Journal of Pure and Applied Algebra 38 (1985) 285-286
North-Holland

A PROPERTY OF INDEPENDENT ELEMENTS

Christer LECH
Department of Mathematics, University of Stockholm, P.O. Box 6701, S-113 85 Stockholm, Sweden

Communicated by C. Löfwall
Received 25 January 1985

Dedicated to Jan-Erik Roos on his 50-th birthday

We shall present a theorem on independent elements, a concept we once introduced in [2]. The theorem has at least one application: it can be used as a tool for proving in a simple manner a certain inequality for local flat morphisms of Noetherian rings (see the corollary of the theorem). This inequality was first proved by Vasconcelos [3, Theorem 2.1] after a partial result by ourselves [2, Theorem 2]. Since then it has been generalized in terms of deviations. The most comprehensive result so far is due to Avramov [1, Theorem 2.7]. Another line of generalization, in terms of Hilbert functions, has been suggested by ourselves [2, the introduction].

Definition. Let R be a commutative ring with unity element. A finite set $\{x_1, \dots, x_r\}$ of elements in R is called independent w.r.t. a (unitary) R-module M if, for any system $m_1, \dots, m_r$ of elements in M, it is true that

$$x_1 m_1 + \cdots + x_r m_r = 0 \Rightarrow m_1, \dots, m_r \in (x_1, \dots, x_r)M.$$

(Cf. [2, p. 77].) The qualification 'w.r.t. M' may be suppressed if $M = R$.

Let us note one immediate property of this concept of independence: If $\{x_1, \dots, x_r\}$ is independent w.r.t. R and M is R-flat, then $\{x_1, \dots, x_r\}$ is independent also w.r.t. M (cf. [2, p. 78]).

Theorem. *A set of independent elements in a ring cannot be contained in a proper ideal generated by fewer elements.*

Proof. Let $\{x_1, \dots, x_r\}$ be a set of independent elements in a ring R, and let $y_1, \dots, y_s$ be further elements in R such that $(x_1, \dots, x_r) \subseteq (y_1, \dots, y_s)$. Form the Koszul algebra $K(R) = R\langle Y_1, \dots, Y_s;\ \mathrm{d}Y_1 = y_1, \dots, \mathrm{d}Y_s = y_s\rangle$ of R w.r.t. $\{y_1, \dots, y_s\}$. Choose linear forms $X_1, \dots, X_r$ in $K(R)$ such that $\mathrm{d}X_1 = x_1, \dots, \mathrm{d}X_r = x_r$. This is possible as $(x) \subseteq (y)$ (where (x) means $(x_1, \dots, x_r)$, etc). Suppose that $r > s$. Then the product $X_1 \cdots X_r$ equals 0, and so certainly

$$X_1 \cdots X_r \in (x)K(R) + \mathrm{d}K(R),$$

where $\mathrm{d}K(R)$ denotes the submodule of boundaries in $K(R)$. Application of the boundary operator yields

$$x_1 X_2 \cdots X_r - \cdots \pm x_r X_1 \cdots X_{r-1} \in (x)\mathrm{d}K(R),$$

and, using the independence of $\{x_1, \dots, x_r\}$ w.r.t. the R-module $K(R)$ which is free and hence flat, we get

$$X_1 \cdots X_{r-1} \in (x)K(R) + \mathrm{d}K(R).$$

Repeating the procedure now performed, we finally obtain

$$1 \in (x)K(R) + \mathrm{d}K(R) \subseteq (y)K(R).$$

Thus, if $r > s$, the ideal (y) cannot be proper.

Corollary. *For any local flat morphism $A \to B$ of Noetherian rings, the embedding dimension of A does not exceed that of B.*

Proof. Observe that any minimal set of generators of the maximal ideal in a Noetherian local ring is independent.

Remark. The reasoning in the proof of the theorem can be extended to give the following more detailed result. Assume that $(y_1, \dots, y_s) \neq (1)$, and let $HK(R/(x))$ be the homology ring of the Koszul algebra $K(R/(x)) = R/(x)\langle Y_1, \dots, Y_s;\ \mathrm{d}Y_1 = \bar{y}_1, \dots, \mathrm{d}Y_s = \bar{y}_s \rangle$. Then there exists a (homogeneous) subring A of $HK(R/(x))$ such that $HK(R/(x))$ is a free strictly commutative extension of A in r variables represented by $X_1, \dots, X_r$. A similar result can be derived for $HK(R/(x)^n)$ $(n > 1)$ under proper assumptions about $\{x_1, \dots, x_r\}$ – n-independence!

References

[1] L. Avramov, Local algebra and rational homotopy, Astérisque 113–114 (1984) 15–43.
[2] Chr. Lech, Inequalities related to certain couples of local rings, Acta Math. 112 (1964) 69–89.
[3] W. Vasconcelos, Ideals generated by R-sequences, J. Algebra 6 (1967) 309–316.