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A remarkable property of the (co) syzygy modules of the residue field of a nonregular local ring

Alex Martsinkovsky*

Department of Mathematics, Northeastern University, Boston, MA 02115, USA

In memory of Maurice Auslander

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Abstract

Let R be a commutative noetherian local ring with residue field k. We introduce two new invariants of R-modules and compute them for the module k. As a consequence, we show, when R is nonregular, that no direct sum of the syzygy modules of k surjects onto a nonzero module of finite projective dimension. A dual statement is also true: no direct sum of the cosyzygy modules of k (in an injective resolution) contains a nonzero module of finite injective dimension with nonzero scale.

1. Introduction

Let $(R, \underline{m}, k)$ be a commutative noetherian local ring. In the early days of homological study of local rings, Kaplansky conjectured that if some power of the maximal ideal is nonzero and of finite projective dimension then R is regular.

This conjecture was proved by G. Levin in his thesis. In fact his result was even stronger:

Theorem 1 (See [4]). *If M is a finitely generated R-module such that $\underline{m}M \neq 0$ and p.d. $\underline{m}M < \infty$, then R is regular.*

More recently, Dutta, in his work on the homological conjectures (see [2]), proved the following

* E-mail: alexmart@nuhub.bitnet

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Theorem 2. *Suppose that, for some n, the nth syzygy module of k has a free summand. Then R is regular.*

The purpose of this paper is to show that the above theorems are special cases of a much more general result. Moreover, this general result has a dual companion. The proofs presented here are very simple, but they rely on two new invariants also introduced here.

2. Main theorem and proof

Let $(R, \underline{m}, k)$ be a commutative noetherian local ring, and M a finitely generated R-module. Let $P_M \to M$ and $P_k \to k$ be minimal projective resolutions of M and k, respectively.

Definition 3. Let $V(M)$ denote the subspace of the vector space $Hom(M, k)$ consisting of all maps $f: M \to k$ that satisfy the following condition: there exists a lifting of f that is homotopic to a bounded chain map $P_M \to P_k$.

It is easy to see that the space $V(M)$ depends only on M and not on the chosen minimal resolutions of M and k.

Definition 4. Let $\xi^i(M) := \dim_k V(\Omega^i(M))$, where $\Omega^i(M)$ is the ith syzygy module of M.

We now record the first straightforward properties of the new invariants.

Lemma 5. (1) $\xi^i(M) = \xi(\Omega^i M)$ *(henceforth* $\xi(-)$ *will stand for* $\xi^0(-)$*).*
(2) $0 \leq \xi^i(M) \leq \beta_i(M)$ *for all i, where* $\beta_i(M)$ *is the ith betti number of M.*
(3) $\xi^i(M \coprod N) = \xi^i(M) + \xi^i(N)$ *for all i.*
(4) *If p.d.* $M < \infty$ *then* $\xi^i(M) = \beta_i(M)$ *for all i.*
(5) *If* $M \to N$ *is a surjection then* $\xi(M) \geq \xi(N)$.

The proof is obvious (in the last part one should use the left-exactness of $Hom(-, k)$).
We now state the main result of this paper. Over Gorenstein rings this result, stated in terms of Auslander's $\delta -$ invariant (related to Cohen-Macaulay approximations), was originally proved by Auslander [1].

Theorem 6. *If the ring R is nonregular, then* $\xi^i(k) = 0$ *for all i.*

Proof. Suppose $\xi^i(k) \neq 0$ for some i. Then there are a map $\theta: \Omega^i k \to k$ and a number N such that in a certain lifting $\theta_.$ of θ we have that $\theta_n = 0$ for all $n \geq N$. We shall now view θ as a degree i element of the Yoneda algebra $Ext^*(k, k)$ of R. Then for any

$\phi \in Ext^*(k, k)$ with $\deg \phi \geq N$ we have that the Yoneda product $\phi\theta$ is zero. Thus θ is annihilated by all elements of sufficiently high degree.

On the other hand, the Yoneda algebra $Ext^*(k, k)$ of R is a universal enveloping algebra UL of a uniquely defined graded Lie algebra L [7]. It was shown in Lemma 3.1(i) of [3] that if UL is infinite-dimensional then no element of UL can be annihilated by the ideal of elements of positive degrees. The same argument shows that no element of UL can be annihilated by the ideal $UL_{\geq N}$ consisting of elements of degree at least N. It goes as follows. If $(UL_{\geq N})\theta = 0$ then, by the graded Poincaré–Birkhoff–Witt theorem, the algebra L must be concentrated in odd degrees, thus making UL into an exterior algebra. If UL were infinite-dimensional, then, by the Poincaré–Birkhoff–Witt theorem again, no element of UL could be annihilated by $UL_{\geq N}$. Thus we conclude that UL must be finite-dimensional. Since R was assumed to be nonregular, we have that UL is infinite-dimensional. The obtained contradiction finishes the proof of the theorem. □

As an immediate application, we have the following

Proposition 7. *Let* $f: \coprod_{i \in I} (\Omega^i k)^{j_i} \to L$ *be a surjective homomorphism of finitely generated R-modules, where* $L \neq 0$ *and p.d.* $L < \infty$. *Then R is regular.*

Proof. Suppose R is nonregular. On one hand, by property (4) above, $\xi(L) = \beta_0(L) \neq 0$. On the other hand, by properties (5), (1) and (2), and by the preceding theorem, we have that $\xi(L) = 0$. Contradiction. □

Now we can deduce the aforementioned results of Levin and Dutta. The latter is the particular case of the preceding proposition with $j_i = 1$ when $i = n$ and $j_i = 0$ otherwise, and f being the canonical projection from the nth syzygy module onto its free direct summand. The former result corresponds to the case $j_i = \beta_0(M)$ when $i = 1$ and $j_i = 0$ otherwise, and $f = \underline{m}p$, where p is the projective cover $R^{\beta_0(M)} \to M$. To see that just multiply the last map by $\underline{m}$.

Surprisingly, with very little effort, the last proposition can be substantially strengthened.

Proposition 8. *Let* $f: \coprod_{i \in I} (\Omega^i k)^{j_i} \to L$ *be a homomorphism of finitely generated R-modules, where* $L \neq 0$ *and p.d.* $L < \infty$. *If* $f \otimes R/m \neq 0$ *then R is regular.*

This result is an immediate consequence of the following

Lemma 9. *Let* $(R, \underline{m}, k)$ *be a commutative noetherian local ring, M and N finitely generated R-modules such that* $\xi(M) = 0$ *and p.d.* $N < \infty$, *and* $f: M \to N$ *a homomorphism of R-modules. Then* $f \otimes R/m = 0$.

Proof. Suppose $f \otimes R/m \neq 0$. Then there exists a surjective composition $M \xrightarrow{f} N \to N/\underline{m}N \to k$, where the map in the middle is the natural surjection. But then, contrary to the assumption, we would have that $\xi(M) \neq 0$ since p.d. $N < \infty$. □

3. Dual results

Replacing now projective resolutions with injective ones, reversing the directions of arrows, and replacing the syzygy modules in projective resolutions by cosyzygy modules in injective resolutions, we arrive at another series of invariants $\chi^i(M)$. They enjoy properties completely analogous to those of $\xi^i(M)$. However, the betti numbers should be replaced by the Bass numbers, finite projective dimension should be replaced by finite injective dimension, and surjective homomorphisms in property (5) of Lemma 5 should be replaced by injections (the new property just says that $\chi = \chi^0$ does not decrease under injections). In addition, to accommodate infinitely generated cosyzygy modules, we want to allow χ to take on the value ∞. We can now state a result dual to Theorem 6.

Theorem 10. *If the ring R is nonregular then* $\chi^i(k) = 0$ *for all i.*

The proof is completely analogous to that of the prototype.

As an application we have the following

Proposition 11. *Let* $g: L \to \coprod_{i \in I} (\Omega^{-i} k)^{j_i}$ *be an injective homomorphism of R-modules, where* $Soc\, L \neq 0$ *and i.d.* $L < \infty$. *Then R is regular. (Here* Ω^{-i} *stands for the ith cosyzygy module and* $Soc\, L$ *denotes the socle of L, i.e., the maximal submodule of R annihilated by the maximal ideal* $\underline{m}$.*)*

A partial strengthening of the last result is given by the following

Proposition 12. *Let* $g: L \to \coprod_{i \in I} (\Omega^{-i} k)^{j_i}$ *be a homomorphism of R-modules, where* $Soc L \neq 0$ *and i.d.* $L < \infty$. *If* $g(Soc L) \neq 0$ *then R is regular.*

This result is an immediate consequence of

Lemma 13. *Let* $(R, \underline{m}, k)$ *be a commutative noetherian local ring, M and N R-modules such that* $\chi(N) = 0$ *and i.d.* $M < \infty$, *and* $f: M \to N$ *a homomorphism of R-modules. Then* $f(Soc M) = 0$.

Proof. Suppose $f(Soc M) = 0$. Then there exists a composite monomorphism $k \to Soc M \to M \xrightarrow{f} N$, where the middle map is the canonical injection. But then we would have, contrary to the assumption, that $\chi(N) \neq 0$ since i.d. $M < \infty$. □

Remark. The invariants introduced in this paper are of cohomological nature. This connection along with other applications will be explained in the forthcoming papers [5] and [6].

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