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A note on relative dimensions of rings and conductors in function fields[1]

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Abstract

Let $\mathscr{R}$ be the local ring at a curve singularity and let $\mathscr{S}$ be a ring such that $\mathscr{R} \subseteq \mathscr{S} \subseteq \tilde{\mathscr{R}}$, where $\tilde{\mathscr{R}}$ denotes the integral closure of $\mathscr{R}$ in its field of fractions. Let $(\mathscr{R}:\mathscr{S})$ denote the conductor of $\mathscr{S}$ in $\mathscr{R}$. We compare here the dimensions (over the base field) of $\mathscr{S}/\mathscr{R}$ and $\frac{\mathscr{R}}{(\mathscr{R}:\mathscr{S})}$. We relate this with the intersection numbers of branches at the singularity. © 1997 Elsevier Science B.V.

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Let $\mathscr{R}$ be the local ring of a curve singularity and let $\mathscr{K}$ be its field of fractions; i.e., $\mathscr{K}$ is the field of rational functions on the curve. Let $\tilde{\mathscr{R}}$ denote the integral closure of $\mathscr{R}$ in the field $\mathscr{K}$. For a ring $\mathscr{S}$ with $\mathscr{R} \subseteq \mathscr{S} \subseteq \tilde{\mathscr{R}}$, we want to compare the following dimensions (dim means here dimension of vector spaces over the field of constants of $\mathscr{K}$):

$$\dim\left(\frac{\mathscr{S}}{\mathscr{R}}\right) \quad \text{and} \quad \dim\left(\frac{\mathscr{R}}{(\mathscr{R}:\mathscr{S})}\right),$$

where $(\mathscr{R}:\mathscr{S}) = \{\alpha \in \mathscr{K} \mid \alpha \cdot \mathscr{S} \subseteq \mathscr{R}\}$ is the *conductor ideal* of $\mathscr{S}$ in $\mathscr{R}$.

When the ring $\mathscr{R}$ is *Gorenstein* (i.e., when $\dim(\tilde{\mathscr{R}}/\mathscr{R}) = \dim\mathscr{R}/(\mathscr{R}:\tilde{\mathscr{R}})$), we have that $\dim(\mathscr{S}/\mathscr{R}) = \dim\mathscr{R}/(\mathscr{R}:\mathscr{S})$ for any $\mathscr{R}$-fractional ideal $\mathscr{S}$ containing $\mathscr{R}$. In general, one has that $\dim(\tilde{\mathscr{R}}/\mathscr{R}) \geq \dim\mathscr{R}/(\mathscr{R}:\tilde{\mathscr{R}})$ (see [3] or [4]). We show that the inequality

$$\dim\frac{\mathscr{R}}{(\mathscr{R}:\mathscr{S})} \leq \dim\frac{\mathscr{S}}{\mathscr{R}}$$

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holds whenever $\dim\mathscr{R}/(\mathscr{R}:\mathscr{S}) \le 4$, and that it may fail when $\dim\mathscr{R}/(\mathscr{R}:\mathscr{S}) = 5$. We also show (see Section 3) that this inequality (for certain rings $\mathscr{S}_P$) is equivalent to an inequality relating intersection numbers of branches at the singularity. This result was the motivation for investigating the relative dimensions of rings and conductors. We end up by giving examples of three-branch singularities where the inequality above fails.

1. The main result

Theorem 1. *Let $\mathscr{R}$ and $\mathscr{S}$ be as above and suppose that* $\dim\mathscr{R}/(\mathscr{R}:\mathscr{S}) \le 4$. *Then*

$$\dim\frac{\mathscr{R}}{\mathscr{R}:\mathscr{S}} \le \dim\frac{\mathscr{S}}{\mathscr{R}}.$$

Proof. It will be clear from the proof below that we do not need to assume that the ring $\mathscr{R}$ lives inside a function field $\mathscr{K}$ and, moreover, we will just use that $\mathscr{S}$ is stable under multiplication by elements of the ring $\mathscr{R}$.

Fix $n \in \{1,2,3,4\}$. We show that if $\dim\mathscr{R}/(\mathscr{R}:\mathscr{S}) = n$, then $\dim(\mathscr{S}/\mathscr{R}) \ge n$. The case $n = 4$ is the most complicated one and contains all arguments used in the other cases. We will then just consider the case $n = 4$. For an element $Y \in \mathscr{S}$, we consider the linear map φ_Y of vector spaces

$$\varphi_Y\colon \frac{\mathscr{R}}{(\mathscr{R}:\mathscr{S})} \longrightarrow \frac{\mathscr{S}}{\mathscr{R}}, \qquad \overline{X} \longmapsto \overline{X\cdot Y},$$

where $\bar{\alpha}$ means the equivalence class of α in the (correspondent) quotient space. Note that $\varphi_Y \not\equiv 0$ if and only if $Y \in (\mathscr{S}\backslash\mathscr{R})$.

We consider the following cases:

Case 1: $\exists Y \in (\mathscr{S}\backslash\mathscr{R})$ with φ_Y injective.

Case 2: $\exists Y \in (\mathscr{S}\backslash\mathscr{R})$ with $\dim(\operatorname{Ker}\varphi_Y) = 1$.

Case 3: $\forall Y \in (\mathscr{S}\backslash\mathscr{R})$, $\dim(\operatorname{Ker}\varphi_Y) \ge 2$, and $\exists Y_1 \in (\mathscr{S}\backslash\mathscr{R})$ with $\dim(\operatorname{Ker}\varphi_{Y_1}) = 2$.

Case 4: $\forall Y \in (\mathscr{S}\backslash\mathscr{R})$, $\dim(\operatorname{Ker}\varphi_Y) = 3$.

Since $\dim\mathscr{R}/(\mathscr{R}:\mathscr{S}) = 4$, these are all the cases to be considered. Also, there is nothing to prove in Case 1.

Case 2: Choose $Y_1 \in (\mathscr{S}\backslash\mathscr{R})$ with $\operatorname{Ker}\varphi_{Y_1}$ unidimensional. Take $\overline{X_4} \ne 0$ in $\operatorname{Ker}\varphi_{Y_1}$. This means that $X_4 \in \mathscr{R}$, $X_4 \notin (\mathscr{R}:\mathscr{S})$ and $X_4\cdot Y_1 \in \mathscr{R}$. Since $X_4 \notin (\mathscr{R}:\mathscr{S})$, take $Y_4 \in \mathscr{S}$ with $X_4 \cdot Y_4 \notin \mathscr{R}$.

Since $\dim(\operatorname{Im}\varphi_{Y_1}) = 3$, we just have to exhibit an element of $(\mathscr{S}/\mathscr{R})$ not belonging to $\operatorname{Im}\varphi_{Y_1}$. We claim that $\overline{Y_4} \in (\mathscr{S}/\mathscr{R})$ is such an element. In fact, suppose $\overline{Y_4} = \overline{X\cdot Y_1}$ for some $X \in \mathscr{R}$; i.e., suppose $(Y_4 - X\cdot Y_1) \in \mathscr{R}$ for some $X \in \mathscr{R}$. Multiplying by X_4, we would get

$$X_4\cdot Y_4 - X\cdot X_4\cdot Y_1 \in \mathscr{R}.$$

Since $X_4\cdot Y_1 \in \mathscr{R}$, we would then conclude that $X_4\cdot Y_4 \in \mathscr{R}$, a contradiction.

Case 3: Choose $Y_1 \in (\mathscr{S}\backslash\mathscr{R})$ with

$$\dim(\text{Ker } \varphi_{Y_1}) = 2.$$

Then, $\dim(\text{Im}\varphi_{Y_1}) = 2$ and we take X_1 and X_2 in $\mathscr{R}$ so that $\overline{X_1 \cdot Y_1}$ and $\overline{X_2 \cdot Y_1}$ are linearly independent elements of $(\mathscr{S}/\mathscr{R})$ (i.e., $\overline{X_1 \cdot Y_1}$ and $\overline{X_2 \cdot Y_1}$ form a basis for $\text{Im } \varphi_{Y_1}$). Choose now $\overline{X}_4 \neq 0$ in Ker φ_{Y_1}. This means, as before, $X_4 \in \mathscr{R}$, $X_4 \notin (\mathscr{R} : \mathscr{S})$ and $X_4 \cdot Y_1 \in \mathscr{R}$. Since $X_4 \notin (\mathscr{R} : \mathscr{S})$, we can choose $Y_4 \in \mathscr{S}$ with $X_4 \cdot Y_4 \notin \mathscr{R}$. We consider two subcases.

Case 3.1: There exists a choice of Y_1,X_4 and Y_4 as above such that

$$(\text{Ker } \varphi_{Y_1}) \cap (\text{Ker } \varphi_{Y_4}) \neq (0).$$

Case 3.2: For all such choices of Y_1,X_4 and Y_4 we have

$$(\text{Ker } \varphi_{Y_1}) \cap (\text{Ker } \varphi_{Y_4}) = (0).$$

In Case 3.1 we choose $\overline{X}_3 \neq 0$ in the intersection $(\text{Ker } \varphi_{Y_1}) \cap (\text{Ker } \varphi_{Y_4})$. As before, we can choose $Y_3 \in \mathscr{S}$ such that $X_3 \cdot Y_3 \notin \mathscr{R}$. We claim that the elements $\overline{X_1 \cdot Y_1}$, $\overline{X_2 \cdot Y_1}$, $\overline{Y}_3$ and $\overline{Y}_4$ of $(\mathscr{S}/\mathscr{R})$ are linearly independent. In fact, suppose we have a linear equation (α_i belonging to the constant field):

$$\alpha_1 X_1 \cdot Y_1 + \alpha_2 X_2 \cdot Y_1 + \alpha_3 Y_3 + \alpha_4 Y_4 \in \mathscr{R}.$$

Multiplying it by X_3 and using $X_3 \cdot Y_1 \in \mathscr{R}$ and $X_3 \cdot Y_4 \in \mathscr{R}$, we get $\alpha_3 = 0$. Then, the linear equation is

$$\alpha_1 X_1 \cdot Y_1 + \alpha_2 X_2 \cdot Y_1 + \alpha_4 Y_4 \in \mathscr{R}.$$

Multiplying it by X_4 and using $X_4 \cdot Y_1 \in \mathscr{R}$, we obtain $\alpha_4 = 0$. We now conclude that $\alpha_1 = \alpha_2 = 0$, since $\overline{X_1 \cdot Y_1}$ and $\overline{X_2 \cdot Y_1}$ are linearly independent in $(\mathscr{S}/\mathscr{R})$.

We then consider the situation in Case 3.2. We must have that $\dim(\text{Ker } \varphi_{Y_4}) = 2$, since if it were equal to three we would have that $(\text{Ker } \varphi_{Y_1}) \cap (\text{Ker } \varphi_{Y_4}) \neq (0)$. This is so because the ambient vector space $\mathscr{R}/(\mathscr{R} : \mathscr{S})$ is four-dimensional. Let $\{\overline{X}_1,\overline{X}_2\}$ be a basis for Ker φ_{Y_4} and let $\{\overline{X}_3,\overline{X}_4\}$ be a basis for Ker φ_{Y_1}. Then $\overline{X}_1,\overline{X}_2,\overline{X}_3$ and $\overline{X}_4$ constitute a basis for $\mathscr{R}/(\mathscr{R} : \mathscr{S})$ and, moreover, $\{\overline{X_1 \cdot Y_1}, \overline{X_2 \cdot Y_1}\}$ is a basis for $\text{Im } \varphi_{Y_1}$ and $\{\overline{X_3 \cdot Y_4}, \overline{X_4 \cdot Y_4}\}$ is a basis for $\text{Im } \varphi_{Y_4}$. We claim that the elements $\overline{X_1 \cdot Y_1}$, $\overline{X_2 \cdot Y_1}$, $\overline{X_3 \cdot Y_4}$ and $\overline{X_4 \cdot Y_4}$ of $(\mathscr{S}/\mathscr{R})$ are linearly independent. In fact, suppose we have a linear relation

$$\alpha_1 X_1 \cdot Y_1 + \alpha_2 X_2 \cdot Y_1 + \alpha_3 X_3 \cdot Y_4 + \alpha_4 X_4 \cdot Y_4 \in \mathscr{R}.$$

Multiplying it by X_1 and using $X_1 \cdot Y_4 \in \mathscr{R}$, we get

$$X_1 \cdot (\alpha_1 X_1 \cdot Y_1 + \alpha_2 X_2 \cdot Y_1) \in \mathscr{R}.$$

Similarly, multiplying it by X_2, we get

$$X_2 \cdot (\alpha_1 X_1 \cdot Y_1 + \alpha_2 X_2 \cdot Y_1) \in \mathscr{R}.$$

Let $\tilde{Y} = (\alpha_1 X_1 \cdot Y_1 + \alpha_2 X_2 \cdot Y_1)$ and consider the associated linear map $\varphi_{\tilde{Y}}$. We have $\varphi_{\tilde{Y}}(\overline{X}_1) = \varphi_{\tilde{Y}}(\overline{X}_2) = \varphi_{\tilde{Y}}(\overline{X}_3) = \varphi_{\tilde{Y}}(\overline{X}_4) = 0$. This means that $\varphi_{\tilde{Y}} \equiv 0$ or, equivalently, $\tilde{Y} \in \mathscr{R}$. We then conclude that $\alpha_1 = \alpha_2 = 0$, since $\overline{X_1 \cdot Y_1}$ and $\overline{X_2 \cdot Y_1}$ are linearly independent elements of $(\mathscr{S}/\mathscr{R})$. The linear relation then reduces to $\alpha_3 X_3 \cdot Y_4 + \alpha_4 X_4 \cdot Y_4 \in \mathscr{R}$ and, similarly, we get $\alpha_3 = \alpha_4 = 0$.

Case 4. Take $Y_1 \in (\mathscr{S} \setminus \mathscr{R})$ and let $\overline{X}_4 \neq 0$ be an element of Ker φ_{Y_1}. Choose $Y_4 \in \mathscr{S}$ such that $X_4 \cdot Y_4 \notin \mathscr{R}$. We have that

$$\dim \left[(\text{Ker } \varphi_{Y_1}) \cap (\text{Ker } \varphi_{Y_4})\right] \geq 2,$$

since $\dim(\text{Ker } \varphi_{Y_1}) = 3$, $\dim(\text{Ker } \varphi_{Y_4}) = 3$ and $\dim \mathscr{R}/(\mathscr{R} : \mathscr{S}) = 4$. We then see that

$$\dim \left[(\text{Ker } \varphi_{Y_1}) \cap (\text{Ker } \varphi_{Y_4})\right] = 2,$$

since $\overline{X}_4 \in \text{Ker } \varphi_{Y_1}$ and $\overline{X}_4 \notin \text{Ker } \varphi_{Y_4}$. Let $\overline{X}_2 \neq 0$ be an element of $(\text{Ker } \varphi_{Y_1}) \cap (\text{Ker } \varphi_{Y_4})$, and choose $Y_2 \in \mathscr{S}$ such that $X_2 \cdot Y_2 \notin \mathscr{R}$. We have that

$$W = (\text{Ker } \varphi_{Y_2}) \cap (\text{Ker } \varphi_{Y_1}) \cap (\text{Ker } \varphi_{Y_4}) \neq (0),$$

since $\dim(\text{Ker } \varphi_{Y_2}) = 3$ and $\dim \left[(\text{Ker } \varphi_{Y_1}) \cap (\text{Ker } \varphi_{Y_4})\right] = 2$.

Take $\overline{X}_3 \neq 0$ in the intersection W above and choose $Y_3 \in \mathscr{S}$ such that $X_3 \cdot Y_3 \notin \mathscr{R}$. We claim now that $\overline{Y}_1, \overline{Y}_2, \overline{Y}_3$ and $\overline{Y}_4$ are linearly independent in $(\mathscr{S}/\mathscr{R})$. In fact, suppose we have a linear combination

$$\alpha_1 Y_1 + \alpha_2 Y_2 + \alpha_3 Y_3 + \alpha_4 Y_4 \in \mathscr{R}.$$

Multiplying it by X_3, we get $\alpha_3 = 0$. The linear combination then reduces to

$$\alpha_1 Y_1 + \alpha_2 Y_2 + \alpha_4 Y_4 \in \mathscr{R}.$$

Multiplying it now by X_2, we get $\alpha_2 = 0$. The linear combination then takes the form $(\alpha_1 Y_1 + \alpha_4 Y_4) \in \mathscr{R}$. Multiplying it by X_4, we get $\alpha_4 = 0$ and then $\alpha_1 = 0$.

This concludes the proof of the theorem. □

Remark. The proof when $\dim \mathscr{R}/(\mathscr{R} : \mathscr{S}) = 2$ only involves Cases 1 and 2. The proof when $\dim \mathscr{R}/(\mathscr{R} : \mathscr{S}) = 3$ involves Cases 1, 2 and 3.1.

2. The example with $\dim \mathscr{R}/(\mathscr{R} : \mathscr{S}) = 5$

By a numerical semigroup G we mean a subset G of the natural numbers with finite complement and stable under addition. The associated semigroup ring $k[[G]]$ (k is the constant field) is the subring of the power series ring $k[[t]]$ given below:

$$k[[G]] = \left\{ \left(\sum_j a_j t^j \right) \in k[[t]] \mid a_j = 0 \text{ if } j \notin G \right\}.$$

Given two numerical semigroups G and H with $G \subseteq H$, we denote $\mathscr{R} = k[[G]]$ and $\mathscr{S} = k[[H]]$. It is easy to check that

$$\dim \frac{\mathscr{S}}{\mathscr{R}} = \sharp(H \backslash G)$$

and

$$(\mathscr{R} : \mathscr{S}) = \left\{ \left(\sum_j a_j t^j \right) \in \mathscr{R} \,|\, a_j = 0 \text{ if } (j + H) \not\subseteq G \right\}.$$

Let $\{\ell_1 < \ell_2 < \ldots < \ell_m\} = (H \backslash G)$. Then,

$$\dim \mathscr{R}/(\mathscr{R} : \mathscr{S}) = \sharp\{\alpha \in G \mid \alpha = (\ell_j - \ell_i), \text{ for some } 1 \le i \le j \le m\}.$$

In order to find an example satisfying $\dim(\mathscr{S}/\mathscr{R}) < \dim \mathscr{R}/(\mathscr{R} : \mathscr{S})$, we will construct numerical semigroups $G \subseteq H$ satisfying

$$\sharp(H \backslash G) < \sharp\{\alpha \in G \mid \alpha = (\ell_j - \ell_i), \text{ for some } 1 \le i \le j \le m\}.$$

We are going to exhibit such G and H with the set at the left in the above inequality having 4 elements and the one at the right having 5 elements.

Let G be the semigroup generated by the natural numbers 10, 12, 14, 16, 17, 18, 19 and 21. Take now $H = G \cup \{\ell_1, \ell_2, \ell_3, \ell_4\}$, where $\ell_1 = 9$, $\ell_2 = 13$, $\ell_3 = 23$ and $\ell_4 = 25$. One easily checks that H is also a semigroup. We have $\sharp(H \setminus G) = 4$ and, moreover, $(\ell_1 - \ell_1) = 0 \in G$; $(\ell_4 - \ell_2) = 12 \in G$; $(\ell_4 - \ell_1) = 16 \in G$; $(\ell_3 - \ell_2) = 10 \in G$ and $(\ell_3 - \ell_1) = 14 \in G$.

The associated rings

$$\mathscr{R} = k[[t^{10}, t^{12}, t^{14}, t^{16}, t^{17}, t^{18}, t^{19}, t^{21}]]$$

and

$$\mathscr{S} = k[[t^9, t^{10}, t^{12}, t^{13}, t^{14}, t^{16}, t^{17}]]$$

then satisfy

$$\dim \frac{\mathscr{S}}{\mathscr{R}} = 4 \quad \text{and} \quad \dim \frac{\mathscr{R}}{\mathscr{R} : \mathscr{S}} = 5.$$

Remark. This example is also good in the sense that one cannot find (monomial) semigroup rings $\mathscr{R}$ and $\mathscr{S}$ with $\dim(\mathscr{S}/\mathscr{R}) = 3$ and $\dim \mathscr{R}/(\mathscr{R} : \mathscr{S}) > 3$. In fact, if such rings $\mathscr{R}$ and $\mathscr{S}$ existed and denoting as before $\{\ell_1 < \ell_2 < \ell_3\}$ the complementary set $(H \setminus G)$, we would have that $0, (\ell_2 - \ell_1), (\ell_3 - \ell_1)$ and $(\ell_3 - \ell_2)$ would be four distinct elements of G. Consider then the element $(\ell_1 + \ell_3 - \ell_2)$, which belongs to H.

We have $\ell_1 < (\ell_1 + \ell_3 - \ell_2) < \ell_3$ and, also, $(\ell_1 + \ell_3 - \ell_2) \neq \ell_2$. Hence, we must have $(\ell_1 + \ell_3 - \ell_2) \in G$. Now, since $(\ell_2 - \ell_1)$ belongs to G, we would have

$$(\ell_1 + \ell_3 - \ell_2) + (\ell_2 - \ell_1) = \ell_3 \in G,$$

a contradiction.

3. Intersection numbers of branches

Here $\mathscr{R}$ will denote the completion of the local ring at a curve singularity. We denote $\mathscr{P}_1, \mathscr{P}_2, \ldots, \mathscr{P}_r$ the minimal prime ideals of $\mathscr{R}$; i.e., the branches of the curve at the singular point. If A is a subset of $\{1, 2, \ldots, r\}$, we denote $\mathscr{P}_A = \bigcap_{j \in A} \mathscr{P}_j$. If A and B are two disjoint subsets of $\{1, 2, \ldots, r\}$ we denote

$$\mathscr{I}_{A,B} = \dim \frac{\mathscr{R}}{\mathscr{P}_A + \mathscr{P}_B},$$

the *intersection number* of the branches in A with those in B. For a three-set partition $P = \{A, B, C\}$ of the set $\{1, 2, \ldots, r\}$, we put $\mathscr{S}_P = \mathscr{R}_A \times \mathscr{R}_B \times \mathscr{R}_C$, where $\mathscr{R}_A = \mathscr{R}/\mathscr{P}_A$. Clearly, $\mathscr{R}$ can be identified with the diagonal of $\mathscr{S}_P$.

Theorem 2. *For a partition $P = \{A, B, C\}$ of the set of branches at a curve singularity, the following assertions are equivalent:*

(1) $\dim(\mathscr{S}_P/\mathscr{R}) \geq \dim \mathscr{R}/(\mathscr{R} : \mathscr{S}_P)$.

(2) $\mathscr{I}_{A,B\cup C} \leq \mathscr{I}_{A,B} + \mathscr{I}_{A,C}$.

Proof. The proof is essentially contained in [1, Theorem 4.1]. Clearly, the first assertion is equivalent to the following inequality:

$$\dim \frac{\mathscr{S}_P}{(\mathscr{R} : \mathscr{S}_P)} \leq 2 \cdot \dim \frac{\mathscr{S}_P}{\mathscr{R}}. \tag{1}$$

From [1, Proof of Theorem 3.9], we have

$$\dim \frac{\mathscr{S}_P}{(\mathscr{R} : \mathscr{S}_P)} = \mathscr{I}_{A,B\cup C} + \mathscr{I}_{B,A\cup C} + \mathscr{I}_{C,A\cup B}.$$

Ordering the subsets as B, A, C, we have (from [2, Proposition 1])

$$\dim \frac{\mathscr{S}_P}{\mathscr{R}} = \mathscr{I}_{A,B} + \mathscr{I}_{C,A\cup B}.$$

Ordering the subsets as C, A, B, we have (from [2, Proposition 1])

$$\dim \frac{\mathscr{S}_P}{\mathscr{R}} = \mathscr{I}_{A,C} + \mathscr{I}_{B,A\cup C}.$$

So, $2.\dim(\mathscr{S}_P/\mathscr{R}) = \mathscr{I}_{A,B} + \mathscr{I}_{A,C} + \mathscr{I}_{C,A\cup B} + \mathscr{I}_{B,A\cup C}$ and hence the inequality (1) is also equivalent to the second assertion. □

We consider now three-branch singularities ($A = \{1\}, B = \{2\}$ and $C = \{3\}$). If the three branches are all non-singular, then we have that the ring $\mathscr{S}_P$ coincides with the integral closure of $\mathscr{R}$ and, consequently, we have that the inequality below holds:

$$\mathscr{I}_{1,\{2,3\}} \leq \mathscr{I}_{1,2} + \mathscr{I}_{1,3}.$$

The example of the three axes in the three-dimensional space shows that the inequality can be strict, since we have in this case $\mathscr{I}_{1,\{2,3\}} = \mathscr{I}_{1,2} = \mathscr{I}_{1,3} = 1$. We end up by giving examples of three-branch singularities where

$$\mathscr{I}_{1,\{2,3\}} > \mathscr{I}_{1,2} + \mathscr{I}_{1,3}.$$

We write $\alpha = (\Sigma\alpha_j t^j)$ for an element α in the power series ring $k[[t]]$. We consider the linear subspace $\mathscr{R}$ of $k[[t]] \times k[[t]] \times k[[t]]$ consisting of elements (α, β, γ) satisfying the following relations:

$$\left\{\alpha_0 = \beta_0 = \gamma_0,\right.$$

$$\begin{cases} \alpha_1 = \beta_1 = \gamma_1 = 0, \\ \vdots \quad \vdots \quad \vdots \\ \alpha_n = \beta_n = \gamma_n = 0, \end{cases}$$

$$\begin{cases} \alpha_{n+1} = 0 \text{ and } \beta_{n+1} = \gamma_{n+1}, \\ \vdots \qquad\qquad \vdots \quad \vdots \\ \alpha_{n+r} = 0 \text{ and } \beta_{n+r} = \gamma_{n+r}, \end{cases}$$

$$\begin{cases} \alpha_{n+r+1} - \beta_{n+r+1} + \gamma_{n+r+1} = 0, \\ \vdots \qquad \vdots \qquad \vdots \\ \alpha_{n+r+s} - \beta_{n+r+s} + \gamma_{n+r+s} = 0. \end{cases}$$

One can check that if $s \leq (n+1)$, then $\mathscr{R}$ is actually a local ring with maximal ideal $\mathscr{M}$ given below,

$$\mathscr{M} = \{(\alpha, \beta, \gamma) \in \mathscr{R} \mid \alpha_0 = 0\}.$$

The minimal prime ideal $\mathscr{P}_1$ (resp. $\mathscr{P}_2$ and $\mathscr{P}_3$) of the ring $\mathscr{R}$ has as elements those elements in $\mathscr{R}$ having first (resp. second and third) coordinate equal to zero. Explicitly,

$$\begin{aligned} \mathscr{P}_1 &= \{(0,\beta,\gamma) \in k[[t]]^3 \mid \beta \equiv 0 \mod t^{n+1} \quad \text{and} \quad \gamma \equiv \beta \mod t^{n+r+s+1}\}, \\ \mathscr{P}_2 &= \{(\alpha,0,\gamma) \in k[[t]]^3 \mid \alpha \equiv 0 \mod t^{n+r+1} \quad \text{and} \quad \gamma \equiv -\alpha \mod t^{n+r+s+1}\}, \\ \mathscr{P}_3 &= \{(\alpha,\beta,0) \in k[[t]]^3 \mid \alpha \equiv 0 \mod t^{n+r+1} \quad \text{and} \quad \beta \equiv \alpha \mod t^{n+r+s+1}\}. \end{aligned}$$

Clearly, $\mathscr{P}_2 \cap \mathscr{P}_3 = \{(\alpha, 0, 0) \in k[[t]]^3 \mid \alpha \equiv 0 \bmod t^{n+r+s+1}\}$. One can check that

$$\mathscr{P}_1 + \mathscr{P}_2 = \mathscr{P}_1 + \mathscr{P}_3 = \mathscr{M}$$

and

$$\mathscr{P}_1 + \mathscr{P}_2 \cap \mathscr{P}_3 = \{(\alpha,\beta,\gamma) \in \mathscr{R} \mid \alpha \equiv 0 \mod t^{n+r+s+1}\}.$$

We then conclude that $\mathscr{I}_{1,2} = \mathscr{I}_{1,3} = 1$ and $\mathscr{I}_{1,\{2,3\}} = (1+s)$.

Taking any $2 \leq s \leq (n+1)$, we have that $\mathscr{I}_{1,\{2,3\}} > \mathscr{I}_{1,2} + \mathscr{I}_{1,3}$ or, equivalently, we have that

$$(s+r+2) = \dim \frac{\mathscr{S}_P}{\mathscr{R}} < \dim \frac{\mathscr{R}}{(\mathscr{R} : \mathscr{S}_P)} = (2s+r+1).$$

Similarly, one can also check that the other intersection numbers are given by

$$\mathscr{I}_{2,3} = (r+1) \quad \text{and} \quad \mathscr{I}_{2,\{1,3\}} = \mathscr{I}_{3,\{1,2\}} = (s+r+1).$$

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