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A REPRESENTATION OF LOCALLY ARTINIAN CATEGORIES

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The main result of this work is the following explicit description of locally artinian categories.

Theorem 1. *A Grothendieck category $\mathscr{C}$ is locally artinian iff it is naturally equivalent with the category of all discrete right R-modules over a right topologically artinian ring R.*

Note that R is not, in general, an object of $\mathscr{C}$. In the Popescu–Gabriel representation of Grothendieck categories the ground rings are always objects of them. Hence our representation is of a different type. Recall that a Grothendieck category $\mathscr{C}$ is said to be *locally artinian* if it has a set of generators which are artinian objects. For the necessary definition, concepts and results we refer to [2], [4] and [5]. Since the word 'strict' is used for linearly compact modules in two different meanings, we make the following convention: a complete topologically coherent R-module M is called *strict* if any topologically coherent closed submodule N of M, such that M/N endowed with the discrete topology is coherent, is open. (This is 'strict' in the sense of Oberst [4].)

A linearly compact module M is called *toplogically artinian* if each discrete factor of M is artinian. (In this case M is usually said to be *strictly linearly compact.* In the present paper, we have to use both notions, and in order to avoid misunderstandings, we shall use the term 'topologically artinian' in the latter case.) We give the proof in several steps. First we describe the dual of $\mathscr{C}$.

Theorem 2. *The dual of a locally artinian category $\mathscr{C}$ is just the category of strict linearly compact topologically noetherian left S-modules over a strict linearly compact topologically noetherian ring S.*

Proof. By Lemma 1.3 and Proposition 1.5 in [4] we can choose a full, skeletal-small, finitely closed and generating subcategory $\mathscr{A}$ of $\mathscr{C}$ in which every object is artinian and an injective cogenerator E of $\mathscr{C}$ such that for each $N \in \mathscr{N}$ there is a short exact sequence

$$0 \to N \to E^k$$

for some natural k. Let $S = \mathscr{C}(E, E)$. If $A \in \mathscr{C}$, then $\mathscr{C}(A, E)$ is a left S-module, the scalar multiplication being given by composition. If $A \in \mathscr{C}$ and $B \subseteq A$ is a subobject of A we identify $\mathscr{C}(A/B, E)$ with its image under the canonical inclusion

$$\mathscr{C}(A/B, E) \to \mathscr{C}(A, E)$$

i.e.,

$$\mathscr{C}(A/B, E) = \{f : A \to E, f|_B = 0\}.$$

By Lemma 3.1 in [4] the left ideals $\mathscr{C}(E/N, E)$, $N \subseteq E$ and $N \in \mathscr{N}$ induce a left linear ring topology on S. Lemma 3.2 in [4] shows that for any $A \in \mathscr{C}$ the left S-module $\mathscr{C}(A, E)$ is a linearly topologized S-module whose open submodules are $\mathscr{C}(A/N, E)$, $N \subseteq A$, $N \in \mathscr{N}$. By [4, Proposition 3.3] the functor $A \mapsto \mathscr{C}(A, E)$ induces a duality between $\mathscr{N}$ and the category Coh(Dis S) of all discrete, coherent left S-modules. Since a finitely generated submodule of a coherent discrete module is again coherent, we have immediately that all coherent discrete left S-modules are noetherian. Therefore they are linearly compact by Lemma 4.10 in [4], hence by Theorem 6.2 in [4] the dual of $\mathscr{C}$ is the category of all strict linearly compact topologically noetherian left S-modules over S which as a left S-module is again strict linearly compact topologically noetherian, and such one is the dual of a locally artinian category. This completes the proof. □

Now we are able to verify the validity of Theorem 1

Proof of Theorem 1. By Theorem 2, the dual of a locally artinian category $\mathscr{C}$ is the category $\mathscr{S}$ of all strict linearly compact topologically left S-modules over a strict linearly compact topologically noetherian ring S. Let ${}_S\mathsf{U}$ be a discrete injective cogenerator in the category of all linearly topologized left S-modules which is an essential extension of its socle, and let $R = \mathrm{Hom}_S({}_S\mathsf{U}, {}_S\mathsf{U})$ endowed with the weak $*$ topology. By Theorem 2.5 in [2] the discrete bimodule ${}_S\mathsf{U}_R$ induces a topological Morita duality for S and R. Since S is toplogically noetherian, R is topologically artinian on the right. In what follows, by the dual X^* of a linearly topologized left S- or right R-module X we mean the right R- or left S-module $\mathrm{Hom}_S(X, {}_S\mathsf{U})$ or $\mathrm{Hom}_R(X, \mathsf{U}_R)$ of all continuous homomorphism $f : X \to \mathsf{U}$, respectively.

I. If X is a strict linearly compact topologically noetherian, left S-module, then for each continuous homomorphism $f : X \to {}_S\mathsf{U}$ one can easily see that Xf is a finitely generated submodule of ${}_S\mathsf{U}$, i.e., f is annihilated by an open right ideal of R. Therefore X^* can be considered as a discrete right R-module.

In the case when X is a discrete right R-module, we endow X^* with the topology an open basis for which is the set of all annihilators $\mathrm{ann}_{X^*}Y$, where a submodule

Y of X is isomorphic to a submodule of a finite direct sum of duals of discrete factors of S. It is clear that this topology is finer than the weak $*$ topology on X^*.

Since R is topologically artinian, X^* is obviously linearly compact and topologically noetherian. We show now that X^* is strict. For, if Z is any closed submodule of X^* such that X^*/Z endowed with the discrete topology is coherent, then the dual of X^*/Z with the induced topology is obviously $\mathrm{ann}_X Z$, which equals the dual of the discrete module X^*/Z, as is easy to verify. On the other hand, $\mathrm{ann}_X Z$ is isomorphic to a submodule of a finite direct sum of duals of discrete factors of S, because X^*/Z is noetherian. Hence by Proposition 2.1 in [2], $Z = \mathrm{ann}_{X^*}\mathrm{ann}_X Z$ is open in X^*, i.e., X^* is strict.

II. If X is a discrete right R-module, then by Theorem 4.3 in [2] the evaluation map $\omega : x \to x^{**}$ is an isomorphism.

In the case $X \in \mathscr{S}$, Proposition 4.2 in [2] implies only that the evaluation map $\omega : X \to X^{**}$ is an algebraic isomorphism. We show that ω is a topological isomorphism, too. If L is any open submodule of X^{**}, then $L = \mathrm{ann}_X \mathrm{ann}_{X^*} L$ is closed in X and X/L is coherent.

This implies that L is open in X. Conversely, if L is any open submodule of X, then $L = \mathrm{ann}_{X^{**}}\mathrm{ann}_{X^*} L$ and $\mathrm{ann}_{X^*} L$, as the dual of X/L, is isomorphic to a submodule of a finite direct sum of duals of discrete factors of S, and thus L is open in X^{**}.

III. For any two linearly topologized modules X and Y and any continuous homomorphism $f : X \to Y$, we have an induced continuous homomorphism $f^* : Y^* \to X^*$. Now it is routine to see that $f^{**} = f$ if X and Y are objects in $\mathscr{S}$ or discrete right R-modules. This shows that in these cases the map $* : \mathrm{Hom}(X, Y) \to \mathrm{Hom}(Y^*, X^*)$ is an isomorphism.

IV. The calculations (I), (II) and (III) establish that the dual of $\mathscr{S}$ is the category of all discrete right R-modules, where R is a topologically artinian ring. On the other hand, the latter category is clearly locally artinian and hence the proof is complete. $\square$

The theorem of Hopkins says that every artinian ring is at the same time a noetherian ring, too. This statement can be formulated as follows: for a ring R, if the category Mod-R of right R-modules is locally artinian, then Mod-R is locally noetherian. However, it is well known that there are locally artinian categories which are not locally noetherian. As an application of Theorem 1 we can prove

Corollary 3. *A locally artinian category $\mathscr{C}$ is locally noetherian if and only if all of its objects have height at most $\aleph_0$.*

Before proving this statement, let us recall the definition of height. For any object X of a Grothendieck category and any ordinal α a subobject X_α is defined by transfinite induction as follows: $X_0 = \mathrm{So}(X)$, the socle of X; if $\alpha = \beta + 1$, then X_α

is defined so that $X_\alpha/X_\beta = \mathrm{So}(X/X_\beta)$; if α is a limit ordinal, then $X_\alpha = \bigcup X_\beta$. We shall say that the height of X is defined and equals α if α is the smallest ordinal with $X = X_\alpha$. It is clear the every object of a locally artinian category has a height.

Proof of Corollary 3. If a locally artinian category $\mathscr{C}$ is locally notherian, then it is obviously locally finite. Therefore every object of $\mathscr{C}$ has trivially a height at most $\aleph_0$. Conversely, by theorem 1 $\mathscr{C}$ is naturally equivalent to the category of discrete right R-modules over a toplogically artinian ring R. If L is any open right ideal of R, then R/L has a height at most $\aleph_0$. Since R/L is finitely generated, it must be finite, i.e., R/L is of finite length. This shows that $\mathscr{C}$ is locally finite and hence it is obviously locally noetherian.

A right toplogically artinian ring R is said to be *pseudo-compact* if each discrete right R-factor module of R is of finite length. Now we can describe locally finite categories as follows.

Corollary 4. *A Grothendieck category $\mathscr{C}$ is locally finite iff it is naturally equivalent with the category of all discrete modules over a pseudo-compact ring.*

Proof. It is clear that the category of all discrete modules over a pseudo-compact ring is locally finite. The converse statement follows trivially from the proof of Corollary 3.

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