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A PERIODIC MODULE OF INFINITE VIRTUAL PROJECTIVE DIMENSION

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A finitely generated module M over a local ring R (with infinite residue field) is said to have finite virtual projective dimension, if the completion $\hat{R}$ of R can be presented in the form $Q/(\boldsymbol{x})$, where $\boldsymbol{x}$ is a regular sequence in the local ring Q, and the projective dimension of $\hat{M}$, viewed as a Q-module, is finite. It is known that if the Betti numbers of M are bounded, then the finiteness of its virtual projective dimension implies that its minimal free resolution eventually becomes periodic of period 2. The purpose of this note is to construct examples which show that the converse does not hold in general.

1. Introduction

In this note $(R, \mathfrak{m}, k)$ denotes a noetherian local ring R with maximal ideal $\mathfrak{m}$ and residue field $k = R/\mathfrak{m}$. If M is a finitely generated R-module, and $\mathbb{F}$ is its minimal R-free resolution, then the rank of F_n is called the *nth Betti number* of M over R, and is denoted $b_n^R(M)$; it can also be defined as $\dim_k \operatorname{Ext}_R^n(M,k)$.

If $d \geq 0$ is the smallest integer, for which there is a real number $\gamma > 0$, such that $b_n^R(M) \leq \gamma n^{d-1}$ holds for all large enough n, we say M has *complexity d*, and write $\operatorname{cx}_R M = d$; if no such d exists, set $\operatorname{cx}_R M = \infty$. In order to define the *virtual projective dimension* of M, $\operatorname{vpd}_R M$, we introduce the notation $\tilde{R}$ for the $\mathfrak{m}$-adic completion $\hat{R}$ of R, in case k is infinite; when k is finite, we set $\tilde{R} = (R[Y]_{\mathfrak{m}R[Y]})\hat{}$, where Y is an indeterminate; in either case, $\tilde{M} = \tilde{R} \otimes_R M$. A local ring $(Q, \mathfrak{n}, k)$ is called a deformation of $\tilde{R}$, if there is a Q-regular sequence $\boldsymbol{x}$, such that $\tilde{R} \cong Q/(\boldsymbol{x})$; when $(\boldsymbol{x}) \subset \mathfrak{n}^2$, the deformation is called embedded. Now set $\operatorname{vpd}_R M = \min_Q \{\operatorname{pd}_Q \tilde{M} \mid Q$ is an embedded deformation of $\tilde{R}\}$.

The notions described in the preceding paragraph were introduced in [2], where it was proved that

$$(*) \qquad \operatorname{vpd}_R M = \operatorname{depth} R - \operatorname{depth} M + \operatorname{cx}_R M,$$

provided the left-hand side is finite. Furthermore, in an early version of that paper it was conjectured that equality holds always, i.e. that finite complexity implies finite virtual projective dimension.

Our purpose here is to show this is not always the case, by means of Example 2.1.

2. Example and remarks

2.1. Example. Let k be a field of characteristic $\neq 2$. Set $R = k[X_1, X_2, X_3, X_4]/I$, where the X_i's are indeterminates, and I is the ideal generated by the seven quadratic forms:

$$X_1^2, \qquad X_1X_2 - X_3^2, \qquad X_1X_3 - X_2X_4, \qquad X_1X_4,$$
$$X_2^2 + X_3X_4, \qquad X_2X_3, \qquad X_4^2.$$

Denote by x_i the image of X_i in R, and set

$$M = R^2/((x_1, x_2), (x_3, x_4)).$$

The R-module M is periodic of period 2, i.e. M is isomorphic to its second syzygy, *while the ring $\tilde{R}$ has no embedded deformation $Q \not\twoheadrightarrow \tilde{R}$.*

In particular, $\operatorname{cx}_R M = 1$, *and* $\operatorname{vpd}_R M = \infty$.

2.2. Remarks. (i) Löfwall has noted that the preceding construction does not produce an example in characteristic 2, and has proposed the following modification, which works over arbitrary fields: I is generated by the seven quadratic forms

$$X_1^2, \qquad X_1X_2 - X_3X_4, \qquad X_1X_2 - X_4^2, \qquad X_1X_3 - X_2X_4,$$
$$X_1X_4 - X_2^2, \qquad X_1X_4 - X_2X_3, \qquad X_1X_4 - X_3^2;$$

the R-module M has the same presentation as above. One treats this example by the same arguments as used in Section 3 below.

(ii) If the ring R has $\mathfrak{m}^2 = 0$, then either $\dim_k \mathfrak{m} \leq 1$, R is a quotient of a discrete valuation ring and $\operatorname{vpd}_R M \leq 1$ holds trivially for every R-module M, or $\dim_k \mathfrak{m} \geq 2$ and then every non-free M has exponentially growing Betti numbers.

If the ring R has embedding dimension, $\operatorname{edim} R = \dim_k \mathfrak{m}/\mathfrak{m}^2$, less than or equal to 3, then it is proved in [2, (1.6)] that every R-module of finite complexity has finite virtual projective dimension.

Thus, the 'smallest' k-algebra R, over which a module of finite complexity and infinite virtual projective dimension can exist, has to be of the form $k[X_1, \ldots, X_4]/I$, for an ideal I satisfying $(X_1, \ldots, X_4)^3 \subseteq I \subseteq (X_1, \ldots, X_4)^2$. In this case, by [7, Theorem B and Proposition 3.9], I has to be generated by seven quadratic forms.

(iii) The module M has the minimal possible complexity, since $\operatorname{cx}_R M = 0$ means precisely that M has finite projective dimension over R, and then $\operatorname{vpd}_R M = \operatorname{pd}_R M$, cf. [2, (3.2.2)], so that in this case (*) holds and coincides with a well known formula of Auslander and Buchsbaum.

(iv) Set $N = \mathrm{Hom}_k(M,k)$, and give it the canonical R-module structure. One sees N is coperiodic of period 2, i.e. is isomorphic to the second cosyzygy in its minimal injective resolution. In particular, its Bass numbers $\mu_R^n(N) = \dim_k \mathrm{Ext}_R^n(k,M)$ are bounded (in fact constant, cf. Section 3 below), so that it has plexity 1 in the terminology of [2, (5.1)], but its virtual injective dimension (loc. cit.) is infinite. Thus, one sees that the equality

$$\mathrm{vid}_R N = \mathrm{depth}\, R + \mathrm{px}_R N,$$

established in [2, (5.2)] under the assumption that $\mathrm{vid}_R N < \infty$, does not extend to arbitrary modules. Here again, the results of [3] show that both $\mathrm{edim}\, R$ and $\mathrm{px}_R N$ are the least possible for such an example.

3. Proof

Consider the endomorphisms φ and ψ of the R-module R^2, which in the standard basis are given respectively by the matrices

$$\begin{pmatrix} x_1 & x_3 \\ x_2 & x_4 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} x_4 & -x_3 \\ -x_2 & x_1 \end{pmatrix}.$$

We shall show the sequence

$$(**) \qquad R^2 \xleftarrow{\varphi} R^2 \xleftarrow{\psi} R^2 \xleftarrow{\varphi} R^2 \longleftarrow \cdots$$

is exact; since $\mathrm{Coker}\,\varphi = M$, this will turn out to be a minimal free resolution of M, thus showing it is periodic of period 2.

That (**) is a complex follows from the equalities $\psi\varphi = \varphi\psi = (x_1x_4 - x_2x_3)\mathrm{id}_{R^2} = 0$. In order to show exactness, note first R is graded (by setting $\deg x_i = 1$), and $R_j = 0$ for $j \geq 3$ (i.e. $\mathfrak{m}^3 = 0$). Consider now φ and ψ as endomorphisms of the graded k-space R^2, and let φ_{ij}, and ψ_{ij} denote their components, which map $(R^2)_i$ to $(R^2)_j$. Obviously, $\varphi_{ij} = 0 = \psi_{ij}$ unless (i,j) is either $(0,1)$ or $(1,2)$, and $\mathrm{rank}_k\, \varphi_{01} = 2 = \mathrm{rank}_k\, \psi_{01}$. Choose in R_1 (resp. in R_2) the basis x_1, x_2, x_3, x_4 (resp. x_1x_2, x_1x_3, x_3x_4), and use this to write down a basis for $(R^2)_1$ (resp. $(R^2)_2$). The matrices of φ_{12} and ψ_{12} are particularly simple in these bases, and readily show that both maps have rank 6 if k is of characteristic $\neq 2$. Thus we have $\mathrm{rank}_k\, \varphi = 8 = \mathrm{rank}_k\, \psi$, which in view of the equality $\dim_k(R^2) = 16$ implies the complex (**) is exact.

Next we turn to the proof that $\tilde{R}$ has no nontrivial embedded deformations. First note that when k is infinite, one has $\tilde{R} = R$, while for finite k it holds that $\tilde{R} = k(Y) \otimes_k R$. Thus we have to show that the rings constructed in Example 2.1 starting from an infinite field k do not admit embedded deformations $Q \not\equiv R$. We shall do this by checking $\mathrm{Ext}_R^*(k,k)$ for central elements with respect to the algebra structure defined by the Yoneda multiplication.

More precisely, recall that $\mathrm{Ext}_R^*(k,k)$ is the universal enveloping algebra of a graded Lie algebra, $\pi^*(R)$, called the homotopy Lie algebra of R (cf., e.g. [1, Section 1]). If Q is an embedded deformation of R, then the canonical map $\pi^*(R)\to\pi^*(Q)$ is surjective, with kernel a $(\dim Q-\dim R)$-dimensional subspace of $\pi^2(R)$, consisting of central elements (cf. [2, (6.1)]).

To prove our assertion on R we shall show that $\pi^2(R)$ contains no non-zero central elements.

Since the socle of R is $\mathfrak{m}^2$, our example falls under the hypotheses of Lescot in [7, Theorem B]. Since R has a finitely generated module with bounded Betti numbers, this asserts that

$$P_k^R(t) = 1/(1-\dim_k R_1 t+\dim_k R_2 t^2).$$

From Löfwall's thesis we know that this equality implies $\mathrm{Ext}_R^*(k,k)$ is generated as a k-algebra by $\mathrm{Ext}_R^1(k,k)\cong \mathrm{Hom}_k(\mathfrak{m}/\mathfrak{m}^2,k)$ [8, Theorem 1.2], hence is a homomorphic image of the free associative algebra $k\langle T_1,T_2,T_3,T_4\rangle$. Furthermore, applying [8, Corollary 1.3], we can read off the generators of the kernel from the defining equations of R: in our case, they are $(T_1T_2+T_2T_1)+T_3^2$, $(T_1T_3+T_3T_1)+(T_2T_4+T_4T_2)$ and $T_2^2-(T_3T_4+T_4T_3)$.

Returning to $\pi^*(R)$, we see this is a graded Lie algebra on four generators of degree 1: t_1, t_2, t_3 and t_4, subject to the degree 2 relations, $[t_1,t_2]=-t_3^{(2)}$, $[t_1,t_3]=-[t_2,t_4]$ and $t_2^{(2)}=[t_3,t_4]$: we use $[\,,]$ to denote Lie brackets, and $(\ )^{(2)}$ for the quadratic operator defined on elements of odd degree (cf. [1, p. 18]). Consider the following elements, which obviously form a basis of $\pi^2(R)$:

$$u_1=t_1^{(2)};\qquad u_2=[t_1,t_2];\qquad u_3=[t_1,t_3];\qquad u_4=[t_1,t_4];$$
$$u_5=[t_2,t_3];\qquad u_6=[t_3,t_4];\qquad u_7=t_4^{(2)}.$$

A straightforward computation shows that a basis of $\pi^3(R)$ is given by

$$v_i=[t_1,u_{i+1}]\quad \text{for } 1\le i\le 6;$$
$$v_7=[t_4,u_2];\qquad v_8=[t_4,u_3].$$

Let now $t=\sum_{i=1}^7 a_iu_i$ be a central element of $\pi^2(R)$. Then $0=[t_1,t]=\sum_{i=2}^7 a_iv_{i-1}$ shows that $t=a_1u_1$, and $0=[t_2,t]=-a_1v_1$ brings us down to $t=0$, which was to be proved.

4. Concluding comments

(i) The center of $\pi^*(R)$ (for R as in (2)) is actually trivial, since by a result of Bøgvad [4, Corollary, p. 165] the non-trivial central elements could have been located in degree 2 only.

(ii) For the ring and module of Example 2.1, $\mathrm{Ext}_R^*(M,k)$ is finitely generated over $\mathrm{Ext}_R^*(k,k)$, the module structure being that defined by Yoneda products. Indeed, a direct computation shows that multiplication by (t_1+t_4) provides isomorphisms $\mathrm{Ext}_R^n(M,k) \xrightarrow{\cong} \mathrm{Ext}_R^{n+1}(M,k)$, for all $n \geq 0$. This should be compared to the fact that over any R, $\mathrm{vpd}_R N < \infty$ implies the finite generation of $\mathrm{Ext}_R^*(N,k)$ over a polynomial subalgebra of $\mathrm{Ext}_R^*(k,k)$, generated by finitely many *central* elements of degree 2: cf. [2, (6.1.2)] and the theorem of Gulliksen, reproduced in [2, (2.1)].

(iii) Clearly, the complex (**) *does not* arise from a matrix factorization, in the sense of Eisenbud [5, Section 5], of a non zero divisor x in a local ring Q, such that $Q/(x)=R$: to our knowledge, this is the first example of a minimal complex, which is periodic of period 2, but does not come from Eisenbud's construction.

In this connection, we also note that the second and third named authors have produced examples of periodic modules of (minimal) periods different from 2, and of non-periodic modules with bounded Betti numbers, thus providing counterexamples to a conjecture of Eisenbud [5, p. 37]. These constructions are given in [6].

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References

[1] L.L. Avramov, Local algebra and rational homotopy, Astérisque 113–114 (1984) 15–43.
[2] L.L. Avramov, Modules of finite virtual projective dimension, Invent. Math. 96 (1989) 71–101.
[3] L.L. Avramov, Homological asymptotics of modules over local rings, in: M. Hochster, C. Huneke and J.D. Sally, eds., Commutative Algebra, M.S.R.I. Publications 15 (Springer, Berlin, 1989) 33–62.
[4] R. Bøgvad, Some elementary results on the cohomology of graded Lie algebras, Astérisque 113–114 (1984) 156–166.
[5] D. Eisenbud, Homological algebra on a complete intersection, with an application to group representations, Trans. Amer. Math. Soc. 260 (1980) 35–64.
[6] V.N. Gasharov and I.V. Peeva, Boundedness versus periodicity over commutative local rings, Trans. Amer. Math. Soc., to appear.
[7] J. Lescot, Asymptotic properties of Betti numbers of modules over certain rings, J. Pure Appl. Algebra 38 (1985) 287–298.
[8] C. Löfwall, On the subalgebra generated by the one-dimensional elements in the Yoneda Ext-algebra, in: J.-E. Roos, ed., Algebra, Algebraic Topology, and their Interactions, Lecture Notes in Mathematics 1183 (Springer, Berlin, 1986) 291–338.