A Surprising Property of the Least Eigenvalue of a Graph*

Michael Doob
Department of Mathematics
University of Manitoba
Winnipeg, Manitoba, Canada

Submitted by W. G. Bridges

ABSTRACT

Let $\lambda(G)$ be the least eigenvalue of a graph G. A real number r has the *induced subgraph property* provided $\lambda(G)<r$ implies G has an induced subgraph H with $\lambda(H)=r$. It is shown that the only numbers with the induced subgraph property are 0, -1, $-\sqrt{2}$, and -2.

1. PRELIMINARIES

In various studies of the relationships between a graph and the eigenvalues of its 0-1 adjacency matrix, many early works considered graphs with least eigenvalue equal to -2 [4–8, 10–18]. One reason for this was that line graphs were some of the graphs with least eigenvalue bounded from below by -2. Other graphs that satisfy this property include the cocktail party graph, $\mathrm{CP}(n)$ (K_{2n} with a one factor removed) and the generalized line graph. The concept of a generalized line graph is due to A. J. Hoffman [16]. Given a graph G with n vertices and nonnegative integers $a_1, a_2, \dots, a_n$, the generalized line graph $L(G; a_1, \dots, a_n)$ consists of disjoint copies of $\mathrm{CP}(a_1), \dots, \mathrm{CP}(a_n)$ and $L(G)$ with additional edges that join a vertex in $L(G)$ with one in $\mathrm{CP}(a_i)$ if the corresponding edge in G has v_i as an end point. Many interesting properties of line graphs, spectral and otherwise, carry over to generalized line graphs (cf. [3]); one of these is the lower bound of -2 for the least eigenvalue. The original methods for finding all graphs with least eigenvalue bounded below by -2 involved forbidden subgraph techniques [2, 6, 7, 18]. This involved the construction of certain special graphs whose least eigenvalue equals -2. One method of finding such graphs

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52 Vanderbilt Ave., New York, NY 10017 0024-3795/82/050001+07$02.75

was to take an arbitrary graph with least eigenvalue less than -2 and to consider the induced subgraphs contained within it. Since the adjacency matrix of an induced subgraph is a principal submatrix of the original adjacency matrix, the least eigenvalue of the induced subgraph is not less than that of the original graph. Hence this was considered a reasonable technique for finding the type of graph desired. It was noted that this method had remarkable success. One result in this paper (Theorem 2.2) will show why this is so; it will show that this technique will always find graphs with least eigenvalue equal to -2.

We wish to investigate the following phenomenon: Let $\lambda(G)$ be the least eigenvalue of G. For some real numbers r, any graph G with $\lambda(G)<r$ contains an induced subgraph H such that $\lambda(H)=r$. We shall say that such a real number possesses the *induced subgraph property*. At first it might seem surprising that any nontrivial r would exist, but in fact $r=-2$ does possess this property. Knowing that -2 possesses this property, it might be hoped that other real numbers might possess it also. We shall show that the set of reals with the induced subgraph property is precisely $\{0,-1,-\sqrt{2},-2\}$. To understand why this is so, we need to look at a few examples, and to do so we use an old construction of Finck and Grohmann [9]. Given graphs G_1 and G_2, the graph $G_1 \bigtriangledown G_2$ consists of disjoint copies of G_1 and G_2 plus edges joining each vertex in G_1 with every vertex of G_2. If the two graphs are regular, the spectrum of $\bigtriangledown$-product is easily obtained.

THEOREM 1.1 [9]. *If G_1 and G_2 are regular graphs with n_1 and n_2 vertices and with degrees r_1 and r_2 respectively, then the eigenvalues of $G_1 \bigtriangledown G_2$ consist of the subdominant eigenvalues of G_1, the subdominant eigenvalues of G_2, and the roots of the polynomial $p(\lambda)=(\lambda-r_1)(\lambda-r_2)-n_1n_2$.*

COROLLARY 1.2. $\lambda(K_{m,n})=-(mn)^{1/2}$.

Let G^c denote the complement of the graph G.

COROLLARY 1.3. $\lambda(K_n \bigtriangledown K_m^c)=-\frac{1}{2}\left(n-1-\left[(n-1)^2+4mn\right]^{1/2}\right)$.

Note that if m is fixed and n increases, then the least eigenvalue of $K_n \bigtriangledown K_m^c$ monotonically approaches $-m$ from above. In fact, $\lambda(K_n \bigtriangledown K_m^c)<-m+\varepsilon$ if and only if $n>(m-\varepsilon)(m-\varepsilon-1)/2\varepsilon$.

2. THE REAL NUMBERS THAT POSSESS THE INDUCED SUBGRAPH PROPERTY

In this section we show that if $r \in \{0, -1, -\sqrt{2}, -2\}$ then $\lambda(G)$ less than r implies that there is an induced subgraph H of G such that $\lambda(H) = r$.

LEMMA 2.1.

$\lambda(G) = 0$ *iff G is a graph with no edges,*

$\lambda(G) = -1$ *iff G is a graph with at least one edge and every component complete, and*

$\lambda(G) \leqslant -\sqrt{2}$ *otherwise.*

Proof. Any connected graph that is not a complete contains $K_{1,2}$ as an induced subgraph. Hence by Corollary 1.2, any such graph has $\lambda(G) \leqslant -\sqrt{2}$, and the proof of the theorem is complete. Thus 0, -1, and $-\sqrt{2}$ clearly possesses the induced subgraph property. ■

We shall acheived our goal for this section upon the proof of the following:

THEOREM 2.2. *Let G be a graph with* $\lambda(G) < -2$. *Then there exists an induced subgraph H of G such that* $\lambda(H) = -2$.

In order to prove this theorem, we need to use the elegant characterization of graphs G with $\lambda(G) \geqslant -2$ given by Cameron, Goethals, Seidel, and Shult [1]. One form of their characterization yields the following result:

THEOREM 2.3 [1]. *Let G be a graph with n vertices and* $\lambda(G) \geqslant -2$. *Then either*

(i) *G is a generalized line graph, or*
(ii) *there exists an* $n \times 8$ *matrix B such that* $BB^T = A(G) + 2I$.

COROLLARY 2.4. *If G is a graph with at least* 9 *vertices, and G is not a generalized line graph, then* $\lambda(G) \leqslant -2$.

Proof. Suppose $\lambda(G) \geqslant -2$. Then, according to Theorem 2.3, B has more rows than columns, and hence BB^T is singular. Thus $A(G) + 2I$ is singular and $\lambda(G) = -2$. ■

REMARK. This result also follows from the characterization of graphs with $\lambda(G) > -2$ given by Doob and Cvetković [8], since the 573 graphs with $\lambda(G) > -2$ that are not generalized line graphs (found there by computer search) have no more than 8 vertices.

We next note that for graphs with nine or fewer vertices, Theorem 2.2 is valid. This was determined by an exhaustive computer search carried out by Brendan McKay.

LEMMA 2.5 [19]. *Suppose G has $n \leq 9$ vertices and $\lambda(G) < -2$. Then G contains an induced subgraph H with $\lambda(H) = -2$.*

Cvetković, Doob, and Simić [3] have found all minimal nongeneralized line graphs. There are 31 of them, each of which has 5 or 6 vertices; 20 of them have least eigenvalue greater than -2. Of the remaining 11, three contain $K_{2,2}$ as an induced subgraph, another three contain $K_{1,4}$ as an induced subgraph, and the remaining four contain $K_2 \nabla K_3^c$. Hence we see the following:

LEMMA 2.6. *If G is a minimal nongeneralized line graph with $\lambda(G) < -2$, then there exists an induced subgraph H with $\lambda(H) = -2$.*

We have now assembled the tools necessary for the completion of Theorem 2.2. Suppose G is a graph with $\lambda(G) < -2$. Then G is not a generalized line graph, and hence there is an induced subgraph G' which is a minimal nongeneralized line graph. If $\lambda(G') < -2$, then we're done by Lemma 2.6. If $\lambda(G') = -2$, we are obviously done. If $\lambda(G') > -2$ but G has eight or fewer vertices, we're done by Lemma 2.5. Finally, suppose $\lambda(G') > -2$, and we may find a connected graph H' with nine vertices that contains G' and is contained in G. Since G' is not a generalized line graph, H' is not one either. Hence, by Corollary 2.4 we have $\lambda(H') \leq -2$. If $\lambda(H') = -2$, then by letting $H = H'$, we're done. If $\lambda(H') < -2$, then by Lemma 2.5 we have a graph H contained in H' (and hence in G) such than $\lambda(H) = -2$. Hence the proof of Theorem 2.2 is now complete.

3. THE NONEXISTENCE OF OTHER REALS WITH THE INDUCED SUBGRAPH PROPERTY

A number r does not have the induced subgraph property if there exists a graph G such that for every induced subgraph H we have $\lambda(G) < r < \lambda(H)$. Hence the following Lemma is obvious:

LEMMA 3.1. *Let $\mathcal{G}$ be a family of graphs with the following property: any induced subgraph of an element of $\mathcal{G}$ is also in $\mathcal{G}$. Let r satisfy $\lambda(G)<r<0$ for some G in $\mathcal{G}$. Then r can have the induced subgraph property only if r is the least eigenvalue of some graph in $\mathcal{G}$.*

LEMMA 3.2. *If r has the induced subgraph property, then $r=-\sqrt{n}$ for some integer n.*

Proof. Consider the family $K_{1,m}$, $m=1,2,\dots$, and apply Lemma 3.1. ■

LEMMA 3.3. *If r has the induced subgraph property, then $r=-\frac{1}{2}\left(n-1-\left[(n-1)^2+4mn\right]^{1/2}\right)$.*

Proof. Consider the family $K_n \nabla K_m^c$, $m=1,2,\dots$, $n=1,2,\dots$, and apply Lemma 3.1. ■

LEMMA 3.4. *If $r=(a-\sqrt{b})/c=-\sqrt{n}$, where a, b, c, and n are nonzero integers, then r is an integer.*

Proof. $\sqrt{b}=(a^2+b-nc^2)/2a$ is rational and hence integral. Thus $-\sqrt{n}$ is rational and hence integral. ■

COROLLARY 3.5. *If r has the induced subgraph property, $r<-2$, then r is integral.*

Proof. Apply Lemma 3.4 to the result of Lemma 3.3. ■

Now consider $K_{t+2,t-1}$ with $t>2$. Then clearly we have that $\lambda(K_{t+2,t-1}) < -t < \lambda(K_{t+1,t-1}) < \lambda(K_{t+2,t-2})$, and hence $-t$ does not possess the induced subgraph property.

LEMMA 3.6. *The integer $-t$ does not possess the induced subgraph property for $t=3,4,\dots$.*

The graph C_4, a circuit of length 4, shows that $-\sqrt{3}$ does not possess the induced subgraph property. Hence from Lemma 3.2, Corollary 3.5, and Lemma 3.6 we see that the only possible reals with the property are 0, -1, $-\sqrt{2}$ and -2:

THEOREM 3.7. *If $r\notin\{0,-1,-\sqrt{2},-2\}$, then r does not possess the induced subgraph property.*

4. A REMARK ON THE LARGEST EIGENVALUE OF A GRAPH

It is natural to ask the corresponding question for the largest eigenvalue of a graph, i.e., if $\Lambda(G)$ is the largest eigenvalue of a graph, do there exist real numbers r such that $\Lambda(G) > r$ implies that there is an induced subgraph H of G such that $\Lambda(H) = r$? The numbers $r = 0$ and $r = 1$ trivially satisfy this property. But in this case there is also a nontrivial real number that satisfies the property. By considering complete graphs, any other such r must be an integer, and, as in the last section, by considering $K_{t+2,t-1}$, $t > 2$, we see that the only further possible value is $r = 2$. Now if G is a graph with $\Lambda(G) > 2$, and G contains a cycle, then G contains a simple circuit, and such a subgraph has 2 as its maximum eigenvalue. Otherwise G is a tree and hence bipartite. It is well known that the spectrum of a bipartite graph is symmetric with respect to 0, and thus $\lambda(G) < -2$ [2]. Hence by Theorem 2.2 there is a subgraph H with $\lambda(H) = -2$, and since H is also bipartite, $\Lambda(H) = 2$. Thus we see that 2 has the corresponding induced subgraph property for the largest eigenvalues, and we have proved the following theorem:

THEOREM 4.1. *A real number r has the induced subgraph property for largest eigenvalues if and only if $r \in \{0, 1, 2\}$.*

It is interesting to note the ease of the proof of Theorem 4.1 once Theorem 2.2 has been established. Theorem 4.1 can be proved independently by using the characterization of graphs with $\Lambda(G) \leqslant 2$ by J. Smith [20], but the proof is somewhat more involved.

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