A New Criterion for H-Stability of Complex Matrices*

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Generalizing some work of Arrow and McManus [1], Ostrowski and Schneider in [5] defined two strong notions of matrix stability which they called H-stability and H-semistability. They gave necessary and sufficient criteria for a matrix to be H-stable or H-semistable; another criterion for H-stability was given by this author and Schneider [3]. In this note we modify their definitions and give yet another criterion for H-stability. We derive some consequences of our criterion, and conclude by listing some criteria for the stability or semistability of a matrix.

1. DEFINITIONS

We define the inertia of a complex n by n matrix A to be $\operatorname{In} A = (\pi(A), \nu(A), \delta(A))$, where $\pi(A)$, $\nu(A)$, and $\delta(A)$ are respectively the numbers of eigenvalues (counting multiplicities) of A with positive, negative, and zero real parts. While matrix stability is usually defined in terms of eigenvalues with negative real parts, we shall deal exclusively with "positive" stabilities. We define A to be stable if $\pi(A) = n$ and semistable if $\nu(A) = 0$. We shall always use H to denote a hermitian matrix. We define A to be H-stable if AH is stable whenever $H > 0$ (positive definite) and H-semistable if AH is semistable whenever $H > 0$. For real A, we define A to be real H-stable or real H-semistable if the appropriate

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condition holds for all real symmetric H. A real matrix is real H-stable (real H-semistable) if and only if it is H-stable (H-semistable) [5].

2. EQUIVALENCE OF DEFINITIONS

Using continuity, it is easy to see that if AH is semistable for all $H > 0$, then AH is also semistable for all $H \geqq 0$ (positive semidefinite); thus our definition of H-semistability and that in [5] are equivalent. That our definition of H-stability is equivalent to that in [5] follows from

REMARK 1. *If AH is stable whenever $H > 0$, then $H > 0$ whenever AH is stable.*

Proof. Clearly A is H-semistable, and thus by Theorem 3 of [5], $R(A) = (A + A^*)/2 \geqq 0$. By Corollary 4 of [5], $\operatorname{In} AH \leqq \operatorname{In} H$ (i.e., $\pi(AH) \leqq \pi(H)$, $\gamma(AH) \leqq \nu(H)$) for every H. If AH is stable, then $n = \pi(AH) \leqq \pi(H) \leqq n$ and $H > 0$.

The proofs of equivalence for the definitions of real H-stability and real H-semistability follow *mutus mutandi.*

3. A NEW CRITERION FOR H-STABILITY

Our criterion is derived from that given in [3]. We define $I(A) = (A - A^*)/2i$.

LEMMA. *Given a matrix A. Suppose*

(i) $R(A) \geqq 0,$

and let S be any nonsingular matrix for which $S^*R(A)S = P_{11} \oplus 0$, where $P_{11} > 0$. If $S^*I(A)S = Q$ is partitioned conformably, then $Q_{22} = 0$ if and only if

(ii) $x^*R(A)x = 0 \Rightarrow x^*I(A)x = 0.$

Proof. Let $y = S^{-1}x$ be partitioned conformably. Then $x^*R(A)x = y_1^*P_{11}y_1 = 0$ if and only if $y_1 = 0$; and $x^*I(A)x = y_2^*Q_{22}y_2 = 0$ for all such x if and only if $Q_{22} = 0$.

COROLLARY 1. *Suppose* (i) *and* (ii) *hold; then no principal submatrix of A has any nonzero imaginary eigenvalues.*

Proof. Suppose A had a nonzero imaginary eigenvalue βi. Then, as $R(A) \geqq 0$, by Theorem 2 of [5], for some unitary U, $U^*AU = B_{11} \oplus \beta i$. This, we see by the lemma, would violate (ii). As each principal submatrix of A inherits (i) and (ii) from A, the corollary follows.

THEOREM. *The matrix A is H-stable if and only if*

(i) $R(A) \geqq 0$,

(ii) $x^*R(A)x = 0 \Rightarrow x^*I(A)x = 0$,

and

(iii) *A is nonsingular.*

Proof. By Theorem VII of [3], A is H-stable if and only if (i), (iii), and, for any S of the form described in the lemma, $Q_{22} = 0$. The theorem now follows from the lemma.

The notion of H-stability can be generalized to operators on any Hilbert space. We pose a question: is the characterization given in our theorem still valid?

Suppose now that A is real. We can give a criterion for real H-stability in terms of A and real vectors ξ and η. For any complex vector x, we have $x = \xi + i\eta$, where ξ and η are real. Now if $R(A) \geqq 0$,

$$x^*R(A)x = 0 \Leftrightarrow R(A)x = 0 \Leftrightarrow R(A)\xi = R(A)\eta = 0.$$

Now assume that $x^*R(A)x = 0$. We have

$$x^*I(A)x = 0 \Leftrightarrow x^*Ax = 0 \Leftrightarrow \xi'A\xi + \eta'A\eta + i(\xi'A\eta - \eta'A\xi) = 0.$$

As $x^*R(A)x = 0$ and $iI(A) = A - R(A)$ is real and skewsymmetric,

$$\eta'A\xi = i\eta'I(A)\xi = - i\xi'I(A)\eta = - \xi'A\eta.$$

Thus

$$x^*I(A)x = 0 \Leftrightarrow \xi'A\xi + \eta'A\eta + 2i\xi'A\eta = 0.$$

Now it is clear that we have

COROLLARY 2. *Suppose A is real. Then A is real H-stable if and only if* (i) *and* (iii) *hold, and*

(ii′) *for each pair of real vectors ξ, η for which $R(A)\xi = R(A)\eta = 0$, we have $\xi' A\eta = 0$.*

COROLLARY 3. *Suppose A is real. Then A is real H-stable if it is nonsingular and $R(A) \geqq 0$, of rank $n - 1$.*

Proof. If ξ and η are eigenvectors for zero of $R(A)$, we must have $\eta = \lambda\xi$ for some real scalar λ. Now

$$\xi' A\eta = \lambda\xi' A\xi = \lambda\xi' R(A)\xi = 0.$$

4. SOME CONSEQUENCES

Let $F(A) = \{\alpha\colon\ \alpha = x^*Ax,\ x^*x = 1\}$ denote the field of values of A, and let P denote the open right half-plane.

COROLLARY 4. *The matrix is H-stable if and only if A is nonsingular and $F(A) \subseteq P \cup \{0\}$.*

Proof. The proof is immediate as $x^*Ax = x^*R(A)x + ix^*I(A)x$.

COROLLARY 5. *If A is normal, then A is H-stable if and only if A is stable.*

Proof. Clearly A is stable if A is H-stable. If A is normal, the field of values of A is the convex hull of the eigenvalues of A (cf. [4, p. 168]); if A is also stable, then A is nonsingular and $F(A) \subset P$.

We cannot in general replace $F(A)$ by the convex hull of the eigenvalues of A in Corollary 4; the matrix

$$A = \begin{bmatrix} 1+4i & 1+2i \\ 1+2i & 1+i \end{bmatrix}$$

is stable, has $R(A) \geqq 0$, yet is not H-stable.

Ostrowski and Schneider proved in [5] that if $R(A) > 0$, then $\operatorname{In} AH = \operatorname{In} H$ for each H. Now if $\operatorname{In} AH = \operatorname{In} H$ for each $H > 0$, A is H-stable. We can also give a more precise converse to their result:

REMARK 2. *If* $\operatorname{In} AH = \operatorname{In} H$ *for each* $H \geqq 0$, *then* $R(A) > 0$.

Proof. The matrix A is H-stable. If $R(A) \geqq 0$ and singular, then there exists a unitary U such that, as in the lemma,

$$U^*AU = \begin{bmatrix} P_{11} + iQ_{11} & iQ_{12} \\ iQ_{21} & \cdot \end{bmatrix}, \quad P_{11} > 0; \qquad \text{for} \quad U^*HU = \begin{bmatrix} \cdot & \cdot \\ \cdot & I \end{bmatrix} \geqq 0,$$

we have $\operatorname{In} AH = \operatorname{In} U^*AHU = (0, 0, n) \neq \operatorname{In} H$.

5. STABILITY AND SEMISTABILITY CRITERIA

We have

REMARK 3. *The following are equivalent:*

(iv) *A is stable;*
(v) *there exists an $H > 0$ for which AH is H-stable;*
(vi) *there exist a B, $R(B) > 0$, and an $H > 0$ for which $A = BH$.*

The proof is trivial using Theorem 1 of [5].

REMARK 4. *The following are equivalent:*

(vii) *A is semistable, and all elementary divisors associated with imaginary eigenvalues of A are linear;*
(viii) *there exists an $H > 0$ for which AH is H-semistable;*
(ix) *there exist a B, $R(B) \geqq 0$, and an $H > 0$ for which $A = BH$.*

That (vii) $\Leftrightarrow$ (viii) is Corollary III.1 of [3]; the rest of the proof is trivial.

Our final remark is tedious but not difficult to prove:

REMARK 5. *The following are equivalent:*

(x) *A is semistable;*
(xi) *there exists an $H \geqslant 0$ for which $R(AH) \geqslant 0$, and* rank $R(AH) =$ rank $H = \pi(A) + \nu(A)$.

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