A Remark on Hilbert's Matrix*

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ABSTRACT

We exhibit a Jacobi matrix T which has simple spectrum and integer entries, and commutes with Hilbert's matrix. As an application we replace the computation of the eigenvectors of Hilbert's matrix (a very ill-conditioned problem) by the computation of the eigenvectors of T (a nicely stable numerical problem).

1. INTRODUCTION

The purpose of this note is to exhibit a tridiagonal matrix $T_n(\theta)$ with integer coefficients which commutes with Hilbert's matrix

$$(H_n(\theta))_{ij} = (i+j+\theta)^{-1}, \qquad 1 \leqslant i, j \leqslant n.$$

The matrix $T_n(\theta)$ has a simple spectrum, and thus its eigenvectors are eigenvectors of the Hilbert matrix. In the same vein, $H_n(\theta)$ can be written as a polynomial in $T_n(\theta)$. Both of these points are taken up below.

The class of Hankel matrices, i.e. those satisfying

$$M_{i,j} = r(i+j),$$

that have a tridiagonal matrix—with simple spectrum—in their commutator is very limited, and thus the result reported here is quite surprising. The situation resembles that of Toeplitz matrices, among which a complete description of those with a tridiagonal matrix in their commutator has been given in [2].

*Supported by NSF Grant MCS 78-06718.

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As a general reference on the Hilbert matrix one can consult [3] and the references mentioned there.

Here we just mention the central role of Hilbert's matrix in performing polynomial least squares approximation in $[0,1]$. This leads to a system of equations involving $H_n(-1)$ which is numerically intractable due to the large condition number of $H_n(-1)$. One circumvents this by introducing orthogonal polynomials. The result reported here may offer an alternative.

2. THE TRIDIAGONAL MATRIX

If

$$(T_n(\theta))_{ii} = a_i,$$

$$(T_n(\theta))_{i,i+1} = (T_n(\theta))_{i+1,i} = b_i,$$

and

$$(T_n(\theta))_{ij} = 0 \qquad \text{if} \quad |i-j|>1,$$

we have

$$a_i = -2(n-i)(n+i+\theta)\left[i^2+\theta(i-1)-n\right], \qquad 1\leqslant i\leqslant n,$$

$$b_i = i(n-i)(\theta+i+1)(\theta+n+i+1), \qquad 1\leqslant i\leqslant n-1,$$

and one can check the relation

$$T_n(\theta)H_n(\theta) = H_n(\theta)T_n(\theta)$$

for any n and any θ.

A proof of this relation comes about by a lengthy but straightforward computation. A convenient way to cut the labor in half is to notice that, at least for $2\leqslant i, j\leqslant n-1$, we have that

$$(TH)_{ij} = T_{i,i-1}H_{i-1,j} + T_{i,i}H_{i,j} + T_{i,i+1}H_{i+1,j}$$

is given by the ratio of two polynomials in θ, namely

$$(n-1)\theta^4+\{3n^2+[2(i+j)-2]n-2(i+j)-1\}\theta^3$$

$$+\{2n^3+[6(i+j)-1]n^2+[(j^2+i^2)+2ij-3(i+j)-3]n$$

$$-(i^2+j^2+4ij+i+j)\}\theta^2$$

$$+\{4(i+j)n^3+[(3i^2+3j^2)+6ij-(i+j)-4]n^2$$

$$-(i^2+j^2+4ij+i+j)n-2ij(i+j)-2ij\}\theta$$

$$+[(2i^2+2j^2+4ij-2)n^3-2ijn^2-2ijn-2i^2j^2]$$

and

$$\theta^3+3(i+j)\theta^2+(3i^2+3j^2+6ij-1)\theta$$

$$+[j^3+i^3+3ij^2+3ji^2-(i+j)].$$

Notice that these expressions are plainly symmetric in (i,j) and thus we have shown that

$$(TH)_{ij}=(TH)_{ji}, \qquad 2\leqslant i,j\leqslant n-i.$$

If we had proved this relation for all pairs (i,j) we would be done, since T and H are symmetric and thus

$$TH=HT$$

is equivalent to

$$TH=(TH)^*.$$

Extending the proof of $(TH)_{ij}=(TH)_{ji}$ so as to include the whole range $1\leqslant i,j\leqslant n$ requires nothing new and can be handled exactly as above. We do not do it here.

Two features worth emphasizing are the integer character of a_i and b_i and the nonvanishing of b_i.

Clearly we want $\theta\neq-2,-3,\ldots.$

3. A SIMPLE NUMERICAL EXPERIMENT

Take $\theta = -1$. The matrices $H_n(-1)$ are nonsingular but have a cluster of very small eigenvalues. For instance, for $n=6$ the eigenvalues are (to 2 significant figures)

$$1.6,\ 2.4\times 10^{-1},\ 1.6\times 10^{-2},\ 6.2\times 10^{-4}\ 1.3\times 10^{-5},\ 1.1\times 10^{-7}$$

One can thus expect that the determination of eigenvectors of $H_n(-1)$ corresponding to very small eigenvalues, if done by using a standard method —like the Q-R algorithm—will present problems. On the other hand, applying the Q-R algorithm to $T_n(-1)$ should give the same eigenvectors and (hopefully) better answers.

For comparison purposes we applied the Q-R process both to $H_6(-1)$ and to the matrix

$$\frac{T_6(-1)}{275}.$$

Scaling has the effect of setting $b(6) = (T_6(-1))_{5,6} = 1$. Our computations were performed on a TRS 80 and then compared with the (much higher precision) results given in [1].

All the eigenvectors computed from $T_6(-1)$ were in exceedingly good agreement with [1]. Those computed from $H_6(-1)$ and corresponding to the two smallest eigenvalues were completely off.

The eigenvalues of $T_n(-1)$ are well separated, particularly those corresponding to the cluster of small eigenvalues of $H_n(-1)$. We have found this to be true for all values of n up to 20, and we illustrate here the case of $n=6$. The eigenvalues of $T_6(-1)/275$ are given by

$$1.16,\ 1.05,\ 7.21\times 10^{-1},\ 8.70\times 10^{-2},\ -9.62\times 10^{-1},\ -2.54.$$

The nth eigenvalue in this list shares the same eigenvector with the nth element in the list of eigenvalues of $H_6(-1)$ given above.

4. $H_N(\theta)$ AS A POLYNOMIAL IN $T_N(\theta)$

Since $T_n(\theta)$ is tridiagonal and all of its off diagonal elements are nonzero, one gets a simple recursive relation to determine the coefficients z_i in the expression

$$H_n(\theta) = z_0 I + z_1 T_n(\theta) + z_2 T_n^2(\theta) + \cdots + z_{n-1} T_n^{n-1}(\theta). \qquad (1)$$

This comes about by looking at the entries $(1,k)$, $k=n,n-1,\dots,1$, for the right hand side of (1). In this way we obtain

$$(H_n(\theta))_{1,n}=b_1b_2\cdots b_{n-1}z_{n-1},$$

$$(H_n(\theta))_{1,n-1}=b_1\cdots b_{n-2}[z_{n-2}+(a_1+\cdots+a_{n-1})z_{n-1}],$$

etc. In other words the vectors

$$(H_n(\theta))_{1,1},\dots,(H_n(\theta))_{1,n}$$

and

$$(z_0,\dots,z_{n-1})$$

are connected by an upper triangular matrix R whose diagonal elements are given by

$$R_{ii}=b_1b_2\dots b_{i-1}$$

with the convention $b_0=1$.

Using the expressions given earlier for a_i and b_i, one can thus find z_k.

5. THE CONTINUOUS VERSION

By rewriting $T_n(1)$ in the self-adjoint form

$$D^-AD^+ + B,$$

one can easily conjecture that the limiting second order differential operator

$$D_2\equiv\frac{d}{dx}\left((1-x^2)x^2\frac{d}{dx}\right)-2x^2$$

commutes with the integral operator in $L^2[0,1]$ whose kernel is the Cauchy kernel

$$\frac{1}{x+y}.$$

This can be easily verified. We conclude by noticing that, very much as in the work of Slepian, Pollak, and Landau (see [4] and its references), one can use this "accident" as a way of computing the eigenfunctions of the Cauchy kernel.

REFERENCES

1 H. Fettis and J. Caslin, Eigenvalues and eigenvectors of Hilbert matrices of order 3 through 10, *Math. Comp.*, 1967, pp. 431–441.
2 F. A. Grünbaum, Toeplitz matrices commuting with a tridiagonal matrix, *Linear Algebra Appl.* 40:25–36 (1981).
3 J. Todd, Computational problems concerning the Hilbert matrix, *J. Res. Nat. Bureau Standards Sect. B*65B:19–22 (1961).
4 D. Slepian, Prolate spheroidal wave functions, Fourier analysis and uncertainty V, *Bell System Tech. J.* 57(5):1371–1430 (1978).

Received 30 March 1981; revised 6 July 1981