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A Quillen–Gersten type spectral sequence for the K-theory of schemes with endomorphisms

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Abstract

A Quillen–Gersten type spectral sequence is drawn for the K-theory of schemes with endomorphisms. We also prove an anology of Gersten's conjecture in the K-theory of schemes with endomorphisms for the equal characteristic case. © 1998 Elsevier Science B.V. All rights reserved.

1. Introduction

Let X be a scheme. Let $\mathscr{E}nd(X)$ denote the category whose objects are all pairs $(\mathscr{F}, f)$ with $\mathscr{F}$ a vector bundle on X and f an endomorphism of $\mathscr{F}$, and the morphisms in $\mathscr{E}nd(X)$ from $(\mathscr{F}, f)$ and $(\mathscr{G}, g)$ are those morphisms from $\mathscr{F}$ to $\mathscr{G}$ which commute with the endomorphisms f and g. $\mathscr{E}nd(X)$ becomes an exact category when we define $(\mathscr{F}, f) \to (\mathscr{G}, g) \to (\mathscr{H}, h)$ to be short exact if and only if $\mathscr{F} \to \mathscr{G} \to \mathscr{H}$ is as vector bundles. By the K-theory of X with endomorphisms we mean the K-theory of the exact category $\mathscr{E}nd(X)$.

The forgetful map $(\mathscr{F}, f) \to \mathscr{F}$ gives a functor from $\mathscr{E}nd(X)$ to the category of all vector bundles over X. This forgetful functor is split by the injection $\mathscr{F} \to (\mathscr{F}, 0)$. Define

$$End_i(X) = \ker(K_i(\mathscr{E}nd(X)) \to K_i(X)).$$

When $X = \mathrm{Spec}(A)$, we write $End_i(A)$ for $End_i(X)$.

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The first main result of this paper is the existence of a Quillen–Gersten type spectral sequence for the K-theory of schemes with endomorphisms.

Theorem 2.4. *Let X be a regular noetherian scheme with a family of ample line bundles. Then we have a Quillen–Gersten type spectral sequence:*

$$E_1^{pq}(X) = \coprod_{x \in X[T]_p - \tilde{X}} K_{-p-q}(k(x)) \Rightarrow End_{-p-q}(X).$$

(The meanings of the notations will be clear in later sections.)

Next we show a theorem which is an anology to Gersten's conjecture for the K-theory of schemes.

Theorem 3.1. *Let* $X = \mathrm{Spec}(A)$ *be an affine scheme where A is a regular semi-local ring obtained by localizing a finite type algebra over a field. Then the sequence*

$$0 \to End_i(A) \to \coprod_{x \in X[T]_1 - \tilde{X}} K_i(k(x)) \to \coprod_{x \in X[T]_2 - \tilde{X}} K_{i-1}(k(x)) \to \cdots$$

is exact for all i.

We also show that the projective space bundle formula also holds for the K-theory of schemes with endomorphisms.

Corollary 2.2 (Projective space bundles theorem). *Let E be a vector bundle of rank r over X, let PE be the associate projective space bundle, and let* $f : PE \to X$ *be the structure map. Then we have the isomorphisms*

$$End_i(PE) \cong \bigoplus_{n=1}^{r} End_i(X)$$

for all i.

2. A Quillen–Gersten type spectral sequence

Given a scheme X, let $\tilde{S}$ be the multiplicatively closed set of all polynomials of the form $f(T) = 1 + a_1T + \cdots + a_nT^n$ where all $a_i \in \Gamma(\mathcal{O}_X, X)$ are global sections on X, i.e., $\tilde{S} = 1 + T\Gamma(\mathcal{O}_X, X)[T] \subset \Gamma(\mathcal{O}_{X[T]}, X[T])$. Here $X[T] = X \times \mathrm{Spec}(Z[T])$. We form a new scheme $\tilde{X} = \tilde{S}^{-1}X[T]$ in the following way: locally for any affine open subscheme U of X, $U = \mathrm{Spec}(A)$, denote by $\tilde{S}_U$ the image of $\tilde{S}$ under the restriction map

$$\Gamma(\mathcal{O}_X, X)[T] \to \Gamma(\mathcal{O}_X, U)[T] = A[T].$$

Let $\tilde{U} = \mathrm{Spec}(\tilde{S}_U^{-1}A[T])$. Clearly these locally defined affine schemes can glue up and form the scheme $\tilde{X}$.

We have the map $\varphi : X \to \tilde{X}$ which is induced locally by the surjective ring map $\tilde{S}_U^{-1}A[T] \to A$ by setting $T=0$. The map φ has a retraction map $\psi : \tilde{X} \to X$ induced locally by the ring embedding $A \to \tilde{S}_U^{-1}A[T]$. Define

$$EK_i(X) = \ker(K_i(\tilde{X}) \xrightarrow{\varphi^*} K_i(X)).$$

When $X = \mathrm{Spec}(A)$, we write $EK_i(A)$ for $EK_i(X)$.

Theorem 2.1. *If X is a quasi-compact scheme with an ample family of line bundles, then with the above notations, we have*

$$End_i(X) \cong EK_{i+1}(X).$$

Proof. See [9]. □

Corollary 2.2 (Projective space bundles theorem). *Let E be a vector bundle of rank r over X, where X is a quasi-compact scheme with an ample family of line bundles. Let PE be the associate projective space bundle and $f : PE \to X$ be the structure map. Then we have the isomorphisms*

$$End_i(PE) \cong \bigoplus_{n=1}^{r} End_i(X)$$

for all i.

Proof. If $X = \mathrm{Spec}(A)$ is affine and E is a trivial bundle on X, then $PE = \mathrm{Proj}(A[x_1, \ldots, x_r])$ and

$$\Gamma(\mathcal{O}_{PE}, PE) = A = f^*(\Gamma(\mathcal{O}_X, X)).$$

For a general nonaffine X, the global sections are just the glueing-ups of the sections over a covering. Thus we see that

$$\Gamma(\mathcal{O}_{PE}, PE) = f^*(\Gamma(\mathcal{O}_X, X)),$$

and

$$\widetilde{PE} = P\psi^*(E)$$

where $\psi : \tilde{X} \to X$ is the map defined above. Then we have

$$\begin{aligned} End_i(PE) &\cong EK_{i+1}(PE) = \ker(K_{i+1}(\widetilde{PE}) \to K_{i+1}(PE)) \\ &= \ker(K_{i+1}(P\psi^*(E)) \to K_{i+1}(PE)) \\ &\cong \ker\left(\bigoplus_{n=1}^{r} K_{i+1}(\tilde{X}) \to \bigoplus_{n=1}^{r} K_{i+1}(X)\right) \\ &= \bigoplus_{n=1}^{r} EK_{i+1}(X) \cong \bigoplus_{n=1}^{r} End_i(X). \end{aligned}$$ □

For a noetherian scheme X, let $Modc(X)$ denote the category of all coherent $\mathcal{O}_X$-modules on X. Let $\underline{\underline{M}}^{\tilde{S}}(X[T])$ denote the subcategory of $Modc(X[T])$ of all coherent $\mathcal{O}_{X[T]}$-modules $\mathcal{F}$ such that there is an

$$s \in \tilde{S} = 1 + T\Gamma(\mathcal{O}_X, X)[T]$$

such that $s\mathcal{F} = 0$. For $s \in \tilde{S}$, let $X[T]_s$ denote the locus of s in $X[T]$, i.e., $X[T]_s = \mathrm{Supp}(\mathcal{O}_{X[T]}/(s))$. We set

$$X[T] - \tilde{X} = \bigcup_{s \in \tilde{S}} X[T]_s.$$

Then $\mathcal{F} \in \underline{\underline{M}}^{\tilde{S}}(X[T])$ if and only if $\mathrm{Supp}(\mathcal{F}) \subset X[T] - \tilde{X}$.

Lemma 2.3. *Assume X is a regular noetherian scheme with a family of ample line bundles. Then we have*

$$End_i(X) \cong K_i(\underline{\underline{M}}^{\tilde{S}}(X[T])).$$

Proof. Let

$$\eta : \tilde{X} = \tilde{S}^{-1}X[T] \to X[T]$$

be the map induced by the localization. Clearly $\eta^*(\underline{\underline{M}}^{\tilde{S}}(X[T])) = 0$. So we have the induced functor

$$\bar{\eta}^* : Modc(X[T])/\underline{\underline{M}}^{\tilde{S}}(X[T]) \to Modc(\tilde{X}).$$

The functor $\bar{\eta}^*$ is in fact an equivalence of categories. This can be checked by the very definition of the quotient category (a quick read about quotient categories is [5, Appendix B]). For any $\mathcal{F} \in Modc(\tilde{X})$, since $\eta_*(\mathcal{F})$ is a quasi-coherent $\mathcal{O}_{X[T]}$-module,

$$\eta_*(\mathcal{F}) = \varinjlim \mathcal{G}$$

where $\mathcal{G}$ runs through all coherent submodule of $\eta_*(\mathcal{F})$. So for some $\mathcal{G}$, we have $\eta^*(\mathcal{G}) = \mathcal{F}$. This shows that every object $\mathcal{F} \in Modc(\tilde{X})$ is isomorphic to $\bar{\eta}^*(\mathcal{G})$ for some $\mathcal{G} \in Modc(X[T])$. The fact that $\bar{\eta}^*$ is full and faithful is also easily checked.

Since X is regular, so are $X[T]$ and $\tilde{X}$. We have $K_i(X) \cong K_i(X[T])$. Then

$$EK_i(X) = \mathrm{coker}(K_i(X) \to K_i(\tilde{X})) \cong \mathrm{coker}(K_i(X[T]) \to K_i(\tilde{X})).$$

Applying Quillen's localization theorem for the K-theory of abelian categories, plus the fact that the map $K_i(X[T]) \to K_i(\tilde{X})$ is splitting injective, the localization long exact sequence breaks into short exact sequences:

$$0 \to K_i(X[T]) \to K_i(\tilde{X}) \to K_{i-1}(\underline{\underline{M}}^{\tilde{S}}(X[T])) \to 0.$$

Thus we have

$$\begin{aligned} End_i(X) &\cong EK_{i+1}(X) = \operatorname{coker}(K_{i+1}(X[T]) \to K_{i+1}(\tilde{X})) \\ &= K_i(\underline{\underline{M}}^{\tilde{S}}(X[T])). \qquad \square \end{aligned}$$

In order to produce a Quillen–Gersten type spectral sequence, we consider the filtration by supports. Let $\underline{\underline{M}}_p^{\tilde{S}}(X[T])$ denote the subcategory of all coherent $\mathcal{O}_{X[T]}$-modules in $\underline{\underline{M}}^{\tilde{S}}(X[T])$ whose supports are of codim $\geq p$. Clearly $\underline{\underline{M}}_p^{\tilde{S}}(X[T])$ is a Serre subcategory of $\underline{\underline{M}}^{\tilde{S}}(X[T])$ and

$$\underline{\underline{M}}^{\tilde{S}}(X[T]) = \underline{\underline{M}}_0^{\tilde{S}}(X[T]) = \underline{\underline{M}}_1^{\tilde{S}}(X[T]) \supseteq \underline{\underline{M}}_2^{\tilde{S}}(X[T]) \supseteq \cdots.$$

We set

$$X[T]_p - \tilde{X} = \{x \in X[T] - \tilde{X} \mid \operatorname{codim}_{X[T]}(x) = p\}.$$

Theorem 2.4. *Let X be a regular noetherian scheme with a family of ample line bundles. Then we have the spectral sequence*

$$E_1^{pq}(X) = \coprod_{x \in X[T]_p - \tilde{X}} K_{-p-q}(k(x)) \Rightarrow End_{-p-q}(X)$$

which is convergent when X has finite dimension.

Proof. For $p \geq 1$, we have the equivalence of categories:

$$\underline{\underline{M}}_p^{\tilde{S}}(X[T]) / \underline{\underline{M}}_{p+1}^{\tilde{S}}(X[T]) \cong \coprod_{x \in X[T]_p - \tilde{X}} \bigcup_n Modf(\mathcal{O}_{X[T],x}/m_x^n).$$

To see this equivalence, we can reduce it to the known case. Since

$$\underline{\underline{M}}_p^{\tilde{S}}(X[T]) = \varinjlim_{s \in \tilde{S}} \underline{\underline{M}}_{p-1}(X[T]_s),$$

$$\underline{\underline{M}}_{p+1}^{\tilde{S}}(X[T]) = \varinjlim_{s \in \tilde{S}} \underline{\underline{M}}_p(X[T]_s),$$

and

$$\underline{\underline{M}}_{p-1}(X[T]_s) / \underline{\underline{M}}_p(X[T]_s) \cong \coprod_{x \in (X[T]_s)_{p-1}} \bigcup_n Modf(\mathcal{O}_{X[T]_s,x}/m_x^n + (s))$$

as is in [4], we have

$$\begin{aligned} \underline{\underline{M}}_p^{\tilde{S}}(X[T]) / \underline{\underline{M}}_{p+1}^{\tilde{S}}(X[T]) &= \varinjlim_{s \in \tilde{S}} \underline{\underline{M}}_{p-1}(X[T]_s) / \underline{\underline{M}}_p(X[T]_s) \\ &= \varinjlim_{s \in \tilde{S}} \coprod_{x \in (X[T]_s)_{p-1}} \bigcup_n Modf(\mathcal{O}_{X[T]_s,x}/m_x^n + (s)) \\ &= \coprod_{x \in X[T]_p - \tilde{X}} \bigcup_n Modf(\mathcal{O}_{X[T],x}/m_x^n). \end{aligned}$$

Applying Quillen's localization theorem for the K-theory of abelian categories, we have the long exact sequences for each $p \geq 1$:

$$\cdots \to K_i(\underline{\underline{M}}^{\tilde{S}}_{p+1}(X[T])) \to K_i(\underline{\underline{M}}^{\tilde{S}}_{p}(X[T])) \to \coprod_{x \in X[T]_p - \tilde{X}} K_i(k(x)) \to \cdots.$$

Then the standard process of producing a spectral sequence gives the stated spectral sequence in the theorem. □

3. An analogy of Gersten's conjecture

The following theorem gives an affirmative answer to an analogy of Gersten's conjecture in the current situation for the equal characteristic case, cf. [4, Theorem 5.11].

Theorem 3.1. *Let $X = \mathrm{Spec}(A)$ be an affine scheme where A is a regular semi-local ring obtained by localizing a finite type algebra over a field. Then*

(i) *the inclusion*

$$\underline{\underline{M}}^{\tilde{S}}_{p+1}(A[T]) \to \underline{\underline{M}}^{\tilde{S}}_{p}(A[T])$$

induces zero maps on the K-theory

$$K_i(\underline{\underline{M}}^{\tilde{S}}_{p+1}(A[T])) \xrightarrow{0} K_i(\underline{\underline{M}}^{\tilde{S}}_{p}(A[T]))$$

for all i and all $p \geq 1$, and

(ii) *we have a resolution (exact sequence) for $End_i(A)$:*

$$0 \to End_i(A) \to \coprod_{x \in X[T]_1 - \tilde{X}} K_i(k(x)) \to \coprod_{x \in X[T]_2 - \tilde{X}} K_{i-1}(k(x)) \to \cdots.$$

Proof. The proof we give here follows the outline of the proof of [4, Theorem 5.11], but some extra effort and care is necessary to make the proof work in our current situation.

(i) According to the assumption, let R be a finite type algebra over a field k, V a finite set of prime ideals of R such that A is the localization of R with respect to V, i.e., $A = (R - V)^{-1}R$. Here we also use V to denote the set of elements $\bigcup_{P \in V} P$.

First, we reduce to the case where R is smooth over the field k. As in [4], there is a subfield k' of k which is finitely generated over the prime field k_0, a finite type k'-algebra R', a finite set V' of prime ideals of R' and a regular semi-local ring $A' = (R' - V')^{-1}R'$, such that $R = k \otimes_{k'} R'$, $A = k \otimes_{k'} A'$ and $V = \{k \otimes_{k'} P' \mid P' \in V'\}$.

Let k_i be any subfield of k which contains k' and is finitely generated over k', $R_i = k_i \otimes_{k'} R', A_i = k \otimes_{k'} A'$ and $V_i = \{k_i \otimes_{k'} P' \mid P' \in V'\}$. Denote $\tilde{S}_i = 1 + TA_i[T]$. Then we have

$$\underline{\underline{M}}^{\tilde{S}}_{p}(A[T]) = \varinjlim_{k_i} \underline{\underline{M}}^{\tilde{S}_i}_{p}(A_i[T])$$

where k_i runs through all subfields of k which contain k' and are finitely generated over k'. So we need to show that for each such k_i, the inclusion

$$\underline{\underline{M}}_{p+1}^{\tilde{S}_i}(A_i[T]) \to \underline{\underline{M}}_p^{\tilde{S}_i}(A_i[T])$$

induces zero maps on their K-theory.

Since R_i is finitely generated over k_i and k_i is finitely generated over k_0, R_i is finitely generated over k_0. Since R_i is regular on V_i and k_0 is a prime field, R_i is smooth on V_i. So R_i is smooth on a neighbourhood of V_i. Then there is an $f \in R_i - V_i$ such that R_{if} is smooth over k_0. R_{if} is still finitely generated over k_0. Therefore we have reduced to the case where R is smooth finitely generated over a field k, V is a finite set of prime ideals of R and $A = (R - V)^{-1}R$.

Since

$$A = \varinjlim_{f \in R-V} R_f,$$

if we define $\tilde{S}^f = 1 + TR_f[T]$, then

$$\tilde{S} = 1 + TA[T] = \varinjlim_{f \in R-V} \tilde{S}^f$$

and

$$\underline{\underline{M}}_p^{\tilde{S}}(A[T]) = \varinjlim_{f \in R-V} \underline{\underline{M}}_p^{\tilde{S}^f}(R_f[T]).$$

So what remains to show is that the localization

$$\underline{\underline{M}}_{p+1}^{\tilde{S}^f}(R_f[T]) \to \underline{\underline{M}}_p^{\tilde{S}}(A[T])$$

induces zero maps on all K_* for all $f \in R - V$ and $p \geq 1$. Write R for R_f and $\tilde{S}^1 = 1 + TR[T]$. Then we need to show that the localization functor

$$\underline{\underline{M}}_{p+1}^{\tilde{S}^1}(R[T]) \to \underline{\underline{M}}_p^{\tilde{S}}(A[T])$$

induces zero maps on all K_* for all $p \geq 1$. It suffices to show that there exists an $f \in R - V$ such that the localization functor

$$\underline{\underline{M}}_{p+1}^{\tilde{S}^1}(R[T]) \to \underline{\underline{M}}_p^{\tilde{S}^f}(R_f[T])$$

induces zero maps on all K_* for all $p \geq 1$.

First we claim that

$$\underline{\underline{M}}_{p+1}^{\tilde{S}^1}(R[T]) = \varinjlim_t \underline{\underline{M}}_p^{\tilde{S}^1}(R/tR[T])$$

where t runs through all regular elements in R.

To see this, let $M \in \underline{\underline{M}}_{p+1}^{\tilde{S}^1}(R[T])$. By definition,

$$\operatorname{codim}(\operatorname{ann}(M)) = \inf\{\operatorname{height}(P) \mid P \supset \operatorname{ann}(M)\} \geq p + 1.$$

For any prime ideal P in $R[T]$, let $Q=P\cap R$. Then height$(Q)=$height(P) or height $(P)-1$ [3, Theorem 149]. So if $P\supset \mathrm{ann}(M)$, then height$(Q)\geq p\geq 1$. Let $\{P_1,\ldots,P_r\}$ be all the minimal prime ideals over $\mathrm{ann}(M)$. Then there is an integer e such that $(P_1\cdots P_r)^e\subset \mathrm{ann}(M)$, so $(Q_1\cdots Q_r)^e\subset \mathrm{ann}(M)$. Since each Q_j has height ≥ 1, $(Q_1\cdots Q_r)^e\neq 0$, i.e., there is a t regular and $t\in \mathrm{ann}(M)$. Therefore M is an $R/tR[T]$-module, and $M\in \underline{\underline{M}}_p^{\tilde{S}^1}(R/tR[T])$.

Now we need to show that for any $t\in R$ regular, the localization functor

$$\underline{\underline{M}}_p^{\tilde{S}^1}(R/tR[T])\rightarrow \underline{\underline{M}}_p^{\tilde{S}^f}(R_f[T])$$

induces zero maps on all K_* for all $p\geq 1$. We need the following results used in the proof of [4, Theorem 5.11]. The result (1) is Lemma 5.12 in [4] and the proof of (2) is given in [4] right after the statement of Lemma 5.12.

(1) Let R be a smooth finite type algebra of dimension r over a field k, t a regular element in R and V a finite set of prime ideals in R. Then there exist elements $x_1,\ldots,x_{r-1}$ in R algebraically independent over k such that if $B=k[x_1,\ldots,x_{r-1}]$, then (i) R/tR is finite over B, and (ii) R is smooth over B at all prime ideals in V.

(2) Let I be the kernel of the morphism $R'=R\otimes_B R/tR\rightarrow B'=R/tR$ (induced by the multiplication in the ring R). Then there exists an $f\in R-V$ such that I_f is a principal ideal of R'_f.

For any $M\in \underline{\underline{M}}_p^{\tilde{S}^1}(R/tR[T])$, since $B'_f[T]$ is flat over $B'[T]$, the following sequence is exact:

$$0\rightarrow I_f[T]\otimes_{B'[T]}M\rightarrow R'_f[T]\otimes_{B'[T]}M\rightarrow B'_f[T]\otimes_{B'[T]}M\rightarrow 0.$$

We claim that

$$R'_f[T]\otimes_{B'[T]}M\in \underline{\underline{M}}_p^{\tilde{S}^f}(R_f[T])$$

and therefore

$$I_f[T]\otimes_{B'[T]}M\in \underline{\underline{M}}_p^{\tilde{S}^f}(R_f[T])$$

since $I_f\cong R'_f$. It suffices to show that there is an $s\in\tilde{S}^f$ such that

$$s(R_f[T]\otimes_{B'[T]}M)=0.$$

Let $s_0\in\tilde{S}^1=1+TR[T]$ such that $s_0M=0$. Then T induces an isomorphism on M as an $R/tR[T]$-module. Since R/tR is finite over B and M is finite over $R/tR[T]$, M is finite over $B[T]$. There is a monic polynomial $h(x)\in B[T][x]$ that annihilates T^{-1}, i.e., $h(T^{-1})M=0$. Assume

$$h(x)=x^n+a_{n-1}(T)x^{n-1}+\cdots+a_0(T)$$

where $a_j(T)\in B[T]$. Then

$$s=T^n h(T^{-1})\in 1+TB[T]\subset\tilde{S}^f$$

and

$$s(R'_f[T]\otimes_{B'[T]}M)=R'_f[T]\otimes_{B'[T]}(sM)=0.$$

So we have a short exact sequence of exact functors

$$0\to I_f[T]\otimes_{B'[T]}(\ )\to R'_f[T]\otimes_{B'[T]}(\ )\to B'_f[T]\otimes_{B'[T]}(\ )\to 0.$$

By the additivity theorem, the functor $B'_f[T]\otimes_{B'[T]}(\quad)$ induces zero maps on K-theory:

$$K_*(\underline{\underline{M}}_p^{\tilde{S}'}(R/tR[T]))\xrightarrow{0}K_*(\underline{\underline{M}}_p^{\tilde{S}'}(R_f[T])).$$

(ii) The same proof as the one for Proposition 5.6 of [4] carries over. □

Given a regular scheme X with a family of ample line bundles, let $\mathscr{E}nd_n$ denote the sheaf associated to the presheaf

$$U\to End_n(U),\quad U\subset X.$$

It is not hard to see that the stalk of this presheaf at a point $x\in X$ is $End_n(\mathscr{O}_{X,x})$. Let $\mathscr{E}^{p,q}$ denote the sheaf associated to the presheaf

$$U\to\coprod_{x\in U[T]_p-\tilde{U}}K_{-p-q}(k(x)),\quad U\subset X,$$

where recall that $\tilde{S}=1+T\Gamma(\mathscr{O}_U,U)[T]$. Then we see that the stalk of this presheaf at $x\in X$ is (here $W=\mathrm{Spec}(\mathscr{O}_{X,x})$)

$$\coprod_{x\in W[T]_p-\tilde{W}}K_{-p-q}(k(x)).$$

By the Theorem 2.4, we see that we have an exact sequence of sheaves:

$$0\to\mathscr{E}nd_n\to\mathscr{E}^{1,-n}\to\mathscr{E}^{2,-n}\to\cdots\to\mathscr{E}^{n,-n}\to 0.$$

But, unlike the case of K-theory of schemes, the presheaf

$$U\to\coprod_{x\in U[T]_p-\tilde{U}}K_{-p-q}(k(x)),\quad U\subset X,$$

is not a flasque sheaf, so we do not have the description for the E_2-term for the Quillen–Gersten type spectral sequence in the Theorem 2.4 as the sheaf homology of $\mathscr{E}nd_n$. Instead we have the following

Corollary 3.2. *Let X be a regular scheme with a family of ample line bundles. Let $\mathscr{F}^{-n}$ denote the chain complex of sheaves*:

$$\mathscr{E}^{1,-n}\to\mathscr{E}^{2,-n}\to\cdots\to\mathscr{E}^{n,-n}.$$

Then we have isomorphisms for all n and p:

$$H^p(X, \mathscr{E}nd_n) \cong hH^p(X, \mathscr{I}^{-n})$$

where hH^p denotes the hyperhomology for the chain complex of sheaves.

Proof. The morphism $\mathscr{E}nd_n \to \mathscr{I}^{-n}$ is a quasi-isomorphism. □

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