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A remark on the homology of cosimplicial spaces[1]

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Abstract

The (second-quadrant) mod p homology spectral sequence determined by a cosimplicial space always converges to zero in negative dimensions. More precisely, for every s and t with $t-s<0$ there is some r such that $E^r_{-s,t}=E^\infty_{-s,t}=0$. © 1998 Published by Elsevier Science B.V. All rights reserved.

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1. Introduction

Theorem. *The (second-quadrant)* mod$p$ *homology spectral sequence determined by a cosimplicial space always converges to zero in negative dimensions. More precisely, for every s and t with $t-s<0$ there is some r such that $E^r_{-s,t}=E^\infty_{-s,t}=0$.*

The proof will show that the "rate of convergence" is independent of the particular cosimplicial space, in the sense that r can be taken to be a function of p, s and t. Of course, that has to be so: if an infinite sequence of cosimplicial spaces required larger and larger values of r then their disjoint union would be a counter-example. However the function $r(p,s,t)$ will not be made explicit.

The theorem is at first glance hardly surprising, since spaces have no homology in negative dimensions and the spectral sequence of the cosimplicial space X is generally viewed as trying to compute the homology of a space (the total space Tot$X$ of X or, if X is not fibrant, the total space of a fibrant replacement).

On the other hand, the theorem is false for rational or integral coefficients. Examples are given in Section 7.

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In the case $p = 2$, the theorem was noticed some time ago by Dwyer, but never published; it follows from the operations constructed in [5].

More recently, Shipley proved it for all primes p, but with the added hypothesis that the group $H_t(X^S; Z/p)$ should be finite for all s and t. In fact, according to Corollary 7.7 of [7] the spectral sequence in that case "pro-converges" in dimension n to the pro-group given by applying the functor H_n to a certain tower of spaces (namely the Tot_s tower of the diagonal of the bicosimplicial space obtained by degreewise Bousfield–Kan p-completion of X). Of course, for $n < 0$ this pro-group is (pro-) trivial.

In each of these cases the vanishing result for negative dimensions was incidental to some larger and deeper piece of work. We offer a simpler proof of the vanishing result alone, without any unneeded hypotheses.

The new idea, if any, is in Section 5, where a universal example is used to reduce the problem to an extremely special case. The special case satisfies Shipley's finite-type hypothesis, but it is so special that a quick proof is possible. Such a proof, based on a small part of Shipley's (and Bousfield's and Dwyer's) work, is outlined in Section 6.

The proof is written for based cosimplicial spaces and reduced homology. The analogous result for unbased cosimplicial spaces and unreduced homology can easily be deduced by adding a basepoint.

Throughout, "space" means "simplicial set".

2. Review of Dold–Kan normalization

A simplicial abelian group G determines a chain complex

$$G_0 \stackrel{\partial}{\leftarrow} G_1 \stackrel{\partial}{\leftarrow} G_2 \leftarrow \cdots \tag{2.1}$$

with $\partial = d_0 - d_1 + \cdots$. Recall that this has a subcomplex NG which is in some ways better. The group N_0G is G_0 while for $t > 0$, N_tG is the intersection of the kernels of the face maps $d_i : G_t \to G_{t-1}$, $0 < i \le t$. Not only does NG have the same homology as the larger complex (2.1), but G can be recovered from NG in the sense that the functor N is an equivalence of categories from simplicial abelian groups to nonnegatively graded chain complexes [8, 8.4]. There is also another description of NG, as a quotient of (2.1) rather than a subcomplex: N_tG is the quotient of G_t by the subgroup generated by the images of all the degeneracy maps $s_i : G_{t-1} \to G_t$, $0 \le i \le t-1$.

In the analogous construction for cosimplicial abelian groups the normalized complex NG is a nonnegatively graded cochain complex, a quotient of the more naive one. The group N^0G is G^0 and otherwise N^sG is the quotient of G^s by the subgroup generated by the images of the coface maps $d^i : G^{s-1} \to G^s$, $0 < i \le s$. Alternatively, N^sG can be described as the intersection of the kernels of the codegeneracies $s^i : G^s \to G^{s-1}$, $0 \le i \le s-1$.

3. The homology tower of a cosimplicial space

Recall the usual construction of the homology spectral sequence of a cosimplicial space (=cosimplicial simplicial set) X. For an abelian group A the cosimplicial simplicial abelian group $A \otimes X$ determines a second-quadrant double chain complex by (simplicial and cosimplicial) Dold–Kan normalization. The group in bidegree $(-s,t)$ is written $N^s N_t(A \otimes X)$. The functor denoted $A \otimes (-)$ here is left adjoint to the functor $\mathrm{Hom}(A,-)$ from abelian groups to *based* sets, so that for a based set X the group $A \otimes X$ is the quotient of a direct sum of X copies of A by one copy of A.

Spectral sequence convergence questions may be reformulated in terms of towers. For each $s \geq 0$ form the truncated double chain complex $N^{\leq s} N_\bullet(A \otimes X)$ coinciding with $N_\bullet N_\bullet(A \otimes X)$ in bidegrees $(-i,j)$ with $0 \leq i \leq s$ and zero in all others. Let $T_s(A \otimes X)$ be its total complex, a chain complex vanishing in degrees $< -s$. There is a surjective chain map $T_s(A \otimes X) \to T_{s-1}(A \otimes X)$ for each s, and the resulting tower $\{T_s(A \otimes X)\}_{s \geq 0}$ of chain complexes yields a tower $\{H_n T_s(A \otimes X)\}_{s \geq 0}$ of abelian groups for each integer n. From one point of view the spectral sequence is determined by an exact couple arising from the exact sequences of chain complexes

$$0 \to \Sigma^{-s} N_\bullet(A \otimes X^s) \to T_s(A \otimes X) \to T_{s-1}(A \otimes X) \to 0.$$

Unwinding the definitions (see [4]) one finds that the following is the same as the theorem stated in the introduction.

Theorem (Second form). *The* mod$p$ *homology tower* $\{H_n T_s((\mathbb{Z}/p) \otimes X)\}_{s \geq 0}$ *of a cosimplicial space* X *is pro-trivial for* $n < 0$.

That is, for any $s \geq 0$ there exists $m > s$ such that the map $H_n T_m(\mathbb{Z}/p) \otimes X) \to H_n T_s((\mathbb{Z}/p) \otimes X)$ in the tower is zero. Again, m is independent of X.

4. A different view of the homology tower

It is worth thinking about what the main theorem of this paper says in the case of a cosimplicial set. In this case the spectral sequence is concentrated on the axis $t = 0$. The statement, which will be proved below, is:

Lemma 4.1. *For any based cosimplicial set* X *the cochain complex* $N_\bullet(A \otimes X)$ *associated with the cosimplicial abelian group* $A \otimes X$ *is acyclic in positive degrees.*

Here the coefficient group A is arbitrary. This lemma and its consequence (4.3) are due to Bousfield [3, 2.2–2.4].

Of course, the lemma implies that for a cosimplicial space the second-quadrant double complex $N^\bullet N_\bullet(A \otimes X)$ is horizontally acyclic, except possibly on the $s = 0$ axis.

At first glance that may appear to prove the theorem. It does not. (It had better not for an arbitrary coefficient group.) This can be explained as follows: A double chain complex, for example, $N^\bullet N_\bullet(A\otimes X)$, determines two spectral sequences [8, 5.6] because it can be filtered either by columns or by rows. When the double complex lies in the second quadrant then these two spectral sequences are related to two different "total" chain complexes. The total complex that is more significant here has as its chain group in degree n the direct product of the groups occurring in bidegrees $(-i,j)$ with $j-i=n$. It is the inverse limit of the quotient complexes $T_s(A\otimes X)$, each of which is the total complex of a double complex $N^{\leq s}N_\bullet(A\otimes X)$ having only finitely many nonzero columns. The other total complex has chain groups that are direct sums instead of products, and it is the direct limit of total complexes of double complexes $N^\bullet N_{\leq t}(A\otimes X)$ having only finitely many nonzero rows. The lemma above gives an E^1 vanishing result for the wrong spectral sequence.

On the other hand, the lemma has something to say about the right spectral sequence. The reason is that in the case of a truncated complex $N^{\leq s}N_\bullet(A\otimes X)$ the two total complexes are equal, making the "wrong" spectral sequence relevant in this case after all. In fact, Lemma 4.1 leads to an alternative description (4.3 below) of the homology tower of a cosimplicial space, and this plays a role in the proof of the theorem.

Proof of Lemma 4.1. The cochain complex $N^\bullet(A\otimes X)$ can be obtained by applying the functor $A\otimes$ to the following diagram of based sets:

$$\cdots \leftarrow X_N^3 \leftarrow X_N^2 \leftarrow X_N^1 \leftarrow X_N^0. \tag{4.1}$$

Here X_N^0 is X^0 while for $s>0$, X_N^s is the based set $X^s/(d^1X^{s-1}\cup\cdots\cup d^sX^{s-1})$. The map from X_N^s to X_N^{s+1} is induced by the coface map d^0. It is well-defined and based because $d^0d^i=d^{i+1}d^0$ for $i>0$, and the "kernel" contains the image because $d^0d^0=d^1d^0$.

The problem is to show that (4.1) is "exact" at X_N^s for each $s>0$ in a certain strong sense, namely,

(a) kernel equals image, and

(b) outside the kernel, the map $X_N^s\to X_N^{s+1}$ is injective.

Statement (b) follows from the injectivity of d^0, which follows from $s^0d^0=1$. For (a), assume that $x\in X^s$ represents an element of the kernel of $X_N^s\to X_N^{s+1}$. Thus, $d^0x=d^iy$ for some $y\in X^s$ and some $i\geq 1$. If $i>1$ then this implies $x=s^0d^0x=s^0d^iy=d^{i-1}s^0y$. If $i=1$ it implies $x=s^0d^0x=s^0d^1y=y$, whence $d^1x=d^1y=d^0x$ and $x=s^1d^1x=s^1d^0x=d^0s^0x$. In either case the proof of Lemma 4.1 is complete, since x belongs to d^jX^{s-1} for some $j\geq 0$. □

Now let X be a (based) cosimplicial space X, so that (4.1) becomes a diagram of spaces. It will simplify matters a little to assume that the map $X_N^0\to X_N^1$ is injective, i.e.

The maximal augmentation of X is a point. (4.2)

Recall that the maximal augmentation of X is the subspace of X^0 defined by the equation $d^0x = d^1x$. In other words, it corresponds to the terminal example of a map to X from a constant cosimplicial space. Recall also that any (based) cosimplicial space Y admits a map $X \to Y$ from some cosimplicial space X satisfying 4.2 such that each $X^s \to Y^s$ is a weak homotopy equivalence; for example, X^s can be $Y^s \times \Delta^s / * \times \Delta^s$.

Define spaces X^s_T by

$$X^0_T = X^0,$$
$$X^s_T = X^s/(d^0X^{s-1} \cup \cdots \cup d^sX^{s-1}), \quad s > 0.$$

For each $s \geq 0$ there is then a sequence

$$* \leftarrow X^s_T \leftarrow X^s_N \leftarrow \cdots \leftarrow X^2_N \leftarrow X^1_N \leftarrow X^0_N \leftarrow *$$

exact in the same sense as 4.1. Apply the functor $N_\bullet(A \otimes -)$ to make an exact sequence of chain complexes

$$0 \leftarrow N_\bullet(A \otimes X^s_T) \leftarrow N^sN_\bullet(A \otimes X) \leftarrow \cdots \leftarrow N^0N_\bullet(A \otimes X) \leftarrow 0.$$

One sees that for the truncated double complex $N^{\leq s}N_\bullet(A \otimes X)$ the result of taking homology first horizontally and then vertically is $H_j(X^s_T;A)$ in bidegree $(-s,j)$ and zero in bidegrees $(-i,j)$ with $i \neq s$. It follows that there is a natural isomorphism

$$H_nT_s(A \otimes X) \approx H_{n+s}(X^s_T;A). \tag{4.3}$$

The tower $\{H_nT_s(A \otimes X)\}_{s \geq 0}$ can therefore be identified with a tower $\{H_{n+s}(X^s_T, A)\}_{s \geq 0}$. The maps in this tower are evidently the connecting homomorphisms associated with the cofibration sequences

$$X^{s-1}_T \to X^s_N \to X^s_T, \quad s \geq 1.$$

Furthermore, the usual comparison map from $H_n(\mathrm{Tot}_s X;A)$ to $H_nT_s(A \otimes X)$ corresponds to a map induced by a rather tautological map of spaces $S^s \wedge \mathrm{Tot}_s X \to X^s_T$ (see [3, 2.4]).

(Without (4.2) the isomorphism (4.3) would have to be replaced by a long exact sequence

$$\cdots \to H_n(aX;A) \to H_nT_s(A \otimes X) \to H_{n+s}(X^s_T;A) \to H_{n-1}(aX;A) \to \cdots,$$

where aX is the maximal augmentation.)

5. Reduction to a special case

Let S^t be the standard simplicial t-sphere, a based simplicial set having a unique nondegenerate simplex apart from the basepoint. The simplicial abelian group $A \otimes S^t$ has $N_t(A \otimes S^t) = A$ as its unique nontrivial normalized chain group. Let $K_t(A)$ be the underlying space of $A \otimes S^t$, an Eilenberg–MacLane space of type (A,t).

Following [3, p. 389], let $K_{s,t}(A)$ be the underlying cosimplicial space of the cosimplicial simplicial abelian group whose unique nontrivial normalized group is $N^sN_tK_{s,t}(A) = A$. In different notation $K_{s,t}(A)$ is $\mathrm{Map}(S^s, K_t(A))$, where $\mathrm{Map}(S^s, Y)$ is the based cosimplicial space that results from S^s by applying the contravariant functor "based maps to Y" from based sets to based spaces.

Lemma 5.1. *Let p be prime and $m \geq s > t \geq 0$, and suppose that in the* mod p *homology tower of $K_{s,t}(\mathbb{Z}/p)$ the map*

$$H_{t-s}T_m((\mathbb{Z}/p)\otimes K_{s,t}(\mathbb{Z}/p)) \to H_{t-s}T_s((\mathbb{Z}/p)\otimes K_{s,t}(\mathbb{Z}/p))$$

is zero. Then in the mod p *homology tower of any cosimplicial space X the map $H_{t-s}T_m((\mathbb{Z}/p)\otimes X) \to H_{t-s}T_s((\mathbb{Z}/p)\otimes X)$ is zero.*

Proof. Without loss of generality, the maximal augmentation of X is a point. Then by (4.3) the problem is to prove that a certain map in mod p homology

$$H_{m+t-s}X^m_T \to H_tX^s_T$$

is zero, or equivalently, that the dual map

$$H^{m+t-s}X^m_T \leftarrow H^tX^s_T \tag{5.1}$$

in mod p cohomology is zero.

The functor $X \to X^s_T$ from based cosimplicial spaces to based spaces has a right adjoint, namely, the functor $Y \to \mathrm{Map}(S^s, Y)$. Because the counit map $\mathrm{Map}(S^s, Y)^s_T \to Y$ of the adjoint functor pair is an isomorphism, the unit map $X \to \mathrm{Map}(S^s, X^s_T)$ induces an isomorphism $X^s_T \to \mathrm{Map}(S^s, X^s_T)^s_T$. Therefore, it is enough to prove that the natural map (5.1) is zero in the special case when $X = \mathrm{Map}(S^s, Y)$ for some space Y. By another naturality argument it is even enough to do so in the case when Y is some fibrant Eilenberg–MacLane space representing the functor $H^t(-;\mathbb{Z}/p)$, for example, $K_t(\mathbb{Z}/p)$. But $\mathrm{Map}(S^s, K_t(\mathbb{Z}/p))$ is $K_{s,t}(\mathbb{Z}/p)$. □

6. The special case

To complete the proof of the theorem it is enough to show that for $s > t \geq 0$ the mod p homology tower of $K_{s,t}(\mathbb{Z}/p)$ in degree $t-s$ is protrivial. In fact, it is pro-trivial in all degrees. Let us put this statement in some perspective.

A fibrant cosimplicial space X is said to be *strongly convergent* for the coefficient group A if the composed tower map

$$\{H_n(\mathrm{Tot}\,X;A)\}_{m\geq 0} \to \{H_n(\mathrm{Tot}_mX;A)\}_{m\geq 0} \xrightarrow{\Phi} \{H_nT_m(A\otimes X)\}_{m\geq 0}$$

is a pro-isomorphism for each integer n. It is said to be *pro-convergent* if the map Φ is a pro-isomorphism. The following is a very special case of Lemma 5.1 of [7].

Lemma 6.1 (Shipley [7]). *$K_{s,t}(A)$ is strongly convergent for* mod p *homology if A is a finite abelian p-group and $s \geq t$.*

Above the line $t - s = 0$ things are easier. Compare the following very special case of [1].

Lemma 6.2 (Anderson [1]). *$K_{s,t}(A)$ is strongly convergent for all coefficient rings and all A if $s < t$.*

Note that for $m \geq s$ the space $\mathrm{Tot}_m K_{s,t}(A)$ is independent of m and is the function space of simplicial maps from S^s to $K_t(A)$. This is an Eilenberg–MacLane space of type $(A, t-s)$ if $s \leq t$. It is a point if $s > t$, so that in that case Lemma 6.1 is indeed saying that in every degree the mod p homology tower of $K_{s,t}(A)$ is pro-trivial. In either case, strong convergence is the same as pro-convergence since $\mathrm{Tot}_m = \mathrm{Tot}$ for large m.

The case $s = 0$ of Lemma 6.2 is a triviality; constant cosimplicial spaces are utterly convergent.

The case $t = 0$ of Lemma 6.1 is more or less trivial, too: by Lemma 4.1 cosimplicial *sets* are strongly convergent for all coefficients.

The case $t = 1$ of Lemma 6.2 is an instance of a classical Eilenberg–Moore spectral sequence (EMSS) converging to the homology of the loop space of a simply connected space. There is a strategy due to Bott and Segal [2] for deducing convergence results like Anderson's from EMSS results. It will be summarized below.

Dwyer [4] extends EMSS convergence to some new cases, including the mod p homology of the loop space of any space whose fundamental group is a finite p-group, so his result covers the case $s = 1 = t$ of Lemma 6.1. Bousfield [3] uses the method of Bott and Segal to extend some of Dwyer's results to more general cosimplicial spaces. In particular, 3.2(ii) of [3] covers the case $s = t > 1$ of Lemma 6.1. Shipley pushes the method beyond degree zero, obtaining Lemma 6.1 in particular.

Extracting the proof of Lemma 6.1 from Shipley's more general arguments, one finds that, essentially, she deduces Lemma 6.1 from Lemma 6.2 by a downward induction on t, following the Bott–Segal strategy and using [4] as the crucial piece of input. For the present narrow purpose one can equally well use upward induction on s instead. In fact, Lemma 6.2 is proved on p. 389 of [3] by induction on s, and using Dwyer's result one can just keep going and get Lemma 6.1 (for any $t > 0$).

The Bott–Segal idea may be summarized as follows. Recall that a fiber square of spaces leads to a cosimplicial space (Rector's geometric cobar construction [6]), whose homology spectral sequence may be taken as the definition of the EMSS. Now suppose that

$$\begin{array}{ccc} M & \rightarrow & X \\ \downarrow & & \downarrow \\ Y & \rightarrow & B \end{array} \tag{6.1}$$

is a fiber square of fibrant cosimplicial spaces and fibrations, and suppose that, for homology with coefficients in some field, the cosimplicial spaces X, Y, and B are all pro-convergent, and suppose further that for every $m \geq 0$ and every large enough s the cobar constructions for the fiber squares

$$\begin{array}{ccc} M^m & \to & X^m \\ \downarrow & & \downarrow \\ Y^m & \to & B^m \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathrm{Tot}_s M & \to & \mathrm{Tot}_s X \\ \downarrow & & \downarrow \\ \mathrm{Tot}_s Y & \to & \mathrm{Tot}_s B \end{array}$$

are strongly convergent (=pro-convergent). Then M is also pro-convergent.

See [3, Proof of 8.4] for the argument, which uses the bicosimplicial space associated to (6.1) by the cobar construction. (Note, however, that there is no need to treat positive and negative degrees separately if one adopts the point of view of Section 4 above.)

In the cases that occur in the proof of Lemma 6.1, Y is a point and $0 \to M \to X \to B \to 0$ is an exact sequence of cosimplicial simplicial abelian groups with $M = K_{s,t}(\mathbb{Z}/p)$, $B = K_{s-1,t}(\mathbb{Z}/p)$, and X cosimplicially contractible. Furthermore, M, X, B, and Y all have $\mathrm{Tot} = \mathrm{Tot}_s$ for large s so that pro-convergence is the same as strong convergence.

7. Assorted examples

(i) *$K_{s,1}(\mathbb{Z})$ and rational coefficients*: Recall that $K_{s,1}(\mathbb{Z})$ is the cosimplicial space $\mathrm{Map}(S^s, K_1(\mathbb{Z}))$. In general, for $X = \mathrm{Map}(Y, K_1(\mathbb{Z}))$ where Y is a based finite simplicial set, there is the following analysis of E^2. For the based finite set Y_i, the space $X^i = \mathrm{Map}(Y_i, K_1(\mathbb{Z}))$ is a product of copies of $K_1(\mathbb{Z}) \approx S^1$. Therefore, $H_j(X^i; A) = A \otimes H_j(X^i)$ and $H_j(X^i)$ is the group of Σ_j-antisymmetric elements in $H_1(X^i)^{\otimes j} = \mathbb{Z} \otimes Y_i^{(j)}$, where $Y^{(j)}$ means jth smash power of Y. The isomorphism is natural in Y_i; this leads to an identification of the E^1 differential and so shows that $E^2_{-i,j}$ is the ith cohomology of the antisymmetric part of $C^*(Y^{(j)}; A)$, the (reduced, normalized) cochain complex associated to the based simplicial set $Y^{(j)}$. If $A = \mathbb{Q}$ then this is the same as the antisymmetric part of the rational cohomology of $Y^{(j)}$. For example, when $Y = S^s$ and s is even then the rational E^2 has just $\mathbb{Q}$ in bidegree $(-s, 1)$. When $Y = S^s$ and s is odd it has $\mathbb{Q}$ in each bidegree $(-sj, j)$.

(ii) *$K_{s,t}(\mathbb{Z})$ and rational coefficients*: The argument above applies to $\mathrm{Map}(Y, K_t(\mathbb{Z}))$ for any odd t, as long as $A = \mathbb{Q}$. It gives $E^2_{-i,j}$ = antisymmetric part of ith cohomology of $Y^{(k)}$ when $j = tk$ and zero when t does not divide j. When t is even the same conclusion holds except that "antisymmetric" is replaced by "symmetric". In fact, for any $s \geq 0$ and $t > 1$ the (unreduced) rational E^2 of the cosimplicial space $K_{s,t}(\mathbb{Z})$ is a polynomial or exterior algebra (according as $t - s$ is even or odd) on a single generator in bidegree $(-s, t)$. (The spectral sequence has a multiplicative structure because $K_{s,t}(\mathbb{Z})$ is a cosimplicial H-space.) In all these cases $E^2 = E^\infty$ necessarily.

(iii) *$K_{s,1}(\mathbb{Z})$ and integral coefficients*: Returning to the argument in (i) but taking $A = \mathbb{Z}$, note that an antisymmetric element of $C^i(Y^{(j)})$ must vanish on the "fat diagonal" D in $Y^{(j)}$, because it must take the value zero on any simplex of $Y^{(j)}$ that is fixed by a transposition. It follows that $E^2_{-i,j}$ can be identified with the cohomology of the antisymmetric compactly supported singular cochains of the topological space $|Y^{(j)}| - |D|$. Identify the orbit space for the free Σ_j-action on $|Y^{(j)}| - |D|$ with the space $F(j, |Y| - *)$ of unordered j-tuples of distinct points in $|Y| - *$. Then $E^2_{-i,j}$ is the compactly supported cohomology group $H^i_c(F(j, |Y| - *); \mathbb{Z}^\varepsilon)$, where $\mathbb{Z}^\varepsilon$ is the twisted coefficient system associated to the sign action of Σ_j on $\mathbb{Z}$. If $|Y|$ is an s-dimensional sphere then this is isomorphic by Poincare duality to $H_{si-j}(F(j, R^s); \mathbb{Z}^\varepsilon)$ if s is even and $H_{si-j}(F(j, R^s); \mathbb{Z})$ if s is odd.

(iv) $K_{2,1}(\mathbb{Z}/2)$ *and* mod 2 *coefficients*: For $X = \mathrm{Map}(Y, K_1(\mathbb{Z}/2))$ and $A = \mathbb{Z}/2$ the same method shows that $E^2_{-i,j}$ is the ith cohomology of the symmetric part of the cochain complex $C^*(Y^{(j)}; \mathbb{Z}/2)$; in other words, $H^i(Y^{(j)}/\Sigma_j; \mathbb{Z}/2)$. If Y is S^2 then the symmetric power $Y^{(j)}/\Sigma_j$ is a $2j$-sphere and we find that E^2 has a $\mathbb{Z}/2$ in each bidegree $(-2j, j)$. In fact, E^2 is a polynomial algebra on one generator, and one could presumably see the theorem in action here by directly checking that the differential d^2 is nontrivial on the generator.

References

[1] D.W. Anderson, A generalization of the Eilenberg–Moore spectral sequence, Bull. AMS 78 (1972) 784–786.
[2] R. Bott and G. Segal, The cohomology of the vector fields on a manifold, Topology 16 (1977) 285–298.
[3] A.K. Bousfield, On the homology spectral sequence of a cosimplicial space, Amer. J. Math. 109 (1987) 361–394.
[4] W.G. Dwyer, Strong convergence of the Eilenberg–Moore spectral sequence, Topology 13 (1974) 255–265.
[5] W.G. Dwyer, Higher divided squares in second-quadrant spectral sequences, Trans. AMS 260 (1975) 607–617.
[6] D.L. Rector, Steenrod operations in the Eilenberg–Moore spectral sequence, Comm. Math. Helv. 45 (1970) 540–552.
[7] B.S. Shipley, Convergence of the homology spectral sequence of a cosimplicial space, Amer. J. Math. 118 (1996) 179–207.
[8] C. Weibel, An Introduction to Homological Algebra (Cambridge Univ. Press, Cambridge, 1994).