

## A SHORT ELEMENTARY PROOF OF GROTHENDIECK'S THEOREM ON ALGEBRAIC VECTORBUNDLES OVER THE PROJECTIVE LINE\*

Michiel HAZEWINKEL

*Erasmus University, Rotterdam, The Netherlands*

Clyde F. MARTIN\*\*

*Department of Mathematics, Case Institute of Technology, Case Western Reserve University,  
Cleveland, OH 44106, USA*

Communicated by S. Eilenberg

Received 16 June 1981

Let  $E$  be an algebraic (or holomorphic) vectorbundle over the Riemann sphere  $\mathbb{P}^1(\mathbb{C})$ . Then Grothendieck proved that  $E$  splits into a sum of line bundles  $E = \bigoplus L_i$  and that the isomorphism classes of the  $L_i$  are (up to order) uniquely determined by  $E$ . The  $L_i$  in turn are classified by an integer (their Chern numbers) so that  $m$ -dimensional vectorbundles over  $\mathbb{P}^1(\mathbb{C})$  are classified by an  $m$ -tuple of integers

$$\kappa(E) = (\kappa_1(E), \dots, \kappa_m(E)), \quad \kappa_1(E) \geq \kappa_2(E) \geq \dots \geq \kappa_m(E), \quad \kappa_i(E) \in \mathbb{Z}.$$

In this short note we present a completely elementary proof of these facts which, as it turns out, works over any field  $k$ .

### 1. Introduction

Let  $E$  be a holomorphic (or algebraic) vectorbundle over the Riemann sphere  $\mathbb{P}^1(\mathbb{C})$ . (By [2] holomorphic and algebraic vectorbundles over  $\mathbb{P}^1(\mathbb{C})$  amount to the same thing). In [1] Grothendieck proved that  $E$  splits into a sum of line bundles  $E = \bigoplus L_i$  and that the isomorphism classes of the  $L_i$  are (up to order) uniquely determined by  $E$ . The line bundles  $L_i$  in turn are classified by an integer (their first Chern number) so that  $m$ -dimensional vectorbundles over  $\mathbb{P}^1(\mathbb{C})$  are classified by an  $m$ -tuple of integers

$$\kappa(E) = (\kappa_1(E), \dots, \kappa_m(E)), \quad \kappa_1(E) \geq \dots \geq \kappa_m(E), \quad \kappa_i \in \mathbb{Z}.$$

\* This work was done in part while the first author was visiting Case Institute of Technology.

\*\* Supported in part by NASA Grant #2384, ONR Contract #N00014-80C-0199 and DOE Contract #DE-AC01-80RA5256.

Below we give a completely elementary proof of these facts, which, as it turns out, works over any field  $k$ . Of course ‘completely elementary’ means that such concepts as ‘degree of a line bundle’ or ‘first Chern number’ or ‘cohomology’ or ‘intersection number’ are not needed or mentioned below. All we use is some linear algebra (or matrix manipulation).

**2. Vectorbundles over  $\mathbb{P}_k^1$**

Let  $k$  be any field. The projective line  $\mathbb{P}_k^1$  over  $k$  can be obtained as follows. Let  $U_1 = \text{Spec}(k[s])$ ,  $U_2 = \text{Spec}(k[t])$ ,  $U_{12} = \text{Spec}(k[s, s^{-1}]) = U_1 \setminus \{0\}$ ,  $U_{21} = \text{Spec}(k[t, t^{-1}]) = U_2 \setminus \{0\}$ . Now glue  $U_1$  and  $U_2$  together by identifying  $U_{12}$  and  $U_{21}$  by means of the isomorphism

$$k[s, s^{-1}] \xrightarrow{\sim} k[t, t^{-1}], \quad s \mapsto t^{-1}.$$

Now let  $E$  be an  $m$ -dimensional vectorbundle over  $\mathbb{P}_k^1$  defined over  $k_1$  and let  $\mathbb{A}^m = \text{Spec}(k[X_1, \dots, X_m])$ . Then  $E|_{U_i}$ ,  $i = 1, 2$ , is trivial, i.e.  $E|_{U_i} \cong U_i \times \mathbb{A}^m$ , so that  $E$  can be viewed (up to isomorphism) as obtained by glueing together  $U_1 \times \mathbb{A}^m$  and  $U_2 \times \mathbb{A}^m$  by identifying  $U_1 \setminus \{0\} \times \mathbb{A}^m$  and  $U_2 \setminus \{0\} \times \mathbb{A}^m$  by means of an isomorphism of the form

$$(s, v) \mapsto (s^{-1}, A(s, s^{-1})v) \tag{2.1}$$

where  $A(s, s^{-1})$  is a matrix with coefficients in  $k[s, s^{-1}]$  which has nonzero determinant for all  $s \neq 0, s^{-1} \neq 0$ . This last fact means that

$$\det(A(s, s^{-1})) = s^n, \quad n \in \mathbb{Z}. \tag{2.2}$$

A vectorbundle automorphism of  $U_1 \times \mathbb{A}^m$  is necessarily of the form  $(s, v) \mapsto (s, U(s)v)$  where  $U(s)$  is a matrix with coefficients in  $k[s]$  with  $\det U(s) \in k \setminus \{0\}$  and similarly an automorphism of  $U_2 \times \mathbb{A}^m$  is given by a matrix  $V(s^{-1})$  with coefficients in  $k[s^{-1}]$  with determinant in  $k \setminus \{0\}$ . Different trivializations of  $E|_{U_i}$  differ by an automorphism of  $U_i \times \mathbb{A}^m$ . It follows that

**Proposition 2.3.** *Isomorphism classes of  $m$ -dimensional algebraic vectorbundles over  $\mathbb{P}_k^1$  correspond bijectively to equivalence classes of polynomial  $m \times m$  matrices  $A(s, s^{-1})$  over  $k[s, s^{-1}]$  such that  $\det A(s, s^{-1}) = s^n, n \in \mathbb{Z}$  where the equivalence relation is the following:  $A(s, s^{-1}) \sim A'(s, s^{-1})$  iff there exist polynomial invertible  $m \times m$  matrices  $U(s), V(s^{-1})$  over  $k[s]$  and  $k[s^{-1}]$  respectively with constant determinant such that*

$$A'(s, s^{-1}) = V(s^{-1})A(s, s^{-1})U(s). \tag{2.4}$$

### 3. A canonical form for matrices over $k[s, s^{-1}]$

Now let us study canonical forms for  $m \times m$  matrices over  $k[s, s^{-1}]$  under the equivalence relation defined in Proposition 2.3 above. The result is

**Proposition 3.1.** *Let  $A(s, s^{-1})$  be an  $m \times m$  matrix over  $k[s, s^{-1}]$  with determinant equal to  $s^n$  for some  $n \in \mathbb{Z}$ . Then there exist polynomial  $m \times m$  matrices  $V(s^{-1})$  and  $U(s)$  with constant nonzero determinant such that*

$$V(s^{-1})A(s, s^{-1})U(s) = \begin{pmatrix} s^{r_1} & & & 0 \\ & s^{r_2} & & \\ & & \ddots & \\ 0 & & & s^{r_m} \end{pmatrix} \tag{3.2}$$

with  $r_1 \geq r_2 \geq \dots \geq r_m$ ,  $r_i \in \mathbb{Z}$ . The  $r_i$  are uniquely determined by  $A(s, s^{-1})$ . Moreover if  $A(s, s^{-1})$  is polynomial in  $s$  then  $r_i \geq 0$ ,  $i = 1, \dots, m$ , and if  $A(s, s^{-1})$  is polynomial in  $s^{-1}$  then  $r_i \leq 0$ ,  $i = 1, \dots, m$ .

**Proof.** Let's prove uniqueness first. Write  $D(r_1, \dots, r_m)$  for the matrix on the right in (3.2). Suppose there were two such matrices equivalent to  $A(s, s^{-1})$ . Then there would be polynomial matrices with constant nonzero determinant  $U(s), V(s^{-1})$  such that

$$V(s^{-1})D(r_1, \dots, r_m) = D(r'_1, \dots, r'_m)U(s).$$

If  $A$  is a matrix let

$$A_{j_1, \dots, j_k}^{i_1, \dots, i_k}$$

denote the minor of  $A$  obtained by taking the determinant of the submatrix of  $A$  obtained by removing all rows with index in  $\{1, \dots, m\} \setminus \{i_1, \dots, i_k\}$  and all columns with index in  $\{1, \dots, m\} \setminus \{j_1, \dots, j_k\}$ . Then of course

$$(AB)_{j_1, \dots, j_k}^{i_1, \dots, i_k} = \sum_{r_1 < \dots < r_k} A_{r_1, \dots, r_k}^{i_1, \dots, i_k} B_{j_1, \dots, j_k}^{r_1, \dots, r_k}.$$

Using this on the equality  $V(s^{-1})D(r_1, \dots, r_m) = D(r'_1, \dots, r'_m)U(s)$  one finds that

$$V_{i_1, \dots, i_k}^{1, 2, \dots, k}(s^{-1})s^{r_{i_1} + \dots + r_{i_k}} = s^{r'_1 + \dots + r'_k} U_{i_1, \dots, i_k}^{1, 2, \dots, k}(s) \tag{3.3}$$

for all  $i_1 < \dots < i_k$ . Now for some  $i_1, \dots, i_k$ ,

$$U_{i_1, \dots, i_k}^{1, 2, \dots, k}(s) \neq 0.$$

Hence  $r'_1 + \dots + r'_k \leq r_{i_1} + \dots + r_{i_k}$  for some  $i_1 < \dots < i_k$ , and hence certainly  $r'_1 + \dots + r'_k \leq r_1 + \dots + r_k$  for all  $k$ . Multiplying with  $V(s^{-1})^{-1}$  on the left and  $U(s)^{-1}$  on the right in  $V(s^{-1})D(r_1, \dots, r_m) = D(r'_1, \dots, r'_m)U(s)$  and repeating the argument gives  $r_1 + \dots + r_k \leq r'_1 + \dots + r'_k$  for all  $k$  and hence  $r_i = r'_i$ ,  $i = 1, \dots, m$ .

It remains to prove existence. First multiply  $A(s, s^{-1})$  with a suitable power  $s^n$ ,  $n \in \mathbb{N} \cup \{0\}$  to obtain a polynomial matrix  $B(s)$ . Then by post multiplication with a suitable  $U(s)$  (column operations) we can find a  $B'(s)$  with  $b'_{11} \neq 0$  and  $b'_{1i} = 0$ ,  $i = 2, \dots, m$  ( $b'_{11}$  is the greatest common divisor of  $b_{11}, \dots, b_{1m}$ ). Of course  $b'_{11} = s^{k_1}$  for some  $k_1 \in \mathbb{N} \cup \{0\}$  because  $\det B(s)$  is a power of  $s$ . Let  $B_2$  be the lower-right  $(m-1) \times (m-1)$  submatrix of  $B$ . By induction we can assume that the proposition holds for  $(m-1) \times (m-1)$  matrices. (The case  $m=1$  is trivial). So there are  $U_2(s), V_2(s^{-1})$  such that  $V_2(s^{-1})B_2U_2(s)$  is of the form of the right hand side of (3.2). Then

$$C(s) = \begin{pmatrix} 1 & 0 \\ 0 & V_2 \end{pmatrix} B \begin{pmatrix} 1 & 0 \\ 0 & U_2 \end{pmatrix} = \begin{pmatrix} s^{k_1} & 0 & \dots & 0 \\ c_2 & s^{k_2} & & 0 \\ \vdots & & \ddots & \\ \vdots & 0 & & \\ c_m & & & s^{k_m} \end{pmatrix} \quad (3.4)$$

for certain  $k_1, k_2, \dots, k_m \in \mathbb{N} \cup \{0\}$  (same  $k_1$  as before) and  $c_i \in k[s, s^{-1}]$ ,  $i = 2, \dots, m$ . Subtracting suitable  $k[s^{-1}]$  multiples of the first row from rows  $2, \dots, m$  (which is premultiplication with a  $V(s^{-1})$ ) we can moreover see to it that  $c_i \in k[s]$ .

Now consider all polynomial matrices of the form (3.4) which are equivalent to  $B(s)$ . Choose one for which  $k_1$  is maximal. Such a one exist because  $k_1 \leq \text{degree}(\det B(s))$  because  $k_2, \dots, k_m \geq 0$ . We claim that then  $k_1 \geq k_i$ ,  $i = 2, \dots, m$ . Indeed suppose that  $k_1 < k_i$ . Subtracting a suitable  $k[s^{-1}]$  multiple of the first row from the  $i$ -th row we find a matrix (3.4) with  $c_i = s^{k_1+1}c'_i(s)$ . Now interchange the first and the  $i$ -th row to find a polynomial matrix  $B'(s)$  such that the greatest common divisor of its first row elements is  $s^{k'_1}$  with  $k'_1 \geq k_1 + 1$ . Now apply to  $B'(s)$  the same procedure as above to  $B(s)$ . This would give a  $C'(s)$  of the form (3.4) with  $k'_1 > k_1$ , a contradiction. We can therefore assume that in (3.4)  $k_1 \geq k_i$ ,  $c_i \in k[s]$ ,  $i = 2, \dots, m$ . Subtracting suitable  $k[s]$ -multiples of the 2-nd,  $\dots$ ,  $m$ -th columns from the first one we find a matrix (3.4) with  $\text{degree}(c_i) \leq k_i$ . But then  $\text{deg}(c_i) < k_1$  so that a suitable  $k[s^{-1}]$  multiple of  $s^{k_1}$  is equal to  $c_i$  so that a further premultiplication with a  $V(s^{-1})$  gives us a matrix (3.4) with  $c_2 = \dots = c_m = 0$ . This proves the first half of the last part of the statement of the proposition and shows that there are  $k_1, \dots, k_n \in \mathbb{N} \cup \{0\}$ ,  $k_1 \geq \dots \geq k_m$  (by permuting columns and rows if necessary) and  $U(s), V(s^{-1})$  of constant nonzero determinant such that

$$V(s^{-1})s^n A(s, s^{-1})U(s) = V(s^{-1})B(s)U(s) = D(k_1, \dots, k_m).$$

Multiplying with  $s^{-n}$  gives  $V(s^{-1})A(s, s^{-1})U(s) = D(r_1, \dots, r_m)$  with  $r_i = k_i - n$ . The second half of the last statement of the proposition is proved as the first half starting with a matrix  $B(s^{-1})$  and using row (resp. column) operations everywhere where we used column (resp. row) operations above. This concludes the proof of Proposition 3.1.

#### 4. Classification of vectorbundles over $\mathbb{P}_k^1$

Let  $O(n)$ ,  $n \in \mathbb{Z}$  be the line bundle over  $\mathbb{P}_k^1$  defined by the glueing matrix  $A(s, s^{-1}) = s^{-n}$ . Obviously then the bundle defined by the glueing matrix  $A(s, s^{-1}) = D(r_1, \dots, r_m)$  is equal to the direct sum  $O(-r_1) \oplus \dots \oplus O(-r_m)$ .

**Theorem 4.1.** *Let  $E$  be an algebraic  $m$ -dimensional vectorbundle over  $\mathbb{P}_k^1$  which is defined over  $k$ . Then  $E$  is isomorphic over  $k$  to a direct sum of line bundles*

$$E \simeq O(\kappa_1) \oplus \dots \oplus O(\kappa_m), \quad \kappa_1 \geq \dots \geq \kappa_m, \quad \kappa_i \in \mathbb{Z}, \quad i = 1, \dots, m,$$

and the  $\kappa_i$  are uniquely determined by the isomorphism class of  $E$ .

**Remarks 4.2.** It is perhaps worth remarking that  $E$  is positive (meaning that all the  $\kappa_i(E) \geq 0$ ) if the glueing matrix  $A(s, s^{-1})$  is polynomial in  $s^{-1}$  and that  $E$  is negative (i.e.  $\kappa_i(E) \leq 0$  all  $i$ ) if  $A(s, s^{-1})$  is polynomial in  $s$ . This follows from the last statement of Proposition 3.1. Also  $E$  contains a summand  $O(n)$  with  $n > 0$  if  $\deg(\det A(S, s^{-1})) < 0$ . Finally it follows that vectorbundles over  $\mathbb{P}_k^1$  have no forms, i.e. if  $E$  and  $E'$  are two vectorbundles over  $k$  which become isomorphic over the algebraic closure  $\bar{k}$  of  $k$  then  $E$  and  $E'$  are also isomorphic over  $k$ . This can of course also be seen by other, more sophisticated, means (e.g. Galois cohomology).

#### References

- [1] A. Grothendieck, Sur la classification des fibrés holomorphes sur la sphère de Riemann, Amer. J. Math. 79 (1957) 121–138.
- [2] J.-P. Serre, Géometrie algébrique et géometrie analytique, Annales Inst. Fourier 6 (1956) 1–42.