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A simplification of the Eilenberg–Steenrod axioms, 2

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Abstract

Since the famous book of Eilenberg and Steenrod (1952) it is well known that a homology theory on the category of pairs of finite triangulable spaces and continuous maps is determined by four simple axioms.

In 1988 Dawson showed that in the much smaller category of pairs of finite simplicial complexes and simplicial maps one can replace this system of axioms by a simpler one, i.e., by replacing the dimension axiom and omitting the homotopy axiom.

In this paper I will show that it is also possible to do the same on the bigger category which was dealt with by Eilenberg and Steenrod. My proof uses ideas of Dawson, but it is easier. The second theorem shows that the homotopy axiom is nearly redundant.

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Let TOP be the well-known category of topological spaces and continuous maps. TOP^2 is the category of topological pairs. Define a functor $R:\mathrm{TOP}^2 \to \mathrm{TOP}^2$, $R[(X,A) \xrightarrow{f} (Y,B)] := [(A,\emptyset) \xrightarrow{f_{|A}} (B,\emptyset)]$. Let SC be the category of finite geometric simplicial complexes with simplicial maps as morphisms, let SC^2 be the category of pairs (K,L) of finite geometric simplicial complexes, i.e., L is a subcomplex of K. Let POL be the category of polyhedrons:

$$\mathrm{Ob(POL)} := \{X \in \mathrm{Ob(TOP)} \mid \exists K \in \mathrm{Ob(SC)} \text{ with } X \simeq |K|\}$$

(this means that there is a homeomorphism from X to the underlying topological space of K). These objects are called polyhedrons or finite triangulable spaces.

$$\mathrm{Mor(POL)} := \{f : X \to Y \mid f \text{ continuous}\}.$$

Hence, POL is a full subcategory of TOP^2.

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There is a corresponding category POL^2 with pairs (X,A) as objects whenever there is a pair $(K,L) \in \mathrm{Ob}(\mathrm{SC}^2)$ with a homeomorphism

$$h : X \xrightarrow{\simeq} |K| \text{ such that } h_{|A} : A \xrightarrow{\simeq} |L| .$$

POL^2 is a full subcategory of TOP^2.

A homology theory on TOP^2 or on a subcategory $\mathscr{K}^2$ of TOP^2 is a pair $(H,\partial) := (H_n,\partial_n)_{n\in\mathbb{Z}}$ with the following properties: For all $n \in \mathbb{Z}, H_n$ is a functor from TOP^2 or $\mathscr{K}^2$ to AB, the category of abelian groups; and ∂_n is a natural transformation from H_n to $H_{n-1} \circ R$. It is common usage to denote $H_n(f)$ of a map f by $f_\sharp$ and omit $f_\sharp$ if f is an embedding. Eilenberg and Steenrod proved in [1] that a homology theory $(H,\partial) := (H_n,\partial_n)_{n\in\mathbb{Z}}$ on the category POL^2 is determined (up to isomorphisms) by their famous four axioms, namely,

Axiom 1 (*Exactness axiom*). If $(X,A) \in \mathrm{Ob}(\mathrm{POL}^2)$ there is an exact sequence

$$\cdots \longrightarrow H_{n+1}(X,A) \xrightarrow{\partial_{n+1}} H_n(A) \longrightarrow H_n(X) \longrightarrow H_n(X,A) \xrightarrow{\partial_n} H_{n-1}(A) \longrightarrow \cdots$$

Axiom 2 (*Excision axiom*). If $(X,A),(X,B) \in \mathrm{Ob}(\mathrm{POL}^2)$ with $\mathrm{int}(A) \cup \mathrm{int}(B) = X$ and $(A,A\cap B) \in \mathrm{Ob}(\mathrm{POL}^2)$, then the embedding $(A,A\cap B) \hookrightarrow (X,B)$ yields $\forall n \in \mathbb{Z}$ isomorphisms

$$H_n(A,A\cap B) \xrightarrow{\simeq} H_n(X,B)$$

Axiom 3 (*Homotopy axiom*). If $f,g : (X,A) \longrightarrow (Y,B), f,g$ continuous with $f \sim g$ (i.e. f homotopic to g) then $f_\sharp = g_\sharp$.

Axiom 4 (*Dimension axiom*). If $\{p\}$ is a one-point space, then the homology groups are trivial for all $n \neq 0$, and $H_0(\{p\}) \simeq \mathbb{Z}$.

I will now show that there is an equivalent but apparently weaker system of axioms. Consider

Axiom 4⁺. Let $q \in \mathbb{N}$, let S be any q-dimensional simplex. Then the homology groups are trivial for all $n \neq 0$ and $H_0(S) \simeq \mathbb{Z}$.

Axiom 2a. If $(K,A),(K,B) \in \mathrm{Ob}(\mathrm{SC}^2)$ with $|A| \cup |B| = |K|$, then the embedding $(|A|, |A| \cap |B|) \hookrightarrow (|K|,|B|)$ yields $\forall n \in \mathbb{Z}$ an isomorphism

$$H_n(|A|,|A|\cap|B|) \xrightarrow{\simeq} H_n(|K|,|B|)$$

Let me define two systems of axioms:

$$\begin{aligned} AXSET\ 1 &:= \{\text{Axiom 1}, \text{Axiom 2}, \text{Axiom 3}, \text{Axiom 4}\} \\ &\qquad \text{(the Eilenberg–Steenrod axioms)}; \\ AXSET\ 2 &:= \{\text{Axiom 1}, \text{Axiom 2a}, \text{Axiom } 4^+\}. \end{aligned}$$

In 1988 Dawson [2] proved that these two systems of axioms are equivalent on the category SC^2. I will show that this also holds in POL^2.

Theorem 1. *Let (H, ∂) be a homology theory on* POL^2. *Then*

$$(H, \partial) \text{ satisfies } AXSET\ 1 \iff (H, \partial) \text{ satisfies } AXSET\ 2.$$

"$\Longrightarrow$": This is obvious, because any simplex is homotopy-equivalent to a one-point space, Axiom 4^+ then follows directly from the homotopy axiom and the dimension axiom, and the excision axiom and the homotopy axiom imply Axiom 2a; see [1, p. 76].

"$\Longleftarrow$": Axioms 1 and 4 are obvious.

On POL^2, Axiom 2a implies Axiom 2; see [1, p. 77].

The difficult part is to verify the homotopy axiom. First consider the case of two maps $s, t : (X, \emptyset) \to (Y, \emptyset); s, t \in \mathrm{Mor}(POL^2)$ with $s \sim t$. Let me now prove that $s_\sharp = t_\sharp$. As $X, Y \in \mathrm{Ob}(POL)$ there are $K, L \in \mathrm{Ob}(SC)$ and homeomorphisms $X \xrightarrow{k} |K|$ and $Y \xrightarrow{j} |L|$. Define $\phi := j \circ s \circ k^{-1}$, $\psi := j \circ t \circ k^{-1}$, thus $s = j^{-1} \circ \phi \circ k$, $t = j^{-1} \circ \psi \circ k$.

It suffices to show that $\phi_\sharp = \psi_\sharp$. As $s \sim t$ we have $\phi \sim \psi$. To prove this case it suffices to show that by fixing $f, g : |K| \to |L|$ continuous, $K, L \in \mathrm{Ob}(SC)$ with $f \sim g$, then $f_\sharp = g_\sharp$. $f \sim g$ means there is a continuous map $H : |K| \times I \to |L|$ and maps $e_0, e_1 : |K| \hookrightarrow |K| \times I, e_0(x) := (x, 0), e_1(x) := (x, 1)$ with $f = H \circ e_0$, $g = H \circ e_1$. Hence, it suffices to show that $(e_0)_\sharp = (e_1)_\sharp$.

Lemma 1. *For any homology theory on* POL^2 *we have*

$$\forall X \in \mathrm{Ob}(POL)\colon (e_0)_\sharp = (e_1)_\sharp \iff \forall X \in \mathrm{Ob}(POL)\colon (e_0)_\sharp \textit{ is an isomorphism.}$$

Proof. "$\Longleftarrow$": Let $(e_0)_\sharp$ be an isomorphism. Because $s : X \times I \to X \times I; s(x, t) := (x, 1-t)$ is a homeomorphism in TOP, $s_\sharp$ is an isomorphism in AB. As $e_1 = s \circ e_0$, $(e_1)_\sharp$ is also an isomorphism. Let $p : X \times I \to X, p(x, t) := x$; then $id_X = p \circ e_0 = p \circ e_1$. Thus, $(e_0)_\sharp = (p_\sharp)^{-1} = (e_1)_\sharp$.

"$\Longrightarrow$": Let $X \in \mathrm{Ob}(POL)$. As $id_X = p \circ e_0$, $(e_0)_\sharp$ is a right inverse. Now let η_0, $\eta_1 : X \times I \hookrightarrow X \times I \times I, \eta_0(x, t) := (x, t, 0), \eta_1(x, t) := (x, t, 1)$. Let $\mathcal{M} : X \times I \times I \to X \times I, \mathcal{M}(x, t, s) := (x, t(1-s))$. So $\mathcal{M} \circ \eta_0 = id_{X \times I}$ and $\mathcal{M} \circ \eta_1 = e_0 \circ p$. In the next section it will be shown that for all $X \in \mathrm{Ob}(POL)$, $X \times I \in \mathrm{Ob}(POL)$. Hence, by assumption, $(\eta_0)_\sharp = (\eta_1)_\sharp$, and $(e_0)_\sharp$ is also the left inverse of $p_\sharp$. $\square$

It remains to be shown that $e_0 : |K| \hookrightarrow |K| \times I$ yields an isomorphism in homology. This can be proved by decomposing the morphism $(e_0)_\sharp$ into a finite number of group morphisms, each of them being an isomorphism. Remember that $K \in \mathrm{Ob}(SC)$. Thus there is an r $\in \mathbb{N}$ and there are simplexes S_i with $K = \{S_0, S_1, \ldots, S_r\}$.

It is rather simple to show that $|K| \times I$ is also a polyhedron: First let there be a q-simplex $S \in K$ with $|K| = S = [v_0, v_1, \ldots, v_q]$; $S \subset \mathbb{R}^n$. Let $\mathbb{R}^n \subset \mathbb{R}^{n+1}$, so for $j = 0, \ldots, q$, $v_j = (x_{j,1}, x_{j,2}, \ldots, x_{j,n}, 0)$ is an $n+1$ vector. Define a vertex set and simplexes:

for all $j = 0, \dots, q$, $w_j := (x_{j,1}, x_{j,2}, \dots, x_{j,n}, 1)$ and the simplex $T_j := [w_0, w_1, \dots, w_j, v_j, v_{j+1}, \dots, v_q]$. It is then easy to prove that

(1) T_j is a simplex;

(2) the set $\{T_0, T_1, \dots, T_q\}$ together with all faces is a simplicial complex;

(3) $\bigcup_{j=0}^{q} T_j = S \times I$.

The more difficult case that $|K|$ is not its biggest simplex is treated in a similar way: $|K| \times I = \bigcup_{i=0}^{r} (S_i \times I)$, and for every q-simplex S_i the set $S_i \times I$ is triangulated as above into $q+1$ simplexes $T_{i,j}$ of dimension $q+1$. Then $|K| \times I = \bigcup_{i=0}^{r} \bigcup_{j=0}^{\dim S_i} T_{i,j}$, and it can be proved that the set $\{T_{i,j} \mid i = 0, \dots, r; j = 0, \dots, \dim S_i\}$ together with all their faces is a triangulation of $|K| \times I$. Refer to [2] for a graphical demonstration.

Now I will decompose the map $e_0 : |K| \hookrightarrow |K| \times I$, $e_0(x) := (x, 0)$. Let the set $K = \{S_0, S_1, \dots, S_r\}$ be ordered by the dimensions of the simplexes. Let $K_{-1} := K$, and for $i = 0, \dots, r$, $K_i := K_{i-1} \cup \{T_{i,0}, T_{i,1}, \dots, T_{i,\dim S_i}\}$ together with all their faces. Hence, $|K_i| = |K_{i-1}| \cup (S_i \times I)$, thus $|K_r| = |K| \times I$. Let for $i = 0, \dots, r$: $g_i : |K_{i-1}| \hookrightarrow |K_i|$ be the canonical injection; thus, $|K| = |K_{-1}| \overset{g_0}{\hookrightarrow} |K_0| \overset{g_1}{\hookrightarrow} |K_1| \overset{g_2}{\hookrightarrow} \cdots \overset{g_{r-1}}{\hookrightarrow} |K_{r-1}| \overset{g_r}{\hookrightarrow} |K_r| = |K| \times I$ with $e_0 = (g_r \circ g_{r-1} \circ \dots \circ g_1 \circ g_0)$.

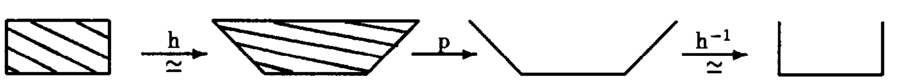
It remains to be shown that every $(g_i)_\#$ is an isomorphism.

First define a "fencing" of a simplex. Let $S = [a_0, a_1, \dots, a_q]$ be a q-simplex, then $F_S := (S \times \{0\}) \cup \bigcup_{i=0}^{q} ([a_0, a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_q] \times I)$; so $F_S \subset S \times I$.

Lemma 2. *For any q-simplex S the embedding* $F_S \overset{i}{\hookrightarrow} S \times I$ *is a section in* TOP, *i.e., there is a map* $S \times I \overset{\gamma}{\rightarrow} F_S$ *with* $\gamma \circ i = id_{F_S}$.

Lemma 3. *For any q-simplex S there is a homeomorphism from* F_S *to S.*

Proof of Lemma 2.



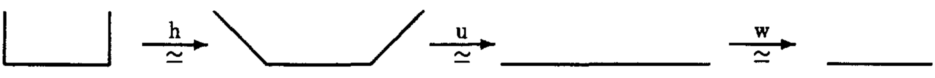
Let S be any q-simplex, and let ρ be a homeomorphism from S to $D^q := \{x \in \mathbb{R}^q \mid ||x|| \leq 1\}$. This map ρ induces homeomorphisms ρ_1, ρ_2 from $S \times I$ to $D^q \times I$, and from F_S to $F_{D^q} := (D^q \times \{0\}) \cup (S^{q-1} \times I)$.

Consider the commutative diagram of maps:

$$\begin{array}{ccccc} F_S & \xrightarrow{\quad i \quad} & S \times I & \xrightarrow{\quad \gamma \quad} & F_S \\ \rho_2 \downarrow \simeq & & \rho_1 \downarrow \simeq & & \rho_2 \downarrow \simeq \\ F_{D^q} & \xrightarrow{\quad i_1 \quad} & D^q \times I & \xrightarrow{\quad \delta \quad} & F_{D^q} \end{array}$$

Let $h : D^q \times I \to \mathbb{R}^{q+1}$, $h(x,t) := ((1+t)x,t)$, and $p : h(D^q \times I) \to h(D^q \times I)$, $p((1+t)x,t) := ((1+t)x,0)$ if $||x|| \leq 1/1+t$ and $p((1+t)x,t) := ((1+t)x,(1+t)\,||x||-1)$ if $1/1+t \leq ||x|| \leq 1$. The map $D^q \times I \xrightarrow{h} h(D^q \times I)$ is a homeomorphism. Let $\delta := h^{-1} \circ p \circ h$, then $\delta \circ i_1 = id_{F_{D^q}}$ holds. With $\gamma := (\rho_2)^{-1} \circ \delta \circ \rho_1$ the diagram commutes, and the Lemma has been proved. □

Proof of Lemma 3.



It remains to be shown that F_{D^q} is homeomorphic to D^q. Let $F_{D^q} \xrightarrow{h} h(F_{D^q}) \xrightarrow{u} 2D^q \times \{0\} \xrightarrow{w} D^q$, where $2D^q := \{x \in \mathbb{R}^q \,|\, ||x|| \leq 2\}$, h is the map defined above and, for all $(x,t) \in F_{D^q}$, let $u \circ h(x,t) = u((1+t)x,t) := ((1+t)x,0)$ and $w(y,0) := \frac{1}{2}y$. Then h, u, w are homeomorphisms: for h and w this is obvious. Let $u^{-1} : 2D^q \times \{0\} \to h(F_{D^q}), u^{-1}(y,0) := (y,0)$ if $||y|| \leq 1$ and $u^{-1}(y,0) := (y, ||y|| - 1)$ if $1 \leq ||y|| \leq 2$. Thus, $w \circ u \circ h$ yields a homeomorphism. □

Corollary. *For all $n \in \mathbb{Z}, q \in \mathbb{N}$ and q-simplexes S, $H_n(S) \simeq H_n(F_S)$.*

Let us now consider the two complexes K_{i-1} and $S_i \times I$; let $G := |K_{i-1}| \cap (S_i \times I)$. There is the following commutative diagram of subspaces:

$$\begin{array}{ccccc}
(G,\emptyset) & \longrightarrow & (|K_{i-1}|,\emptyset) & \longrightarrow & (|K_{i-1}|,G) \\
\downarrow \alpha & & \downarrow g_i & & \downarrow \beta \\
(S\times I,\emptyset) & \longrightarrow & (|K_i|,\emptyset) & \longrightarrow & (|K_i|,S\times I)
\end{array}$$

The embedding β is an excision; thus, because of the axioms, $\beta_\sharp$ is an isomorphism and the diagram commutes in AB:

$$\begin{array}{ccccccccc}
\cdots \longrightarrow H_{n+1}(|K_{i-1}|,G) & \xrightarrow{\partial_{n+1}} & H_n(G) & \longrightarrow & H_n(|K_{i-1}|) & \longrightarrow & H_n(|K_{i-1}|,G) & \xrightarrow{\partial_n} & \cdots \\
\beta_\sharp \downarrow \simeq & & \alpha_\sharp \downarrow & & (g_i)_\sharp \downarrow & & \beta_\sharp \downarrow \simeq & & \\
\cdots \longrightarrow H_{n+1}(|K_i|,S\times I) & \xrightarrow{\partial_{n+1}} & H_n(S\times I) & \longrightarrow & H_n(|K_i|) & \longrightarrow & H_n(|K_i|,S\times I) & \xrightarrow{\partial_n} & \cdots
\end{array}$$

The rows are exact because of the exactness axiom.

To prove that $(g_i)_\sharp$ is an isomorphism use the 5-Lemma; it suffices to show that $\alpha_\sharp$ is an isomorphism.

S_i is a q-simplex. Let T be a real face of S_i. Remember that the set $K = \{S_0, S_1, \ldots, S_r\}$ should be ordered by the dimensions of the simplexes. Thus, there is a $j < i$ with $T = S_j$ and $S_j \times I \in K_{i-1}$. So $|K_{i-1}| \cap (S_i \times I) = F_{S_i}$, and because of Lemma 3 $H_n(|K_{i-1}| \cap (S_i \times I)) \simeq \mathbb{Z}$ for $n = 0$ and is trivial for $n \neq 0$.

By Lemma 2 there is an $S_i \times I \xrightarrow{\gamma} F_{S_i}$ with $\gamma \circ \alpha = id_{F_{S_i}}$. For every q-simplex S there is a $(q+1)$-simplex T with $S \times I \simeq T$; thus, because of Axiom 4^+, all of the spaces S_i, F_{S_i} and $(S_i \times I)$ have the same homology groups.

Hence, for $n = 0$,

$$(\mathbb{Z} \xrightarrow{\alpha_\sharp} \mathbb{Z} \xrightarrow{\gamma_\sharp} \mathbb{Z}) = id_{\mathbb{Z}}.$$

This is only possible with an isomorphism $\alpha_\sharp$. For $n \neq 0$ all groups are trivial, thus all $\alpha_\sharp$ are isomorphisms, thus all $(g_i)_\sharp$ are isomorphisms, and thus $(e_0)_\sharp$ is an isomorphism, and thus the homotopy axiom holds for two homotopic maps $f, g : (X, \emptyset) \to (Y, \emptyset)$.

To prove this for $f, g : (X, A) \to (Y, B)$ show that $e_0 : (X, A) \hookrightarrow (X \times I, A \times I)$ yields an isomorphism $(e_0)_\sharp$ by using the two exact sequences for (X, A) and $(X \times I, A \times I)$ and the 5-Lemma; this finishes the proof of Theorem 1. □

There is an easy way to generalize the result of Theorem 1. In the given proof I use one important property of the group $\mathbb{Z}$: $Is(\mathbb{Z} \xrightarrow{f} \mathbb{Z} \xrightarrow{g} \mathbb{Z}) = id_{\mathbb{Z}}$, then f is already an isomorphism. Define the "quasifinite abelian groups" [3]:

$$\mathcal{Q}f_{\mathrm{AB}} := \{A \in \mathrm{Ob}(\mathrm{AB}) \mid (A \xrightarrow{f} A \xrightarrow{g} A) = id_A, \text{ then } f \text{ is an isomorphism}\}.$$

Many abelian groups belong to this class, for example all finitely generated ones, or for all $n \in \mathbb{N}$: $\mathbb{Q}^n$; but not $\mathbb{R}$. Let

Axiom 4a := For the one point space $\{p\}$ holds $\forall n \in \mathbb{Z} : H_n(\{p\}) \in \mathcal{Q}f_{\mathrm{AB}}$.
Axiom 4a$^+$:= For all $q \in \mathbb{N}$ for all q-simplexes S holds $\forall n \in \mathbb{Z} : H_n(S) \in \mathcal{Q}f_{\mathrm{AB}}$.
AXSET 3 := {Axiom 1, Axiom 2, Axiom 3, Axiom 4a}
AXSET 4 := {Axiom 1, Axiom 2a, Axiom 4a$^+$}

Theorem 2. *Let (H, ∂) a homology theory on* POL^2. *Then one has*

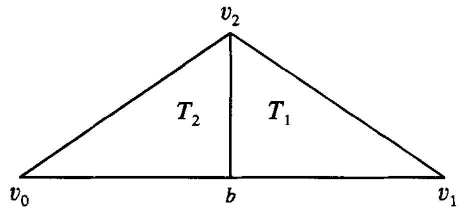
$$(H, \partial) \textit{ satisfies AXSET } 3 \iff (H, \partial) \textit{ satisfies AXSET } 4.$$

Proof.
"$\Longrightarrow$": As in Theorem 1.
"$\Longleftarrow$": Use the following lemma.

Lemma 4. *Let (H, ∂) be a homology theory satisfying AXSET* 4. $\Longrightarrow$: *For all $n \in \mathbb{Z}$, $q \in \mathbb{N}$ and for all q-simplexes S: $H_n(\{p\}) \simeq H_n(S)$.*

Proof. The proof of this lemma follows by induction on the dimension of the simplexes. Assume the lemma holds for all r-simplexes with $0 \leq r \leq q$. Let $S := [v_0, v_1, \ldots, v_q, v_{q+1}]$ be a $(q+1)$-simplex, let $b := \frac{1}{2}(v_0 + v_1)$, let $T_1 := [b, v_1, v_2, \ldots, v_{q+1}]$,

$T_2 := [v_0, b, v_2, \ldots, v_{q+1}]$. Then T_1, T_2 and $T := T_1 \cap T_2 = [b, v_2, \ldots, v_{q+1}]$ are simplexes, and $T_1 \cup T_2 = S$.



There is a commutative diagram of embeddings in POL^2:

$$\begin{array}{ccccc} (T,\emptyset) & \longrightarrow & (T_1,\emptyset) & \longrightarrow & (T_1,T) \\ \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \nu} & & \downarrow{\scriptstyle \xi} \\ (T_2,\emptyset) & \longrightarrow & (S,\emptyset) & \longrightarrow & (S,T_2) \end{array}$$

This yields a commutative diagram in AB:

$$\begin{array}{ccccccccccc} \cdots & \xrightarrow{\partial_{n+1}} & H_n(T) & \longrightarrow & H_n(T_1) & \longrightarrow & H_n(T_1,T) & \xrightarrow{\partial_n} & H_{n-1}(T) & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle \mu_\#} & & \downarrow{\scriptstyle \nu_\#} & & \downarrow{\scriptstyle \xi_\#}\,\simeq & & \downarrow{\scriptstyle \mu_\#} & & \\ \cdots & \xrightarrow{\partial_{n+1}} & H_n(T_2) & \longrightarrow & H_n(S) & \longrightarrow & H_n(S,T_2) & \xrightarrow{\partial_n} & H_{n-1}(T_2) & \longrightarrow & \cdots \end{array}$$

with exact rows. As ξ is an excision, $\xi_\#$ is an isomorphism. The map $T_1 \overset{\nu}{\hookrightarrow} S$ is a section: let $\rho : S \to T_1$, where $\rho_{|T_1} := id_{T_1}$ and $\rho_{|T_2}$ is constructed from the simplicial map m : Vert(T_2) $\to$ Vert(T), $m(v_0) := b, m(a) := a$ otherwise, and $\rho_{|T_2}$ being the continuous extension of the map m. Thus, $\rho_{|T_1}$ and $\rho_{|T_2}$ agree on T and $\rho \circ \nu = id_{T_1}$. Both S and T_1 are simplexes of dimension $q + 1$. Hence, for all $n \in \mathbb{Z} : H_n(S) \simeq H_n(T_1) \in \mathcal{Q}f_{\mathrm{AB}}$; thus, $\nu_\#$ is an isomorphism. Thus, $\mu_\#$ is also an isomorphism, thus, for all $n \in \mathbb{Z} : H_n(\{p\}) \simeq H_n(T) \simeq H_n(T_2)$; this completes the proof of Lemma 4. □

To prove Theorem 2, the reader only has to replace group $\mathbb{Z}$ by groups $H_n(\{p\})$ in the proof of Theorem 1.

References

[1] S. Eilenberg, N. Steenrod, Foundations of Algebraic Topology, Princeton University Press, Princeton, NJ, 1952.

[2] R.M. Dawson, A simplification of the Eilenberg–Steenrod axioms for finite simplicial complexes, J. Pure Appl. Algebra 53 (1988) 257–265.

[3] R.A. Beaumont, R.S. Pierce, Isomorphic direct summands of abelian groups, in: Math. Annalen 153 (1964) 21–37.