D}\mathscr{B})^n} = \mathbf{V}^{(\mathscr{D}\mathscr{B})^\infty}.$$

The same is of course true for the pseudovariety $\mathbf{W}$. From this we get that

$$\mathbf{V} \; D \sqcup B \; \mathbf{V}^{(\mathscr{D}\mathscr{B})^\infty} = \mathbf{W}^{(\mathscr{D}\mathscr{B})^\infty} \; D \sqcup B \; \mathbf{W}$$

and consequently $\mathbf{V} \; D \sqcup B \; \mathbf{W}$. □

Remark. (1) In an obvious way one can use the method of proof of Theorem 3.4 to compute joins like $D_1 \sqcup D_2 \sqcup \cdots \sqcup D_n$ of congruences D_i induced by divisor systems $\mathscr{D}_1, \ldots, \mathscr{D}_n$.

(2) It can be shown that the congruence $D \sqcup B$ itself is induced by a divisor system, say $(\mathscr{D}\mathscr{B})^\infty$, where

$$(\mathscr{D}\mathscr{B})^\infty(S) = \left\{ T \in \bigcup_{n \geq 1} (\mathscr{D}\mathscr{B})^n(S) \mid T \in \mathscr{D}(T) \cap \mathscr{B}(T) \right\}.$$

Later we shall find several examples of divisor systems $\mathscr{D},\mathscr{B}$ with induced congruences D,B such that $D \vee B \neq D \sqcup B$. However, we can formulate a neat criterion which is necessary and sufficient in order that $D \vee B = D \sqcup B$.

Theorem 3.5. *Let $\mathscr{D},\mathscr{B}$ be divisor systems for* **S** *with induced congruences D,B. Then $D \vee B = D \sqcup B$ if and only if for each pseudovariety* **V** *there is $n \in \mathbb{N}$ such that $\mathbf{V}^{(\mathscr{D}\mathscr{B})^n} = \mathbf{V}^{(\mathscr{D}\mathscr{B})^{n+1}} = \cdots = \mathbf{V}^{(\mathscr{D}\mathscr{B})^\infty}$.*

Proof. Notice that, as soon as $\mathbf{V}^{(\mathscr{D}\mathscr{B})^n} = \mathbf{V}^{(\mathscr{D}\mathscr{B})^{n+1}}$ the sequence is already stable: $\mathbf{V}^{(\mathscr{D}\mathscr{B})^n} = \mathbf{V}^{(\mathscr{D}\mathscr{B})^k}$ for all $k \geq n$ and $\mathbf{V}^{(\mathscr{D}\mathscr{B})^n} = \mathbf{V}^{(\mathscr{D}\mathscr{B})^\infty}$. Let us prove the "if" statement first. Suppose, for each pseudovariety **V** there is $n \in \mathbb{N}$ such that $\mathbf{V}^{(\mathscr{D}\mathscr{B})^n} = \mathbf{V}^{(\mathscr{D}\mathscr{B})^\infty}$. Let $\mathbf{V}\ D \sqcup B\ \mathbf{W}$. Then there are $n,m \in \mathbb{N}$ such that $\mathbf{V}^{(\mathscr{D}\mathscr{B})^n} = \mathbf{V}^{(\mathscr{D}\mathscr{B})^\infty}$ and $\mathbf{W}^{(\mathscr{D}\mathscr{B})^m} = \mathbf{W}^{(\mathscr{D}\mathscr{B})^\infty}$. Furthermore, $\mathbf{V}\ D \sqcup B\ \mathbf{W}$ implies $\mathbf{V}^{(\mathscr{D}\mathscr{B})^\infty} = \mathbf{W}^{(\mathscr{D}\mathscr{B})^\infty}$. Since (as has been pointed out in the proof of Theorem 3.4) $\mathbf{V}\ (DB)^n\ \mathbf{V}^{(\mathscr{D}\mathscr{B})^n}$ and $\mathbf{W}^{(\mathscr{D}\mathscr{B})^m}\ (BD)^m\ \mathbf{W}$ we get $\mathbf{V}\ D \vee B\ \mathbf{V}^{(\mathscr{D}\mathscr{B})^n} = \mathbf{W}^{(\mathscr{D}\mathscr{B})^m}\ D \vee B\ \mathbf{W}$, whence $\mathbf{V}\ D \vee B\ \mathbf{W}$. Now to the direct part. Suppose that $D \vee B = D \sqcup B$ and let $\mathbf{V} \in \mathscr{L}(\mathbf{S})$. By Theorem 3.4 we have $\mathbf{V}\ D \sqcup B\ \mathbf{V}^{(\mathscr{D}\mathscr{B})^\infty}$. Since $D \vee B = D \sqcup B$ there exist $\mathbf{V}_1,\ldots,\mathbf{V}_n,\mathbf{W}_1,\ldots,\mathbf{W}_{n-1} \in \mathscr{L}(\mathbf{S})$ such that

$$\mathbf{V} = \mathbf{V}_1\ D\ \mathbf{W}_1\ B\ \mathbf{V}_2\ D\ \mathbf{W}_2 \ldots \mathbf{W}_{n-1}\ B\ \mathbf{V}_n = \mathbf{V}^{(\mathscr{D}\mathscr{B})^\infty}.$$

In particular, we have $\mathbf{V}_i^{\mathscr{D}} = \mathbf{W}_i^{\mathscr{D}}$ and $\mathbf{W}_i^{\mathscr{B}} = \mathbf{V}_{i+1}^{\mathscr{B}}$ for all i. Notice that the operators $\mathbf{U} \mapsto \mathbf{U}^{\mathscr{D}}$ and $\mathbf{U} \mapsto \mathbf{U}^{\mathscr{B}}$ are monotone. Now we have that $\mathbf{V}^{\mathscr{D}} = \mathbf{V}_1^{\mathscr{D}} = \mathbf{W}_1^{\mathscr{D}} \supseteq \mathbf{W}_1$ and thus also $\mathbf{V}^{\mathscr{D}\mathscr{B}} = \mathbf{W}_1^{\mathscr{D}\mathscr{B}} \supseteq \mathbf{W}_1^{\mathscr{B}} = \mathbf{V}_2^{\mathscr{B}} \supseteq \mathbf{V}_2$. Application of $^{\mathscr{D}}$ gives us $\mathbf{V}^{\mathscr{D}\mathscr{B}\mathscr{D}} \supseteq \mathbf{V}_2^{\mathscr{D}} = \mathbf{W}_2^{\mathscr{D}} \supseteq \mathbf{W}_2$. By induction we easily get $\mathbf{V}^{(\mathscr{D}\mathscr{B})^i} \supseteq \mathbf{V}_{i+1}$ and $\mathbf{V}^{(\mathscr{D}\mathscr{B})^i\mathscr{D}} \supseteq \mathbf{W}_{i+1}$ for all possible i. In particular we have $\mathbf{V}^{(\mathscr{D}\mathscr{B})^{n-1}} \supseteq \mathbf{V}_n = \mathbf{V}^{(\mathscr{D}\mathscr{B})^\infty}$. On the other hand, $\mathbf{V}^{(\mathscr{D}\mathscr{B})^k} \subseteq \mathbf{V}^{(\mathscr{D}\mathscr{B})^\infty}$ for all $k \in \mathbb{N}$. Hence $\mathbf{V}^{(\mathscr{D}\mathscr{B})^{n-1}} = \mathbf{V}^{(\mathscr{D}\mathscr{B})^\infty}$, as required. □

Some special cases are described by the next two results.

Corollary 3.6. *Let $\mathscr{D}$, $\mathscr{B}$ be divisor systems for* **S** *with induced congruences D, B on $\mathscr{L}(\mathbf{S})$. If for some $n \in \mathbb{N}$, $(\mathscr{D}\mathscr{B})^n$ is a divisor system (that is, if $(\mathscr{D}\mathscr{B})^n$ satisfies* (D1)*) then $D \sqcup B = (DB)^{2n-1}D = D \vee B$.*

Proof. From Theorem 3.4 we have $\mathbf{V}\ D\ \mathbf{V}^{\mathscr{D}}$, and $\mathbf{V}\ D\ \mathbf{W}$ if and only if $\mathbf{V}^{\mathscr{D}} = \mathbf{W}^{\mathscr{D}}$. Hence $\mathbf{V}^{\mathscr{D}} = \mathbf{V}^{\mathscr{D}\mathscr{D}}$ for each pseudovariety **V** and for each divisor system $\mathscr{D}$. Therefore, if $(\mathscr{D}\mathscr{B})^n$ is a divisor system then $\mathbf{V}^{(\mathscr{D}\mathscr{B})^n} = \mathbf{V}^{(\mathscr{D}\mathscr{B})^{2n}}$ so that $\mathbf{V}^{(\mathscr{D}\mathscr{B})^n} = \mathbf{V}^{(\mathscr{D}\mathscr{B})^{n+1}}$ since $\mathbf{V}^{(\mathscr{D}\mathscr{B})^n} \subseteq \mathbf{V}^{(\mathscr{D}\mathscr{B})^{n+1}} \subseteq \cdots \subseteq \mathbf{V}^{(\mathscr{D}\mathscr{B})^{2n}}$. This holds – for the same n – uniformly in $\mathbf{V} \in \mathscr{L}(\mathbf{S})$. Let $\mathbf{W},\mathbf{V} \in \mathscr{L}(\mathbf{S})$ be such that $\mathbf{V}\ D \sqcup B\ \mathbf{W}$. Then $\mathbf{V}\ (DB)^n\ \mathbf{V}^{(\mathscr{D}\mathscr{B})^n} = \mathbf{W}^{(\mathscr{D}\mathscr{B})^n}\ (BD)^n\ \mathbf{W}$ which gives us $\mathbf{V}\ (DB)^{2n-1}D\ \mathbf{W}$, as required. □

Of particular interest (see the next section) seems to be the following case.

Corollary 3.7. *Let $\mathscr{D}$, $\mathscr{B}$ be divisor systems for* $\mathbf{S}$ *with induced congruences* D *and* B. *If* $\mathscr{D}\mathscr{B}(\mathbf{S}) = \mathscr{B}\mathscr{D}(\mathbf{S})$ *then* $\mathscr{D}\mathscr{B}$ *(and* $\mathscr{B}\mathscr{D}$*) is a divisor system and* $D \sqcup B = DBD = BDB = B \vee D$ *and this equals the congruence induced by* $\mathscr{D}\mathscr{B}$ *(and by* $\mathscr{B}\mathscr{D}$*).*

Proof. We show that $\mathscr{D}\mathscr{B}$ as well as $\mathscr{B}\mathscr{D}$ is a divisor system, the rest follows from Corollary 3.6. It can be easily seen that the operators $\mathscr{D}\mathscr{B}$ and $\mathscr{B}\mathscr{D}$ satisfy the axioms (D2) and (D3) provided $\mathscr{D}$ and $\mathscr{B}$ do. Further, if $S \in \mathscr{D}\mathscr{B}(\mathbf{S})$ then clearly $S \in \mathscr{D}(\mathbf{S})$ so that $S \in \mathscr{D}(S)$ since $\mathscr{D}$ satisfies (D1). Since $\mathscr{D}\mathscr{B}(\mathbf{S}) = \mathscr{B}\mathscr{D}(\mathbf{S})$ in the same way we get $S \in \mathscr{B}(S)$. Application of the operator $\mathscr{D}$ to the relation $S \in \mathscr{B}(S)$ yields $S \in \mathscr{D}(S) \subseteq \mathscr{D}\mathscr{B}(S)$, as required. □

4. Examples of divisor systems

4.1. The divisor systems $\mathscr{T}$, $\mathscr{T}_\ell$, $\mathscr{T}_\imath$, $\mathscr{U}$, $\mathscr{Z}$, $\mathscr{K}_\ell$, $\mathscr{K}_\imath$, $\mathscr{K}$

For any $S \in \mathbf{S}$ denote by $\mu(S), \mu_L(S), \mu_R(S), \nu(S), \zeta(S), \kappa_L(S), \kappa_R(S), \kappa(S)$, respectively, the greatest congruence on S all of whose idempotent classes belong to $E\mathbf{I}, E\mathbf{LZ}, E\mathbf{RZ}, L\mathbf{G}, \mathbf{N}, \mathbf{K}, \mathbf{D}, L\mathbf{I}$, respectively (as defined in Section 2). For existence of these congruences the reader may consult [6] or [5]. From the results therein it follows in particular that each of the following systems

(1) $\mathscr{T}(S) = \{S/\mu\}$,
(2) $\mathscr{T}_\ell(S) = \{S/\mu_L\}$,
(3) $\mathscr{T}_\imath(S) = \{S/\mu_R\}$,
(4) $\mathscr{U}(S) = \{S/\nu\}$,
(5) $\mathscr{Z}(S) = \{S/\zeta\}$,
(6) $\mathscr{K}_\ell(S) = \{S/\kappa_L\}$,
(7) $\mathscr{K}_\imath(S) = \{S/\kappa_R\}$,
(8) $\mathscr{K}(S) = \{S/\kappa\}$,

is a divisor system. We get eight complete congruences on $\mathscr{L}(\mathbf{S})$: $T, T_\ell, T_\imath, U, Z, K_\ell, K_\imath, K$ which have been introduced and studied in [6]. From Lemma 8.1 in [6] and Theorem 3.4 we get

Theorem 4.1.1.

(1) $K_\ell \sqcup K_\imath = K$,
(2) $T \sqcup K_\ell = T_\ell$, $T \sqcup K_\imath = T_\imath$,
(3) $T_\ell \sqcup T_\imath = K_\ell \sqcup T \sqcup K_\imath = T_\ell \sqcup K_\imath = K_\ell \sqcup T_\imath = T \sqcup K = U$.

On the other hand, the following relations trivially hold:

(1) $K \cap T = K_\ell \cap K_\imath = Z$,
(2) $T_\ell \cap T_\imath = T$,
(3) $T_\ell \cap K = K_\ell$, $T_\imath \cap K = K_\imath$.

Consequently, the set $\{T, T_\ell, T_\imath, U, Z, K_\ell, K_\imath, K\}$ forms a sublattice of the lattice of all complete congruences on $\mathscr{L}(\mathbf{S})$ which is isomorphic to the eight-element Boolean lattice. Notice that the above joins do not coincide with the corresponding joins within the lattice of all congruences on $\mathscr{L}(\mathbf{S})$. Indeed, using some results in [16] and/or [19] we can construct examples of bands or completely regular semigroups showing for instance that $\mathbf{S}l^{(K_\ell K_\imath)^n} \neq \mathbf{S}l^K$ and $\mathbf{S}l^{(T_\ell T_\imath)^n} \neq \mathbf{S}l^U$ for each $n \in \mathbb{N}$ where $\mathbf{S}l$ is the pseudovariety of all finite semilattices. The claim then follows from Theorem 3.5.

4.2. *The divisor system $\mathscr{C}$ (and $\mathscr{G}$r, $\mathscr{R}$eg)*

Another important example of a divisor system is defined by the core. More precisely, put $\mathscr{C}(S) = \{\langle E(S)\rangle\}$. It is easily verified that $\mathscr{C}$ is actually a divisor system. The divisor class of $\mathscr{C}$ is the class of all finite idempotent generated semigroups. So $\mathscr{C}$ leads us to the complete congruence C on $\mathscr{L}(\mathbf{S})$ defined by

$\mathbf{V}\ C\ \mathbf{W}$ if and only if $\mathbf{V}$ and $\mathbf{W}$ have the same idempotent generated members

a congruence introduced in [6, 28]. It is well known that this rule defines a complete congruence on the lattice of (pseudo)varieties of completely regular semigroups as well (see [26, 17]). Without giving detailed proofs we mention some similarly defined divisor systems. Put $\mathscr{G}\mathrm{r}(S) = \{\langle \mathrm{Gr}(S)\rangle\}$ and $\mathscr{R}\mathrm{eg}(S) = \{\langle \mathrm{Reg}(S)\rangle\}$. It is easy to establish that $\mathscr{G}$r and $\mathscr{R}$eg are divisor systems. We get two more congruences G and R, associated with the systems $\mathscr{G}$r, $\mathscr{R}$eg. From $E(S) \subseteq \mathrm{Gr}(S) \subseteq \mathrm{Reg}(S)$ we immediately get $R \subseteq G \subseteq C$. Next we present an interesting relationship between these congruences and those of Section 4.1. We use the following lemma, a proof of which can be found in [6] or [5].

Lemma 4.2.1. *Let S be a finite semigroup and $\xi \in \{\mu, \mu_L, \mu_R, \nu, \zeta, \kappa_L, \kappa_R, \kappa\}$. Then for each subsemigroup $S' \in \{\langle E(S)\rangle, \langle \mathrm{Gr}(S)\rangle, \langle \mathrm{Reg}(S)\rangle\}$ of S we have that $\xi(S') = \xi(S)|S'$.*

For the case $S' = \langle \mathrm{Gr}(S)\rangle$ one has to take into account that $\mathrm{Reg}(\langle \mathrm{Gr}(S)\rangle) = \langle \mathrm{Gr}(S)\rangle \cap \mathrm{Reg}(S)$ which follows from some ideas of Johnston and Jones to be found in [13, p. 204]. From this we get that for each substructure

$$\mathrm{Sub}(S) \in \{\langle E(S)\rangle, \langle \mathrm{Gr}(S)\rangle, \langle \mathrm{Reg}(S)\rangle\}$$

and each of the congruences ξ mentioned above,

$$\mathrm{Sub}(S/\xi) \cong \mathrm{Sub}(S)/\xi.$$

Using Corollary 3.7 we thus get the next result.

Theorem 4.2.2. (1) *For each choice of $\mathscr{X} \in \{\mathscr{T}, \mathscr{T}_\ell, \mathscr{T}_\imath, \mathscr{U}, \mathscr{Z}, \mathscr{K}_\ell, \mathscr{K}_\imath, \mathscr{K}\}$ and $\mathscr{Y} \in \{\mathscr{C}, \mathscr{R}\mathrm{eg}, \mathscr{G}\mathrm{r}\}$, $\mathscr{X}\mathscr{Y}(\mathbf{S}) = \mathscr{Y}\mathscr{X}(\mathbf{S})$ and $\mathscr{X}\mathscr{Y}$ is a divisor system.*

(2) *For the associated congruences X and Y, we have* $X \vee Y = XYX = YXY = X \sqcup Y$.

For example, two pseudovarieties $\mathbf{V}$ and $\mathbf{W}$ are $C \vee T$-related if and only if $\mathbf{V}$ and $\mathbf{W}$ contain the same idempotent generated fundamental members (recall that a semigroup S is called fundamental if $\mu(S) = \varepsilon$). Since a (finite) idempotent generated fundamental semigroup can be reconstructed from its biordered set (see [8]), in fact $C \vee T = E$ where the congruence E was found in [4] and is defined by: $\mathbf{V}\ E\ \mathbf{W}$ if and only if the members of $\mathbf{V}$ and $\mathbf{W}$ have the same biordered sets.

4.3. *The divisor systems* $\mathscr{L}\mathrm{oc}, \mathscr{L}\mathrm{oc}_\ell, \mathscr{L}\mathrm{oc}_r$

Putting

(1) $\mathscr{L}\mathrm{oc}(S) = \{eSe \mid e \in E(S)\}$,

(2) $\mathscr{L}\mathrm{oc}_\ell(S) = \{eS \mid e \in E(S)\}$,

(3) $\mathscr{L}\mathrm{oc}_r(S) = \{Se \mid e \in E(S)\}$,

we get three more divisor systems. The associated divisor classes are:

(1) $\mathscr{L}\mathrm{oc}(\mathbf{S})$ the class of all finite monoids,

(2) $\mathscr{L}\mathrm{oc}_\ell(\mathbf{S})$ the class of all finite semigroups having a left identity,

(3) $\mathscr{L}\mathrm{oc}_r(\mathbf{S})$ the class of all finite semigroups having a right identity.

Associated are three complete congruences L, L_ℓ, L_r. For example, L is defined by $\mathbf{V}\ L\ \mathbf{W}$ if and only if $\mathbf{V}$ and $\mathbf{W}$ contain the same monoids. This is a well-established congruence on the lattice of (pseudo)varieties of completely regular semigroups (see [20, 17]). As a congruence on $\mathscr{L}(\mathbf{S})$ it has also been studied by Reilly and Zhang [28]. The idea of considering one sided analogues of L comes from Zhang [30]. Since a semigroup having a left and a right identity must have an identity we get that $\mathscr{L}\mathrm{oc}(\mathbf{S}) = \mathscr{L}\mathrm{oc}_\ell(\mathbf{S}) \cap \mathscr{L}\mathrm{oc}_\ell(\mathbf{S})$. An immediate consequence of Theorem 3.4 then is

Theorem 4.3.1. $L_\ell \sqcup L_r = L$.

Next we establish some relationships with the divisor systems of Section 4.2.

Theorem 4.3.2. *For any choice* $\mathscr{X} \in \{\mathscr{C}, \mathscr{G}\mathrm{r}, \mathscr{R}\mathrm{eg}\}$ *and* $\mathscr{Y} \in \{\mathscr{L}\mathrm{oc}, \mathscr{L}\mathrm{oc}_\ell, \mathscr{L}\mathrm{oc}_r\}$, $\mathscr{X}\mathscr{Y}$ *is a divisor system. For the associated congruences X and Y it follows that*

$$X \vee Y = XYX = X \sqcup Y.$$

Proof. In order to show that $\mathscr{X}\mathscr{Y}$ is a divisor system it suffices to establish property (D1), since (D2) and (D3) are obviously true. This can be done very similarly for the different choices of $\mathscr{X}$ and $\mathscr{Y}$. Let us consider for example $\mathscr{X} = \mathscr{C}$ and $\mathscr{Y} = \mathscr{L}\mathrm{oc}_\ell$. Let $S \in \mathscr{C}\mathscr{L}\mathrm{oc}_\ell(S')$ for some S'. Then $S = \langle E(eS') \rangle$ for some idempotent $e \in E(S')$. Now $e \in S$ and therefore $S \in \mathscr{L}\mathrm{oc}_\ell(S)$. Furthermore, since $\langle E(S) \rangle = S$ we get that $S \in \mathscr{C}\mathscr{L}\mathrm{oc}_\ell(S)$. Consequently $\mathscr{C}\mathscr{L}\mathrm{oc}_\ell$ satisfies condition (D1). The rest is a consequence of Corollary 3.6. □

A similar relationship is obtained between the divisor systems of Section 4.1 and $\mathscr{L}$oc (Theorem 4.3.4). Its proof follows from the next lemma, a proof of which can be found in [5] or [28].

Lemma 4.3.3. *For each* $\xi \in \{\mu, \mu_L, \mu_R, \nu, \zeta, \kappa_L, \kappa_R, \kappa\}$ *and each* $e \in E(S)$, $\xi(S) \mid eSe = \xi(eSe)$.

Theorem 4.3.4. *For any choice* $\mathscr{X} \in \{\mathscr{T}, \mathscr{T}_\ell, \mathscr{T}_\iota, \mathscr{U}, \mathscr{Z}, \mathscr{K}_\ell, \mathscr{K}_\iota, \mathscr{K}\}$ *we have that*
(1) $\mathscr{X}\mathscr{L}\mathrm{oc}(\mathbf{S}) = \mathscr{L}\mathrm{oc}\,\mathscr{X}(\mathbf{S})$ *and* $\mathscr{X}\mathscr{L}$oc *as well as* $\mathscr{L}$oc$\,\mathscr{X}$ *are divisor systems,*
(2) $X \sqcup L = XLX = LXL = X \vee L$.

4.4. Divisor systems defined by regular $\mathscr{J}$-classes

The first of these systems has been suggested to the author by M.V. Volkov. Put $\mathscr{J}_\iota(S)$ to be the set of all principal factors of S which are completely simple or completely 0-simple. From Lemma 2.2 it is not hard to establish that $\mathscr{J}_\iota$ is indeed a divisor system. Its divisor class $\mathscr{J}_\iota(\mathbf{S})$ consists of all finite semigroups which are completely simple or completely 0-simple. The associated congruence J_ι is defined by $\mathbf{V}\,J_\iota\,\mathbf{W}$ if $\mathbf{V}$ and $\mathbf{W}$ contain the same completely (0)-simple members.

A very similar system can be defined as follows: if D is a regular $\mathscr{J}$-class of S then put $\tilde{D} = D$ if D is a subsemigroup of S and $\tilde{D} = D^0$ if D is not a subsemigroup of S (where D^0 denotes the completely 0-simple semigroup with non zero $\mathscr{J}$-class D). Put $\mathscr{D}(S) = \{\tilde{D} \mid D$ is a regular $\mathscr{J}$-class of $S\}$. Again it is not hard to show that $\mathscr{D}$ is a divisor system whose congruence we denote by D. Both systems have almost the same divisor classes, however, $\mathscr{J}_\iota(\mathbf{S})$ comprises all completely 0-simple semigroups whereas $\mathscr{D}(\mathbf{S})$ does not contain completely simple semigroups with zero adjoined. In particular, the corresponding congruences D and J_ι do not coincide as, for example, $\mathbf{I}\,D\,\mathbf{S}l$ yet $(\mathbf{I}, \mathbf{S}l) \notin J_\iota$.

For a third system of this kind the reader is asked to recall the definition of the *blocks* of a finite semigroup according to Pin [23] (this definition varies slightly from the definition of block given in [1]). For a completely 0-simple semigroup S the blocks are the $\mathscr{J}$-classes of $\langle \mathrm{Gr}(S) \rangle$. For an arbitrary finite semigroup S the blocks are the blocks of its regular principal factors. For each block B of S now put, similarly as above, $\tilde{B} = B$ if B is a subsemigroup of S and $\tilde{B} = B^0$ if B is not a subsemigroup. Let $\mathscr{B}(S) = \{\tilde{B} \mid B$ is a block of $S\}$. Again by the use of Lemma 2.2 it is not hard to show that $\mathscr{B}$ is a divisor system.

4.5. Divisor systems associated with certain pseudovarieties of completely simple semigroups

Let $\mathbf{V}$ denote any of the pseudovarieties **CS**, **L*t*G**, **R*t*G**, **G** (completely simple semigroups, left groups, right groups, groups, respectively) and let $\mathscr{D}_\mathbf{V}(S)$ be the set of all divisors of S which are in $\mathbf{V}$. Again from Lemma 2.2 it follows that $\mathscr{D}_\mathbf{V}$ is a divisor system. The divisor class $\mathscr{D}_\mathbf{V}(\mathbf{S})$ is just $\mathbf{V}$ and the mapping $\mathbf{A} \mapsto \mathbf{A} \cap \mathbf{V}$ is a complete lattice morphism $\mathscr{L}(\mathbf{S}) \to \mathscr{L}(\mathbf{V})$. The associated congruences have been mentioned in [6].

4.6. *The divisor system $\mathscr{S}_G$*

It is well known that if a simple group G divides a direct product $S_1 \times S_2$ of two finite semigroups S_1 and S_2 then G must divide S_1 or S_2. This fact ensures that the system $\mathscr{S}_G$, defined by

$$\mathscr{S}_G(S) = \text{the set of all simple groups dividing } S$$

satisfies condition (D2). Consequently $\mathscr{S}_G$ is a divisor system since (D1) and (D3) are obviously fullfilled. The corresponding congruence is given by $\mathbf{V}\ S_G\ \mathbf{W}$ if $\mathbf{V}$ and $\mathbf{W}$ contain the same simple groups. In addition, we could modify $\mathscr{S}_G$ by putting $\mathscr{S}_{G,n}(S)$ to be the set of all simple groups of order less than or equal to n, for some natural number n.

In the following, we mention a few examples of divisor systems for other types of algebras.

4.7. *The divisor system $\mathscr{S}$ for pseudovarieties of multioperator groups and congruence distributive pseudovarieties*

For any finite algebra A put $\mathscr{S}(A)$ to be the set of all simple algebras dividing A. The system $\mathscr{S}$ obviously satisfies the conditions (D1) and (D3). For pseudovarieties $\mathbf{V}$ for which $\mathscr{S}$ in addition satisfies (D2) this yields a divisor system. We have already seen that $\mathscr{S}$ is a divisor system for the pseudovariety of all groups. We give two more examples for which $\mathscr{S}$ is actually a divisor system. Recall from [14] that an algebra $(G,+,-,0,F)$ is a *multioperator group* if

(1) $(G,+,-,0)$ is a (not necessarily commutative) group,

(2) F is a set of operations such that $f(0,\dots,0)=0$ for each non-nullary operation $f \in F$.

Examples of multioperator groups are groups, rings, near rings, etc. A multioperator group enjoys the advantage that each congruence is determined by the congruence class containing 0 (which is a normal subgroup of the underlying group) and which congruence is induced by the partition into the cosets with respect to this normal subgroup (see [14, Proposition 3.21]). Using this, one can prove in the same way as for the ordinary group case:

Lemma 4.7.1. *If a finite simple multioperator group G divides a direct product $S_1 \times S_2$ of finite multioperator groups S_1 and S_2 then G divides at least one of the factors.*

Another example is provided by the next lemma.

Lemma 4.7.2. *Let A_1 and A_2 be members of a congruence distributive pseudovariety* $\mathbf{V}$. *If the simple algebra S divides $A_1 \times A_2$ then S divides A_1 or A_2.*

Proof. Indeed let M be a subalgebra of $A_1 \times A_2$ and ρ be a congruence on M with $M/\rho \cong S$. We assume that $|S| > 1$, the case $|S| = 1$ being trivial. Then $\rho \neq \omega$. Let π_1 and π_2, respectively, be the congruences on M induced by the projections into A_1 and A_2, respectively. Then $\pi_1 \cap \pi_2 = \varepsilon$. Since S is simple there are only two possibilities for the congruence $\pi_i \vee \rho$ namely ω or ρ itself (for $i = 1,2$). If it were true that $\pi_1 \vee \rho = \omega = \pi_2 \vee \rho$ then congruence distributivity would imply that

$$\rho = (\pi_1 \cap \pi_2) \vee \rho = (\pi_1 \vee \rho) \cap (\pi_2 \vee \rho) = \omega,$$

a contradiction to $|S| \neq 1$. Hence $\pi_1 \subseteq \rho$ or $\pi_2 \subseteq \rho$. It follows that S divides M/π_1 or M/π_2 which in turn divides A_1 or A_2. □

Corollary 4.7.3. *The system $\mathscr{S}$ is a divisor system for each congruence distributive pseudovariety and each pseudovariety of multioperator groups.*

4.8. The divisor system induced by n-generated algebras

Let $\mathbf{V}$ be a pseudovariety of any type. For any $n \in \mathbb{N}$ denote by $\mathscr{Q}_n(S)$ the set of all divisors of S generated by at most n elements. It can be readily verified that $\mathscr{Q}_n$ is a divisor system. The associated congruence Q_n is most easily described syntactically: two pseudovarieties are Q_n-related if and only if they satisfy the same pseudoidentities in at most n variables.

4.9. Systems associated with join distributive pseudovarieties

Let $\mathbf{P}$ be a pseudovariety of any type of algebra and let $\mathbf{V}$ be a join distributive element of $\mathscr{L}(\mathbf{P})$, that is, $\mathbf{V}$ satisfies $(\mathbf{A} \vee \mathbf{B}) \cap \mathbf{V} = (\mathbf{A} \cap \mathbf{V}) \vee (\mathbf{B} \cap \mathbf{V})$ for all $\mathbf{A}, \mathbf{B} \in \mathscr{L}(\mathbf{P})$. Then the system

$$\mathscr{D}_{\mathbf{V}}(S) = \text{all divisors of } S \text{ which are in } \mathbf{V}$$

satisfies (D1) and (D3). In general it will not satisfy (D2) (thanks to F. Pastijn for pointing this out to the author). This motivates us to introduce a weaker axiom, (D2″) as follows:

$$(\text{D2}'') \quad \mathscr{D}(S \times T) \subseteq \operatorname{pvar} \bigcup_{n \geq 1} (\mathscr{D}(S^n) \cup \mathscr{D}(T^n))$$

where for any algebra U and any natural number k, U^k denotes the k-fold direct power of U. Clearly, (D2) implies (D2″). By the join distributivity it follows that the system $\mathscr{D}_{\mathbf{V}}$ satisfies the axiom (D2″). It can be shown that systems satisfying (D1), (D2″) and (D3) yield complete congruences on lattices of pseudovarieties similarly as divisor systems do, and divisor systems are particular examples of such systems. More precisely, analogues to Lemma 3.1, Theorem 3.2 and Proposition 3.3 can be obtained provided one defines the class $\mathbf{V}^{\mathscr{D}}$ as follows:

$$\mathbf{V}^{\mathscr{D}} = \{S \in \mathbf{P} \mid (\forall n \in \mathbb{N})\ \mathscr{D}(S^n) \subseteq \mathbf{V}\}.$$

For divisor systems this definition reduces to the definition given in Section 3. The author does not know whether analogues of Theorems 3.4 and 3.5 and Corollaries 3.6 and 3.7 can be proved for such systems. This would be an interesting question for it would allow one to compute the complete join of congruences induced by systems of the form $\mathscr{D}_{\mathbf{V}}$ and by other sytems, for example divisor systems. The use of the systems $\mathscr{D}_{\mathbf{V}}$ solely for establishing complete congruences on $\mathscr{L}(\mathbf{P})$ is rather limited: the divisor class of $\mathscr{D}_{\mathbf{V}}$ is precisely $\mathbf{V}$ and Proposition 3.3 tells us that the mapping $\mathscr{L}(\mathbf{P}) \to \mathscr{L}(\mathbf{V})$, $\mathbf{A} \mapsto \mathbf{A} \cap \mathbf{V}$ is a complete lattice morphism. However, this is already clear from the fact that $\mathbf{V}$ is join distributive. Namely, in each lattice of pseudovarieties each join distributive element is even *completely* join distributive. The latter follows from the fact that given a collection $\{\mathbf{V}_i \mid i \in I\}$ of pseudovarieties then for each $S \in \bigvee_{i\in I} \mathbf{V}_i$ there exist $i_1, \ldots, i_n \in I$ such that $S \in \mathbf{V}_{i_1} \vee \cdots \vee \mathbf{V}_{i_n}$.

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