Journal of Pure and Applied Algebra 26 (1982) 155–157
North-Holland Publishing Company

A SIMPLE PROOF THAT THE UNSTABLE (CO-)HOMOLOGY OF THE BROWN–PETERSON SPECTRUM IS TORSION FREE

Ken CHAN
Bell Laboratories, Lincroft, NJ 07738, USA

Communicated by A. Heller
Received 14 September 1981

Let p be a prime, BP be the Ω-spectrum for the Brown–Peterson cohomology theory for this prime, $\mathbf{BP}_k$ its k-th space, $\mathbf{F}_p$ the integers mod p, and $H_*(-)$ be homology mod p. $\mathbf{BP}_k$ is of course an H-space, so let the superscript (′) denote the component of the identity. Call a Hopf algebra bipolynomial if both it and its dual are polynomial algebras. Its structure is uniquely determined by its generators [2].

Theorem 1. *$H_*\mathbf{BP}'_{\text{even}}$ is a bipolynomial Hopf algebra and $H_*\mathbf{BP}_{\text{odd}}$ is an exterior algebra.*

Remark. W.S. Wilson proved Theorem 1 in [4]. It is also a corollary of one of the proofs in [3]. It follows immediately that $H_*(\mathbf{BP}_k;\mathbf{Z})$ has no torsion. We present here a short proof of Theorem 1.

Proof. Note the following facts: stably, $\pi_*\mathrm{BP}\cong\mathbf{Z}_{(p)}[v_1,v_2,\ldots]$ and $H_*\mathrm{BP}\cong\mathbf{F}_p[b_1,b_2,\ldots]$, $\deg(v_i)=\deg(b_i)=2(p^i-1)$, see [1]. $\pi_0\mathbf{BP}_k=\pi_{-k}\mathrm{BP}$ and from the H-space structure of $\mathbf{BP}_k$, we have isomorphisms of Hopf algebras $H_0\mathbf{BP}_k\cong\mathbf{F}_p[\pi_0\mathbf{BP}_k]$ and $H_*\mathbf{BP}_k\cong H_0\mathbf{BP}_k\otimes H_*\mathbf{BP}'_k$.

The proof employs a technique in [3], namely we compute $H_*\mathbf{BP}_*$ stem by stem using the bar spectral sequence

$$E^2\cong\mathrm{Tor}^{H_*\mathbf{BP}_k}(\mathbf{F}_p,\mathbf{F}_p)\ \Rightarrow\ \text{associated grading of } H_*\mathbf{BP}'_{k+1}.$$

It is done by induction on stems, and on each stem, by induction on k. We show that the number of Hopf algebra generators along each stem remains constant, by comparing $\pi_*\mathrm{BP}$ with $H_*\mathrm{BP}$. Note that $\dim_{\mathbf{Z}_{(p)}}\pi_k\mathrm{BP}=\dim_{\mathbf{F}_p}H_k\mathrm{BP}$. Call this integer $\#(k)$. Note that a knowledge of $H_*\mathbf{BP}_k$ for $*\le q$ enables us to compute $H_*\mathbf{BP}'_{k+1}$ for $*\le q+1$.

All the negative stems are zero by connectivity. For $n\ge 0$, the n-stem starts with $H_0\mathbf{BP}_{-n}$, which is non-zero when $n=2(p-1)m$ for some m, and zero otherwise. To start the induction on each stem, we have Tor of $\mathbf{F}_p[\mathbf{Z}_{(p)}]$ which is an exterior

0022-4049/82/0000–0000/$02.75

algebra. Thus for $n=2(p-1)m$, $\mathrm{QH}_1\mathbf{BP}'_{1-n}$ has a basis with $\#(n)$ generators, a fact that can also be deduced from the Hurewicz Isomorphism Theorem. In addition, we also know that Tor of $H_0\mathbf{BP}_{-n}$ will not have other contributions to $\mathrm{QH}_*\mathbf{BP}'_{1-n}$ for $*>1$. With no extension problems to solve yet, we have $\mathrm{QH}_1\mathbf{BP}'_{1-n}=H_1\mathbf{BP}'_{1-n}$. Let $\#(n,k)=\dim_{\mathbf{F}_p}\mathrm{QH}_{n+k}\mathbf{BP}'_k$, then we have $\#(n)=\#(n,1-n)$.

Note that in the n-stem, $H_{n+k}\mathbf{BP}'_k$ is stable for $k>n$. In the stable range there are no products. Thus $H_{n+k}\mathbf{BP}_k=\mathrm{QH}_{n+k}\mathbf{BP}_k=H_n\mathrm{BP}$ for $k>n$ and therefore, $\#(n)=\#(n,k)$ for $k>n$.

The induction on stems start with the zero stem. We compute $H_1\mathbf{BP}'_1$, which is stable and we are finished with the zero stem. Assume that Theorem 1 is true for the first $n-1$ stems, and true along the n-stem up to $\mathbf{BP}_{k-1}$, i.e. true for $H_p\mathbf{BP}_q$ for $q\geq p-(n-1)$ and ($q=p-n$ and $q\leq k-1$). Note that this implies that all the generators we have computed so far lie in the $2(p-1)m$-stems for some m. We use the bar spectral sequence to compute $\mathrm{QH}_{n+k}\mathbf{BP}_k$ and we solve all the extension problems that arise at $H_{n+k}\mathbf{BP}_k$.

If $n\neq 2(p-1)m$, then $\mathrm{QH}_{n+k-1}\mathbf{BP}_{k-1}=0$ and the spectral sequence computation just repeats the previous computations in the lower stems with one higher degree of nothing added. Thus no generators arise in $H_{n+k}\mathbf{BP}_k$ and this stem remains barren of generators.

If $n=2(p-1)m$ and k is odd, then by hypothesis, $H_*\mathbf{BP}'_{k-1}$ is a polynomial algebra for $*<n+k$, with generators in dimensions $2(p-1)m+k-1$. Thus the first $n+k-1$ rows of $\mathrm{Tor}^{H_*\mathbf{BP}_{k-1}}(\mathbf{F}_p,\mathbf{F}_p)$ form an exterior algebra on the suspension of the polynomial generators. In that range, the generators reside entirely on the 1-column and therefore, the spectral sequence collapses in that range. Together with the fact that $\mathrm{Tor}_{0,n+k}=0$, we see that there are no extension problems even at the prime 2. Thus, for k odd, $H_*\mathbf{BP}_k$ is an exterior algebra for $*<n+k$ and $\sigma:\mathrm{QH}_{n+k-1}\mathbf{BP}_{k-1}\xrightarrow{\cong}\mathrm{QH}_{n+k}\mathbf{BP}_k$ where σ is homology suspension. Hence, $\#(n)=\#(n,k-1)=\#(n,k)$.

If $n=2(p-1)m$ and k is even, then as before, the first $n+k-1$ rows of $\mathrm{Tor}^{H_*\mathbf{BP}_{k-1}}(\mathbf{F}_p,\mathbf{F}_p)$ form a divided power algebra whose primitive generators are the suspension of the generators of $H_*\mathbf{BP}_{k-1}$, i.e. in dimensions $2(p-1)m+k$ for some m. Thus in that range all the generators in the spectral sequence are in even degree and it collapses. We have to solve the extension problem.

The dual of a divided power algebra is a polynomial algebra, which has no extension problems to be solved. Hence, $H_*\mathbf{BP}'_k$ is a polynomial algebra for $*\leq n+k$. The E^∞-term in homology is a tensor product of truncated polynomial algebras all at height one. By hypothesis, all the extension problems are solved in degrees $<n+k$, with no truncated algebras remaining in those degrees. Notice that if a divided power algebra and a polynomial algebra have the same dimension as a vector space, then, by the universal property of a polynomial algebra, it has fewer number of generators unless both are polynomial and therefore have the same number of generators. This remains true in the graded case even if we are dealing only with degrees up to $n+k$ as above. Thus we see that the number of generators in $H_{n+k}\mathbf{BP}_k$ is the same as that in $H_{n+k-1}\mathbf{BP}_{k-1}$ if no truncation occurs at $H_{n+k}\mathbf{BP}_k$.

Otherwise, it definitely increases. Stated differently, $\#(n, k-1) \leq (n, k)$ with equality if and only if no truncation occurs. Thus if we continue the computation up the n-stem, we get a sequence $\#(n) = \#(n, k-1) \leq \#(n, k) \leq \#(n, k+1)$..... But when the stem stabilizes at $k = n+1$, we know that $\#(n, n+1) = \#(n)$. Thus no truncation could have occured anywhere along the stem. And by induction, we have proved Theorem 1.

Acknowledgement

This paper is part of my Ph.D. thesis at the John Hopkins University. I would like to thank my advisor, J.M. Boardman, for his help and encouragement. I would also like to thank W.S. Wilson and to acknowledge the influence of his paper [3].

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