A Note on the Hadamard Product of Matrices

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ABSTRACT

It is shown that the smallest eigenvalue of the Hadamard product $A \times B$ of two positive definite Hermitian matrices is bounded from below by the smallest eigenvalue of AB^T.

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1. NOTATION AND PRELIMINARIES

The Hadamard (or Schur) product of two matrices $A=(a_{ik})$, $B=(b_{ik})$ of the same dimensions is the matrix $A * B=(a_{ik}b_{ik})$. If $C=(c_{ik})$ is a square complex matrix, we denote by $g(C)$ the spectral norm (the matrix norm generated by the Euclidean vector norm), i.e. $\sqrt{\lambda}$, where λ is the maximum eigenvalue of CC^* or C^*C; $N(C)$ will denote the Frobenius norm of C, i.e.

$$N(C)=\left(\sum_{i,k}|c_{ik}|^2\right)^{1/2}=(\operatorname{tr} CC^*)^{1/2}.$$

If C has all eigenvalues real, $m(C)$ will mean the smallest eigenvalue of C. The set of all complex $n \times n$ matrices will be denoted by M_n.

2. RESULTS

We shall prove:

Theorem. *If $A \in M_n$, $B \in M_n$ are both positive definite Hermitian, then*

$$m(A * B) \geqslant m(AB^T)$$

(and both sides exist). Equality is attained iff AB^T is a multiple of I.

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Proof. We shall need two well-known results, formulated as lemmas.

LEMMA 1 [2, p. 42]. *For any matrices* $A \in M_n$, $B \in M_n$,

$$N(AB) \leqslant g(A)N(B).$$

COROLLARY. *For any invertible* $A \in M_n$ *and any* $B \in M_n$,

$$N(AB) \geqslant \left[g(A^{-1})\right]^{-1}N(B).$$

REMARK. It is easily seen that if B is nonsingular, then equality is attained in each of these inequalities iff A is a multiple of a unitary matrix.

LEMMA 2 (Schur [3]). *For any diagonal* $X \in M_n$ *and any invertible* $S \in M_n$,

$$N(S^{-1}XS) \geqslant N(X).$$

To return to the proof, let us denote $x = (x_1, \dots, x_n)^T$, $X = \operatorname{diag}\{x_i\}$. Then, for A, B positive definite,

$$\begin{aligned} m(A * B) &= \min\left\{ \sum_{i,k=1}^{n} a_{ik}\bar{x}_k b_{ik} x_i;\ x^*x = 1 \right\} \\ &= \min\{\operatorname{tr}(AX^*B^TX);\ N(X) = 1\} \\ &= \min\left\{ N^2\left((B^T)^{1/2}XA^{1/2}\right);\ N(X) = 1 \right\} \\ &= \min\left\{ N^2\left((B^T)^{1/2}A^{1/2}\cdot A^{-1/2}XA^{1/2}\right);\ N(X) = 1 \right\} \\ &\geqslant \min\left\{ \left[g\left(\left[(B^T)^{1/2}A^{1/2}\right]^{-1}\right)\right]^{-2} N^2(A^{-1/2}XA^{1/2});\ N(X) = 1 \right\} \\ &\geqslant \left[g\left(\left[(B^T)^{1/2}A^{1/2}\right]^{-1}\right)\right]^{-2} = m(AB^T), \end{aligned}$$

since

$$\begin{aligned} g\left(\left[(B^T)^{1/2}A^{1/2}\right]^{-1}\right) &= \left[m\left((B^T)^{1/2}A^{1/2}A^{1/2}(B^T)^{1/2}\right)\right]^{-1/2} \\ &= \left[m(AB^T)\right]^{-1/2}. \end{aligned}$$

Let equality be attained. Then $(B^T)^{1/2}A^{1/2}$ is a multiple of a unitary matrix, i.e., AB^T is a multiple of the identity matrix. However, in this case equality is attained in

$$m\left(A*(A^T)^{-1}\right)\geqslant 1,$$

since $A*(A^T)^{-1}-I$ is positive semidefinite singular [1]. ∎

REFERENCES

1 M. Fiedler, On some properties of Hermitian matrices (in Czech), *Mat. Fyz. Časopis SAV* 7:168–176 (1957).

2 A. S. Householder, *Principles of Numerical Analysis*, McGraw-Hill, New York, 1953.

3 J. Schur, Über die charakteristischen Wurzeln einer linearen Substitution mit einer Anwendung auf die Theorie der Integralgleichungen, *Math. Ann.* 66:488–510 (1909).

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