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A synthetic Frobenius theorem

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Abstract

A synthetic approach to Frobenius' theorem concerning integrable distributions is discussed.

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1. Introduction

One of the basic theorems of differential geometry is the Frobenius theorem: an integrable distribution defines a foliation. The purpose of this paper is to attempt to generalize this theorem in the context of synthetic differential geometry.

In synthetic differential geometry, the notion of "smooth space" is extended from "C^∞ manifold" to include more general spaces. This extension allows one, for example, to form function spaces and quotient spaces, to form the space of zeroes of arbitrary smooth functions, and to talk of actual infinitesimal elements while remaining in the category of "smooth spaces". Among the most astounding features of this extended notion is that a differential form may be thought of as a quantity, i.e., a differential form on a space M is a map from M into a suitable fixed universal space. As a result, if we consider the distribution in the Frobenius theorem as given as the zeroes of finitely many one-forms, we can consider the forms as maps and take the inverse image of zero. This gives us a subspace of M. Since this subspace is precisely the subspace on which the forms vanish, one could think of the subspace as being the foliation defined by the distribution. Thus this synthetic approach seems to give us the foliation directly.

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It remains to see that the tangent bundle of the foliation is what it ought to be (the restriction of the distribution to the foliation) and that the foliation is a "nice" space.

In as much as this paper is being written for readers with a differential geometry background as much as for those with some familiarity with synthetic differential geometry, it includes a brief and abbreviated introduction to synthetic differential geometry. This introduction is supposed to provide the reader with enough familiarity with the synthetic viewpoint to understand the remainder of the paper, and to provide references to the literature for those interested in learning more about the synthetic approach to differential geometry.

The main results of this paper concern a smooth space M with a finite number of one-forms $\omega^1,\dots,\omega^n$ satisfying the Frobenius integrability condition: there exist n^2 forms ϕ^j_k such that $d\omega^j = \phi^j_k \wedge \omega^k$. Viewing these as real-valued functions on the tangent bundle TM, we let $\mathscr{F}$ be their zeroes. In other words, $\mathscr{F}$ should be thought of as the distribution defined by the forms $\omega^1,\dots,\omega^n$. (Note, however, that $\mathscr{F}$ need not be a distribution in the usual sense, since its rank need not be constant. No assumption on the linear independence of the forms is being made.) Through the magic of the synthetic approach (the "amazing right adjoint" – see Section 2), we may also think of the forms as functions $\hat{\omega}^1,\dots,\hat{\omega}^n$ taking non-classical values, and define a subspace $F \subset M$ as their common zeroes. The first result is that the tangent bundle to F is what it ought to be.

Theorem 1.1. $TF \simeq F \times_M \mathscr{F}$.

The next result ought to be that F is as nice a space as M is. To describe the results here requires some discussion of what a "nice" space is. In general what this ought to mean is that we can do calculus on M just as we can do calculus in $\mathbb{R}^n$. Classically, this means that we assume that M is locally isomorphic to $\mathbb{R}^n$, i.e., that M is a manifold. However, if one recognizes that differential calculus does not involve the full information provided in local descriptions, but rather is done using only infinitesimal information, it makes sense to ask not that the space be locally isomorphic to $\mathbb{R}^n$, but rather that the relationship between the infinitesimals and M is the same as that between the infinitesimals and $\mathbb{R}^n$. One such assumption is that of *microlinearity*. (For a definition, see Section 1.) Alas, in general F will not be microlinear even when M is. Indeed, it seems the only general case when F is microlinear is when $F = M$. (Thus F will *not* be what one thinks of as the foliation, in general.) Fortunately, while F does not share all the infinitesimal properties of Euclidean space, it does have many.

Theorem 1.2. *If M is Frobenius microlinear, then F is Frobenius microlinear.*

This is not the place to define Frobenius microlinearity. (The place is Section 2.) However, it should be mentioned that Frobenius microlinearity is enough to define a module structure on the tangent bundle, and to develop much of the theory of connections.

The outline of the paper is as follows. Section 1 is an expository introduction to synthetic differential geometry, describing how the category of smooth manifolds is imbedded in a larger category of smooth spaces and how this larger category provides us with a different approach to calculus. Section 2 gives a short description of how differential forms are viewed in this larger category. Section 3 contains proofs of the two main theorems and a counterexample to show that F is, in general, not microlinear. The paper concludes with a theoretical discussion in Section 4 of the possible application of the synthetic approach to differential forms to the study of partial differential equations.

2. An introduction to synthetic differential geometry

This section is a brief and very simplified introduction to synthetic differential geometry for those unfamiliar with the subject. No proofs will be given, and, indeed, many definitions will be given in a heuristic rather than an exact manner. For proofs and complete definitions the reader should consult [7, 3] and the references therein.

2.1. Where SDG lives

Modern synthetic differential geometry has at its starting point the category $\mathbb{M}$ of smooth manifolds and smooth maps. There are several difficulties with this category that make it difficult to work with (cf. the introduction to [7]).

(1) $\mathbb{M}$ does not have function spaces. In general, if M and N are smooth manifolds, the space of smooth maps from M to N is not a smooth manifold (being infinite dimensional).

(2) $\mathbb{M}$ has bad limit properties (in the categorical sense). For example (a limit), the zeroes of a smooth real-valued function do not in general form a smooth manifold. As another example (a colimit), if A is a smooth submanifold of M, the quotient space M/A is, in general, not a smooth manifold.

(3) $\mathbb{M}$ has no way of dealing with infinitesimal quantities. The infinitely small is dealt with only by taking limits of finitely small quantities, and not directly.

To correct this, we shall imbed $\mathbb{M}$ into a larger category. As a first step, we consider studying a manifold M by studying its space of real-valued C^∞ functions, $C^\infty(M)$. This space is not only a ring, but a C^∞-ring: Not only do polynomial mappings $\mathbb{R}^n \to \mathbb{R}^m$ give mappings $(C^\infty(M))^n \to (C^\infty(M))^m$, but any smooth map $\mathbb{R}^n \to \mathbb{R}^m$ gives such a map in a natural fashion. (For a complete definition of the notion of C^∞-ring see [7].) Indeed the contravariant functor $M \mapsto C^\infty(M)$ gives a full and faithful embedding of $\mathbb{M}$ into the category of finitely generated C^∞-rings. We let $\mathbb{L}$ be the opposite category of the category of finitely generated C^∞-rings (in other words, the objects are the same, but all the arrows have been reversed), so that the global functions functor is now covariant: $\mathbb{M} \to \mathbb{L}$. The category $\mathbb{L}$ is known as the category of *loci*. (It is worth noting here that we may at this stage consider other subcategories of $\mathbb{L}$, cf. [7].)

Alas, while $\mathbb{L}$ has better limit properties and infinitesimals, it still does not have most function spaces. To fix that, we use the embedding $\mathbb{L} \to \mathrm{Sets}^{\mathbb{L}^{\mathrm{op}}}$ of the category of loci into the category of contravariant functors from $\mathbb{L}$ to the category of sets. This embedding is simply the Yoneda embedding:

$$\ell A \mapsto \hom_{\mathbb{L}}(-, \ell A) \simeq \hom_{C^\infty\text{-rings}}(A, -),$$

where A is a C^∞-ring and ℓA is the same thing thought of as an element of $\mathbb{L}$. The category $\mathrm{Sets}^{\mathbb{L}^{\mathrm{op}}}$ can be thought of as presheaves on $\mathbb{L}$. The analogue most familiar to analysts and geometers might be that of presheaves on a topological space X, which may be viewed as contravariant set-valued functors from the category $\mathcal{O}(X)$ whose objects are the open sets of X and whose morphisms are the inclusion mappings.

We are still not done. In this category $\mathrm{Sets}^{\mathbb{L}^{\mathrm{op}}}$, R is not only not a field, it is not even a local ring, nor is it Archimedean ($\forall x \in R,\ \exists n \in \mathbb{N}$ such that $n > x$). Moreover, the embedding does not preserve the good colimits in $\mathbb{M}$: the open covers. We need to consider not presheaves, but sheaves. This notion is a generalization of the notion of sheaf on a topological space. One can define the concept of a sheaf using open covers. What we need is a suitable notion of "open cover" for the category $\mathbb{L}$. This is the notion of a *Grothendieck site*. There is no room here to discuss this concept (and, indeed, it is not completely necessary for what follows). The reader is referred to [3, 6, 7] for the definition. Suffice it to say that by taking a suitable sub-category $\mathbb{G}$ of $\mathbb{L}$ and a notion of open covering coming directly from that on $\mathbb{R}^n$, one obtains a nice category $\mathcal{G}$ in which to study smooth geometry. (Again, for the definition of $\mathcal{G}$ see [7].) This category has all colimits and all finite limits, is Cartesian closed (and so has function spaces) and has objects corresponding to certain spaces of infinitesimals. Moreover, the embeddings described above give a full and faithful embedding $s : \mathbb{M} \to \mathcal{G}$ which preserves the good limits (transversal intersections) and the good colimits (open covers) of $\mathbb{M}$.

One of the reasons we need not be too concerned with the exact nature of the category $\mathcal{G}$ is that it is possible to speak of the objects in $\mathcal{G}$ as if they were sets with elements. Indeed, an object in $\mathcal{G}$ may be viewed as a "variable set", a set varying over the parameter space $\mathbb{G}$. An "element" of an object of $\mathcal{G}$ is then an element of one of these sets. However, if one speaks of objects of $\mathcal{G}$ as if they were sets, one must limit one's logical capabilities, using only constructionist arguments. The law of the excluded middle does not hold. This may be seen to be reasonable by considering the function $g : R \to R$

$$g(x) = \begin{cases} 1 & \text{if } x \neq 0, \\ 0 & \text{if } x = 0, \end{cases}$$

which should not be a function in this category since it is not smooth. (Here R is the image of $\mathbb{R}$ under the embedding s.)

2.2. Calculus with infinitesimals

The basic object in calculus is the real line $R = s(\mathbb{R})$, where $s : \mathbb{M} \to \mathcal{G}$ is the imbedding described above. Calculus in SDG is based on the following, which may be taken as also being a definition of the derivative of a real-valued function of a real variable.

Kock–Lawvere axiom. Let $f : R \to R$ be a function and let $x \in R$. Then there is a unique $f'(x) \in R$ such that

$$f(x+d) = f(x) + df'(x) \quad \text{for all } d \in D,$$

where $D = \{d \in R : d^2 = 0\}$.

Using this one can get all the standard results (product rule, chain rule) quite easily. Higher derivatives and partial derivatives are obtained via the Generalized Kock–Lawvere axiom. To state that axiom, we need some definitions.

Definition 2.1

$$D_k(n) = \{(x_1, \ldots, x_n) \in R^n : \text{every monomial of degree} > k \text{ vanishes}\}.$$

Definition 2.2. An infinitesimal space is a space $S \subset D_k(n)$ containing the origin and given as the zeroes of finitely many polynomials of degree $\leq k$.

Generalized Kock–Lawvere axiom. Let $S \subset D_k(n)$ be an infinitesimal space. Any map $S \to R$ is given by a polynomial of degree $\leq k$, unique modulo the ideal generated by the polynomials defining S.

For example, one can define the second derivative f'' of $f : R \to R$ by the equation

$$f(x+d) = f(x) + df'(x) + \frac{1}{2}d^2 f''(x) \quad \text{for all } d \in D_2$$

and it is possible to show that this f'' is the derivative of f'.

2.3. The tangent bundle

If we are to speak of differential forms, we must first talk about the tangent bundle. The tangent bundle of a space M is simply M^D, the space of maps from D to M. (The projection $TM = M^D \to M$ is simply evaluation at 0.) Heuristically, one should think of the definition of the tangent bundle of a manifold as equivalence classes of curves, two curves being equivalent if and only if they have the same value and first derivative at the origin. Since the value and first derivative of a function at the origin are determined by its restriction to D (the Kock–Lawvere axiom), it should seem reasonable that TM would be a quotient of M^D. Since the restriction of a function to D is determined by its value and first derivative at the origin (Kock–Lawvere once again), it should seem reasonable that there is no identifications made by the quotient map $M^D \to TM$. Indeed, it can be shown that if $s : \mathbb{M} \to \mathcal{G}$ is the embedding, that $s(M)^D \simeq s(TM)$ for any manifold M. One difficulty that now arises is how to define the addition of two vectors. (The multiplicative action of R is defined via the obvious "change of speed" of the curve.) We need an additional assumption: that the diagram

$$\begin{array}{ccc} M^{D(2)} & \longrightarrow & M^D \\ \downarrow & & \downarrow \\ M^D & \longrightarrow & M \end{array}$$

(where the top arrow is the map $\gamma \mapsto (d \mapsto \gamma(d,0))$, the left arrow the map $\gamma \mapsto (d \mapsto \gamma(0,d))$ and the right and bottom arrows map an infinitesimal curve to its value at 0) is a pullback diagram. Then, given two tangent vectors γ_1, γ_2 at $x \in M$, there is a unique map $\tau : D(2) \to M$ such that $\tau(d,0) = \gamma_1(d)$, $\tau(0,d) = \gamma_2(d)$. We then define $\gamma_1 + \gamma_2$ as $d \mapsto \tau(d,d)$. This indeed is the generalization of the usual addition in the tangent space to a manifold. To prove the associativity of this addition, we need another limit diagram. Indeed, it seems a nice assumption is that M is *microlinear* in the following sense.

Definition 2.3. Let M be any object, and $\mathcal{D}$ a finite co-cone diagram of infinitesimal spaces. $\mathcal{D}$ is said to be an *M-colimit* if the functor $M^{(-)}$ sends $\mathcal{D}$ into a limit diagram.

Definition 2.4. An object M is *microlinear* if every R-colimit is an M-colimit.

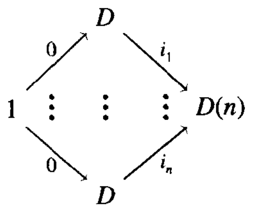
In other words, M is microlinear if, as far as limits properties of maps of infinitesimal spaces into M are concerned, M looks like R. This is a generalization of the notion of manifold. A manifold is a space that "locally looks like R^n", this being a reasonable assumption if one wants to extend calculus to more general spaces. Since differential calculus actually operates at an infinitesimal (not local) level, and since we have, in the synthetic setting, actual infinitesimals, it makes some sense to extend the notion of "space on which one can do calculus". Microlinearity is one way of extending the notion of manifold, and a very nice one at that.

Proposition 2.1. (1) *Every manifold is microlinear.*
(2) *Every limit of microlinear spaces is microlinear.*
(3) *If M is microlinear and X is any space, then M^X is microlinear.*

Proof. See [7, V.1.2, V.7.1]. □

Moreover, using microlinearity, the theory of connections on vector bundles can be developed, including a direct (i.e., using infinitesimals) proof of the Ambrose–Palais–Singer theorem equating sprays and symmetric connections (see [7, Ch. V]).

More general classes of spaces on which calculus can be done may be obtained by restricting the class of diagrams $\mathscr{D}$ considered. Consider, for example, the class IL of diagrams of the form



in which there are n copies of the space D and where $i_k : D \to D(n)$ is the mapping $d \mapsto (0,\dots,0,d,0,\dots,0)$ which imbeds D into $D(n)$ in the kth slot. This is an R-colimit (which is easily checked using the generalized Kock–Lawvere axiom), and the following assumption about a space is sufficient to obtain an R-module structure on the tangent bundle to the space (see [3, I.7.2]).

Definition 2.5. A space M is *infinitesimally linear* if $\mathscr{D}$ is an M-colimit for every $\mathscr{D} \in \mathrm{IL}$.

We shall define later a class of diagrams suitable for studying the Frobenius theorem, and which is perhaps the most suitable for studying spaces defined using one-forms via the "amazing right adjoint".

3. Differential forms

The standard synthetic definition of a differential form is the following.

Definition 3.1. Let M be a space. A *differential n-form on M* is a map

$$M^{D^n} \times D^n \xrightarrow{\omega} R$$

sometimes written as

$$(\gamma, h_1, \dots, h_n) \mapsto \int_{(\gamma, h_1, \dots, h_n)} \omega$$

satisfying the following conditions:

(1) *homogeneity*:

$$\omega(a \cdot_i \gamma, h_1, \dots, h_n) = a \cdot \omega(\gamma, h_1, \dots, h_n),$$

where $a \cdot_i \gamma : D^n \to M$ is defined by

$$a \cdot_i \gamma(x_1, \dots, x_i, \dots, x_n) = \gamma(x_1, \dots, a x_i, \dots, x_n),$$

for every $a \in R$ and infinitesimal n-cube $(\gamma, h_1, \dots, h_n)$;

(2) *alternation*:

$$\omega(\sigma\gamma, h_1, \dots, h_n) = \operatorname{sgn}(\sigma)\omega(\gamma, h_{\sigma(1)}, \dots, h_{\sigma(n)}),$$

where σ is any permutation of $\{1,\dots,n\}$, and $\sigma\gamma$ is γ composed with the coordinate permutation induced by σ, i.e.,

$$\sigma\gamma(x_1,\dots,x_n) = \gamma(x_{\sigma(1)},\dots,x_{\sigma(n)}),$$

and $\operatorname{sgn}(\sigma)$ is the signature of the permutation σ;
(3) *non-degeneracy*: $\omega(\gamma, h_1, \dots, 0, \dots, h_n) = 0$.

It follows from the Kock–Lawvere axiom and non-degeneracy that

$$\omega(\gamma, h_1, \dots, h_n) = h_1 \cdots h_n \cdot \tilde{\omega}(\gamma)$$

for a unique homogeneous, alternating map $\tilde{\omega} : M^{D^n} \to R$, and often the term "differential n-form" will refer to the latter. We let $\Lambda^n(M)$ be the space of n-forms on M.

3.1. The exterior derivative

The exterior derivative is defined by insisting that Stokes' theorem be valid infinitesimally.

Definition 3.2. Let ω be a n-form on M. The *exterior derivative* of ω is the $(n+1)$-form $d\omega$ defined by

$$\int_{(\gamma, h_1,\dots,h_{n+1})} d\omega = \sum_{i=1}^{n+1} \sum_{\alpha=0,1} (-1)^{i+\alpha} \int_{F_{i\alpha}(\gamma, h_1,\dots,h_{n+1})} \omega,$$

where $F_{i\alpha}(\gamma, h_1, \dots, h_{n+1}) \in M^{D^n} \times D^n$ is

$$([(x_1,\dots,x_n) \mapsto \gamma(x_1,\dots,\alpha h_i,\dots,x_n)], h_1,\dots,\hat{h}_i,\dots,h_{n+1}).$$

It is easy to check that $d^2 = 0$ and that d is natural: if $f : N \to M$, then for any n-form ω on M, $d(f^*\omega) = f^*(d\omega)$.

It should be remarked here that these are indeed generalizations of the classical notions, in the sense that given an n-form ω on a manifold M, in other words a multilinear alternating map

$$\omega : TM \times_M \cdots \times_M TM \to \mathbb{R},$$

the embedding $s : \mathbb{M} \to \mathcal{G}$ gives us a space $s(M)$ and a map

$$s(\omega) : s(M)^D \times_{s(M)} \cdots \times_{s(M)} s(M)^D \to R.$$

Since $s(M)$ is microlinear, we may identify $s(M)^D \times_{s(M)} \cdots \times_{s(M)} s(M)^D$ with $s(M)^{D(n)}$. Because M is a manifold, and hence locally (and infinitesimally) like $\mathbb{R}^n$, the multilinear alternating R-values maps on $s(M)^{D(n)}$ are the same as those on $s(M)^{D^n}$. (It should be noted here that in synthetic differential geometry, homogeneity usually implies linearity. See [7, V.1.5]. We will prove a similar result later.) Moreover, $s(d\omega) = d(s(\omega))$. (See [7, IV.3.7]).

3.2. The exterior product

It is possible to define the wedge product of two differential forms. To see that the wedge product is a generalization of the classical notion, we follow closely the classical definition (cf. [8, p. 274ff]). We start with some auxiliary definitions.

Definition 3.3. Let M be a space.

(1) A *classical tensor of degree n* on M is a map $T : M^{D(n)} \to R$ satisfying $T(a \cdot_i \gamma) = a \cdot T(\gamma)$. We let $\mathscr{T}_c^n(M)$ be the space of classical tensors on M.

(2) A *synthetic tensor of degree n* on M is a map $T : M^{D^n} \to R$ satisfying $T(a \cdot_i \gamma) = a \cdot T(\gamma)$. We let $\mathscr{T}_s^n(M)$ be the space of classical tensors on M.

Since there is a natural embedding $D(n) \to D^n$, there is a natural map $\mathscr{T}_s^n(M) \to \mathscr{T}_c^n(M)$, i.e., every classical tensor is a synthetic tensor, however, unlike what happens with differential forms, it is rare that every synthetic tensor on M is a classical tensor. Note that classical tensors are what we obtain using the embedding s from tensors on manifolds. In what follows, we shall use synthetic tensors. The reader may verify that all the definitions made (e.g., of tensor product) take classical tensors to classical tensors and commute (in the obvious sense) with the map $\mathscr{T}_s^n(M) \to \mathscr{T}_c^n(M)$. Thus, the definitions made below are generalizations of the classical definitions. In the following, $\mathscr{T}^n(M)$ can often refer to either of these tensor spaces, but should be taken to refer to $\mathscr{T}_s^n(M)$ when there is any doubt.

Note that $\Lambda^n(M)$ is naturally a subspace of $\mathscr{T}_s^n(M)$.

Definition 3.4. For $T \in \mathscr{T}^p(M)$, $S \in \mathscr{T}^q(M)$ we define their *tensor product* $T \otimes S \in \mathscr{T}^{p+q}(M)$ by

$$T \otimes S(\gamma) = T(\gamma \circ \lambda_p) \cdot S(\gamma \circ \rho_q),$$

where

$$\lambda_p(x_1, \dots, x_p) = (x_1, \dots, x_p, 0, \dots, 0)$$

is a map from D^p to D^{p+q} and

$$\rho_p(x_1, \dots, x_q) = (0, \dots, 0, x_1, \dots, x_p)$$

is a map from D^q to D^{p+q}.

It is straightforward to check that the tensor product is associative, bilinear and the natural generalization of the classical notion of tensor product (cf., e.g. [8, ch. 4]).

Moreover, this tensor product is natural, in the sense that for any $f:N \to M$, $T \in \mathscr{T}^p(M)$, $S \in \mathscr{T}^q(M)$, $f^*(T \otimes S) = f^*(T) \otimes f^*(S)$.

Let S_k be the group of permutations of $\{1,\dots,k\}$. For $\sigma \in S_k$, we define the permutation map $\hat{\sigma}: D^k \to D^k$ by

$$\hat{\sigma}(d_1,\dots,d_k) = (d_{\sigma(1)},\dots,d_{\sigma(k)}).$$

Then for any $T \in \mathscr{T}^k(M)$ we define the "alternation of T" to be the tensor $\mathrm{Alt}(T)$ defined by

$$\mathrm{Alt}(T)(\gamma) = \frac{1}{k!}\sum_{\sigma \in S_k} \mathrm{sgn}(\sigma) T(\gamma \circ \hat{\sigma}).$$

We then have

Proposition 3.1. (1) $T \in \mathscr{T}^k(M) \Rightarrow \mathrm{Alt}(T) \in \Lambda^k(M)$.
(2) $\omega \in \Lambda^k(M) \Rightarrow \mathrm{Alt}(\omega) = \omega$.
(3) $T \in \mathscr{T}^k(M) \Rightarrow \mathrm{Alt}(\mathrm{Alt}(T)) = \mathrm{Alt}(T)$.

Proof. Imitate the arguments given in [8]. □

For $\omega \in \Lambda^p(M)$, $\eta \in \Lambda^q(M)$, let $\omega \wedge \eta \in \Lambda^{p+q}(M)$ be defined by

$$\omega \wedge \eta = \frac{(p+q)!}{p!q!}\mathrm{Alt}(\omega \otimes \eta).$$

The following two results follow easily using arguments analogous to those in [8].

Proposition 3.2. (1) *The wedge product* $\wedge$ *is bilinear over* R.
(2) *The wedge product commutes with pullback: For* $f:N \to M$, $f^*(\omega \wedge \eta) = f^*\omega \wedge f^*\eta$.
(3) *The wedge product is graded commutative: For* $\omega \in \Lambda^p$, $\eta \in \Lambda^q$, $\omega \wedge \eta = (-1)^{pq}\eta \wedge \omega$.

Proposition 3.3. (1) *If* $S \in \mathscr{T}^p(M)$, $T \in \mathscr{T}^q(M)$, *and* $\mathrm{Alt}(S) = 0$, *then* $\mathrm{Alt}(S \otimes T) = \mathrm{Alt}(T \otimes S) = 0$.
(2) $\mathrm{Alt}(\mathrm{Alt}(S \otimes T) \otimes U) = \mathrm{Alt}(S \otimes T \otimes U) = \mathrm{Alt}(S \otimes \mathrm{Alt}(T \otimes U))$.
(3) *If* $\omega \in \Lambda^k(M), \eta \in \Lambda^l(M), \theta \in \Lambda^m(M)$, *then*

$$(\omega \wedge \eta) \wedge \theta = \omega \wedge (\eta \wedge \theta) = \frac{(k+l+m)!}{k!l!m!}\mathrm{Alt}(\omega \otimes \eta \otimes \theta).$$

While classically, the exterior derivative is often defined to be a graded derivation on $\Lambda^*(M)$, and one must then prove Stokes' theorem, synthetically, the exterior derivative has been defined so that (an infinitesimal version of) Stokes' theorem holds, and we must prove that it is a graded derivation.

Proposition 3.4. *For $\omega \in \Lambda^p(M)$, $\eta \in \Lambda^q(M)$,*

$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^p \omega \wedge d\eta.$$

Proof. We need to show that the left and right-hand sides of the equation above take the same values on any infinitesimal $(p+q+1)$-cube $\gamma : D^{p+q+1} \to M$. However, by the naturality of the wedge product and the exterior derivative, it suffices to pullback the forms to D^{p+q+1} via γ and check the result there. Indeed, for any $(p+q+1)$-form Ω, $\gamma^*(\Omega)(I) = \Omega(\gamma)$, where $I : D^{p+q+1} \to D^{p+q+1}$ is the identity $(p+q+1)$-cube.

So we may assume that ω and η are forms on D^{p+q+1}. Using the generalized Kock–Lawvere axiom, we can derive that, e.g.,

$$\omega = \sum \omega_{i_1 \cdots i_p}(x)\, dx^{i_1} \wedge \cdots \wedge dx^{i_p}$$

for $\omega_{i_1 \cdots i_p}(x)$ a first order polynomial in $x \in D^{p+q+1}$. Using the definition of d (or the facts that (1) such a formula defines a form on the classical $\mathbb{R}^{p+q+1}$ and (2) the synthetic d is an extension of the classical d) one can derive that

$$d\omega = \sum \frac{\partial \omega_{i_1 \cdots i_p}}{\partial x^j}(x)\, dx^j \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_p}.$$

The result now follows from the standard calculation. □

3.3. The amazing right adjoint

One of the features of synthetic differential geometry is that for any infinitesimal space S, the functor $(-)^S$ has a right adjoint $(-)_S$ (see [2, Appendix 4; 3, Axiom 3_k, p. 119, Axiom D, p. 214, Theorems III.8.4, III.9.4; 4]). Because an n-form on a space M is a map $\omega : M^{D^n} \to R$, we can associate to any such form ω its right adjunct $\hat{\omega} : M \to R_{D^n}$. Since the multilinearity and symmetry properties of ω are natural, there are subobjects $\Lambda^n \subset R_{D^n}$ with the property that for any n-form ω, $\hat{\omega} : M \to \Lambda^n$, and for any map $\hat{\omega} : M \to \Lambda^n \subset R_{D^n}$, its left adjunct $\hat{\omega} : M \to R_{D^n}$ is an n-form (see [3, I.20]).

Some simple consequences of the existence of this adjoint are

(1) $\omega = \eta$ if and only if $\hat{\omega} = \hat{\eta}$ (which is just the statement that the correspondence between Hom-sets is one-to-one), and

(2) If ω is an n-form on M, and $f : N \to M$ is a morphism (not just an element of the mapping space M^N – see below), then $\hat{\omega} \circ f = \hat{a}\omega \circ f^{D^n} = \hat{a} f^* \omega$ (which follows from the naturality of the adjoint).

As a result, if we consider (as we will later), a finite number of one-forms $\omega^1, \ldots, \omega^n$ on M, and consider their transforms $\hat{\omega}^j$, we can define a subobject F of M as the common zeroes of the $\hat{\omega}^j$, in other words the common equalizer of the maps $\hat{\omega}^j$ and $\hat{0}$ taking M to Λ^1. A map $f : N \to M$ then factors through F if and only if $\hat{\omega}^j \circ f = \hat{0} \circ f$ for all j if and only if $\widehat{f^* \omega^j} = \widehat{f^* 0}$ for all j if and only if $f^* \omega^j = 0$ for all j.

It should be noted that this amazing right adjoint (amazing because it does not happen in the category of sets) is not an internal adjoint, in the sense that, in general, the mapping spaces $M^{(N^S)}$ and $(M_S)^N$ are not isomorphic. For example, take $S = D$, $M = R$, $N = *$, the one-point space (the terminal object of the category). Then

$$M^{(N^S)} = R^{(*^D)} = R^* = R$$

while

$$(M_S)^N = (R_D)^* = R_D$$

and R_D is not isomorphic to R. For this reason, we shall usually work only with forms that are actual morphisms $M^{D^n} \to R$ rather than more general "forms" which would be elements of the mapping space $R^{(M^{D^n})}$.

3.4. Forms on infinitesimal spaces

It is possible to use the generalized Kock–Lawvere axiom to describe the differential forms on infinitesimal spaces. We shall describe this in what follows, limiting ourselves to the case of one-forms for simplicity.

We start by describing the tangent bundle S^D of an infinitesimal space

$$S = \{(x_1, \ldots, x_n) \in D_k(n) : p_1(x) = \cdots = p_l(x) = 0\}.$$

Since $S \subset R^n$, by applying the generalized Kock–Lawvere axiom component-wise, we obtain that, via the mapping

$$(x_1, \ldots, x_n, \dot{x}_1, \ldots, \dot{x}_n) \mapsto (d \mapsto (x_1 + d\dot{x}_1, \ldots, x_n + d\dot{x}_n))$$

we have an isomorphism

$$S^D \cong \left\{ \begin{array}{l} (x_1, \ldots, x_n, \dot{x}_1, \ldots, \dot{x}_n) \in R^n : (x_1, \ldots, x_n) \in S, \\ \displaystyle\sum_{j=0}^{k} x_{i_0} \cdots \dot{x}_{i_j} \cdots x_{i_k} = 0 \text{ for all } (k+1)\text{-tuples } (i_0, \ldots, i_k), \\ \displaystyle\sum_{j=1}^{n} \frac{\partial p_\alpha}{\partial x_j}(x)\dot{x}_j = 0 \end{array} \right\}.$$

We associate to S the infinitesimal space $S' = S^D \cap (S \times D(n))$.

Lemma 3.5. *If $\mu_1, \mu_2 : S^D \to R$ are linear maps and $\mu_1 = \mu_2$ on S', then $\mu_1 = \mu_2$. Thus one-forms on S are determined by their values on S'.*

Proof. For any $d \in D$,

$$d\mu_1(x, \dot{x}) = \mu_1(x, d\dot{x}) = \mu_2(x, d\dot{x}) = d\mu_2(x, \dot{x}).$$

Canceling the universally quantified d, we have the desired result. (In other words, consider both sides as functions of d. By the Kock–Lawvere axiom, the coefficients of d on either side must be equal.) □

The space S' is an infinitesimal space, so by the generalized Kock–Lawvere axiom, any $\mu : S' \to R$ has the form

$$\mu(x, \dot{x}) = \mu_0(x) + \sum_{j=1}^{n} \mu_1^j(x)\dot{x}_j,$$

where the right-hand side is a polynomial, unique modulo the polynomials

$$p_\alpha, \quad \sum_{j=0}^{k} x_{i_0} \cdots \dot{x}_{i_j} \cdots x_{i_k}, \quad \sum_{j=1}^{n} \frac{\partial p_\alpha}{\partial x_j}(x)\dot{x}_j. \tag{3.1}$$

Any linear $\mu : S^D \to R$ has the same form by the lemma. However, linearity also implies that $\mu_0 = 0$, so

$$\mu(x, \dot{x}) = \sum_{j=1}^{n} \mu_1^j(x)\dot{x}_j. \tag{3.2}$$

We have shown the following.

Proposition 3.6. *Any one-form on S has the form* (3.2), *where the right-hand side of* (3.2) *is a polynomial defined modulo the polynomials* (3.1).

3.5. Frobenius microlinearity

A concept of microlinearity we shall use later is defined using differential forms.

Definition 3.5. Let *Frob* be the class of finite co-cone diagrams $\mathscr{D}$ of infinitesimal spaces satisfying the following condition: given any finite set $\{\omega^1, \ldots, \omega^n\}$ of one-forms on the vertex S_0 of $\mathscr{D}$ such that

(1) The forms satisfy the Frobenius integrability condition: there exist forms ϕ_k^j such that $d\omega^j = \phi_k^j \wedge \omega^k$, and,

(2) All the forms vanish when pulled back to any of the other spaces in $\mathscr{D}$,

then the forms $\omega^1, \ldots, \omega^n$ themselves vanish.

A space M is said to be Frobenius microlinear if $M^{\mathscr{D}}$ is a limit diagram for any R-colimit $\mathscr{D} \in$ Frob.

It is possible to describe the forms on any infinitesimal space using the generalized Kock–Lawvere axiom (see above) and thereby show that $IL \subset$ Frob. Hence

microlinear $\Rightarrow$ Frobenius microlinear $\Rightarrow$ infinitesimally linear.

The assumption of Frobenius microlinearity, while strictly weaker than microlinearity (see below), is still quite strong. All of the diagrams used by Moerdijk and Reyes to develop the theory of connections [7, Ch. V] are elements of Frob. (Using the available description of one-forms on infinitesimal spaces it is fairly straightforward to tell whether any particular example of an R-colimit diagram is an element of Frob.)

4. A synthetic Frobenius theorem

In this section we will prove Theorems 1 and 2 and give an example to show that the space F defined in the introduction (and below) is not, in general, microlinear.

Let M be a microlinear space, $\omega^1,\dots,\omega^n$ one-forms on M, $\omega^i : M^D \to R$. Let $\hat{\omega}^i : M \to \Lambda^1 \subset R_D$ be the maps associated to the ω^i via the "amazing right adjoint". Let $F \hookrightarrow M$ be the zeroes of $\hat{\omega}^1,\dots \hat{\omega}^n$, i.e., the equalizer of $(\hat{\omega}^1,\dots,\hat{\omega}^n)$ and $(\hat{0},\dots,\hat{0}) : M \to \Lambda^1 \oplus \cdots \oplus \Lambda^1$. Let $\mathscr{F} \hookrightarrow M^D$ be the zeroes of $\omega^1,\dots,\omega^n$, i.e, the equalizer of $(\omega^1,\dots,\omega^n)$ and $(0,\dots,0) : M^D \to R \oplus \cdots \oplus R$.

Theorem 4.1. *If the ideal $(\omega^1,\dots,\omega^n)$ satisfies the Frobenius condition*

$$d\omega^i \equiv 0 \qquad \mathrm{mod}(\omega^1,\dots,\omega^n),$$

then $F^D \simeq F \times_M \mathscr{F}$. In other words, the tangent bundle to F is the restriction of $\mathscr{F}$ to F.

Proof. We will show that $\hom(A, F^D) \simeq \hom(A, F \times_M \mathscr{F})$, naturally in the arbitrary space A. The result then follows from Yoneda's lemma [5, p. 61].

First,

$$\begin{aligned}\hom(A,F^D) &\simeq \hom(A \times D, F)\\ &\simeq \{\varphi \in \hom(A \times D, M) : \hat{\omega}^j \circ \varphi = \hat{0} \circ \varphi\}\\ &\simeq \{\varphi \in \hom(A \times D, M) : \omega^j \circ \varphi^D = 0\}.\end{aligned}$$

Second, if $i_F : F \hookrightarrow M$ is inclusion and $\pi : \mathscr{F} \to M$ is projection (the restriction to $\mathscr{F}$ of the projection $M^D \to M$), then

$$\begin{aligned}\hom(A, F \times_M \mathscr{F}) &\simeq \{(\psi,\eta) \in \hom(A,F) \times \hom(A,\mathscr{F}) : i_F \circ \psi = \pi \circ \eta\}\\ &\simeq \{(\psi,\eta) \in \hom(A,M) \times \hom(A,M^D) : \psi = \pi \circ \eta,\\ &\qquad \hat{\omega}^i \circ \psi = \hat{0},\ \omega^i \circ \eta = 0\}\\ &\simeq \{\eta \in \hom(A,M^D) : \hat{\omega}^i \circ \pi \circ \eta = \hat{0},\ \omega^i \circ \eta = 0\}\\ &\simeq \{\eta \in \hom(A,M^D) : \omega^i \circ \pi^D \circ \eta^D = 0,\ \omega^i \circ \eta = 0\}.\end{aligned}$$

We shall analyze such η by considering the right adjunct $H \in \hom(A \times D, M)$ of η. Working now synthetically,

$$H(a,d) = \eta(a)(d).$$

The two conditions on η translate as follows:

$$\begin{aligned}\omega^i \circ \eta = 0 &\Leftrightarrow \omega^i(d \mapsto \eta(a)(d)) = 0 \quad \text{for all } a \in A\\ &\Leftrightarrow \omega^i(d \mapsto H(a,d)) = 0 \quad \text{for all } a \in A,\end{aligned} \tag{4.1}$$

$$\begin{aligned}\omega^i \circ \pi^D \circ \eta^D = 0 &\Leftrightarrow \omega^i(d \mapsto \pi(\eta(\gamma(d)))) = 0 \quad \text{for all } \gamma \in A^D\\ &\Leftrightarrow \omega^i(d \mapsto H(\gamma(d),0) = 0 \quad \text{for all } \gamma \in A^D.\end{aligned} \tag{4.2}$$

So let us consider the form $\mu^i = H^*\omega^i$. This is a form on $A \times D$. To study this let us examine first

$$\begin{aligned}(A \times D)^D &\simeq A^D \times D^D\\ &\simeq A^D \times \{(d,b) \in D \times R : d \cdot b = 0\}.\end{aligned}$$

The form $\mu^i : A^D \times D^D \to R$ is homogeneous of degree one. We claim that

$$\mu^i(\gamma,(d,b)) = \mu^i(\gamma,(d,0)) + \mu^i(\gamma(0),(d,b)),$$

where $\gamma \in A^D, (d,b) \in R$ with $db = 0$ and $\gamma(0) \in A^D$ is the constant map $d \mapsto \gamma(0)$. This follows from an argument similar to that used to prove [7, Proposition V.1.5]. It suffices to show that for all $\delta \in D$,

$$\delta\mu^i(\gamma,(d,b)) = \delta(\mu^i(\gamma,(d,0)) + \mu^i(\gamma(0),(d,b))),$$

or

$$\mu^i(\gamma,(d,\delta b)) = \mu^i(\delta\gamma,(d,0)) + \mu^i(\gamma(0),(d,\delta b)).$$

Consider $\varphi, \psi : D(2) \to R$ given, for fixed $\gamma,(d,b)$, by

$$\begin{aligned}\varphi(d_1,d_2) &= \mu^i(d_1\gamma,(d,d_2b)),\\ \psi(d_1,d_2) &= \mu^i(d_1\gamma,(d,0)) + \mu^i(\gamma(0),(d,d_2b)).\end{aligned}$$

If $i_1(d) = (d,0)$ and $i_2(d) = (0,d)$, then $\varphi_k = \psi_k, k = 1,2$. Since

$$\begin{array}{ccc} R^{D(2)} & \xrightarrow{R^{i_1}} & R^D \\ {\scriptstyle R^{i_2}}\downarrow & & \downarrow{\scriptstyle R^0} \\ R^D & \xrightarrow{R^0} & R \end{array}$$

is a pullback, $\varphi = \psi$. Therefore $\varphi(d,d) = \psi(d,d)$, which is what we wanted to show.

By the Kock–Lawvere axiom,

$$\mu^i(\gamma,(d,0)) = \mu^i(\gamma,(0,0)) + v^i(\gamma)d$$

for some $v^i : A^D \to R$.

We now wish to claim that

$$\mu^i(\gamma(0),(d,b)) = g^i(\gamma(0))b$$

for some $g^i : A \to R$. We consider the infinitesimal space $D(2)$. By the generalized Kock–Lawvere axiom, for $(d_1,d_2) \in D(2)$,

$$\mu^i(\gamma(0),(d_1,d_2)) = \mu^i(\gamma(0),(0,0)) + f^i(\gamma(0))d_1 + g^i(\gamma(0))d_2.$$

By the linearity of μ^i in d_2,

$$\begin{aligned}&\delta(\mu^i(\gamma,(0,0)) + f^i(\gamma(0))d_1 + g^i(\gamma(0))d_2)\\&\quad = \mu^i(\gamma,(0,0)) + f^i(\gamma(0))d_1 + \delta g^i(\gamma(0))d_2\end{aligned}$$

for all $\delta \in D$. It follows that $\mu^i(\gamma,(0,0)) = 0$ and $f^i(\gamma(0)) = 0$. But then, for all $\delta \in D$,

$$\begin{aligned}\delta\mu^i(\gamma(0),(d,b)) &= \mu^i(\gamma(0),(d,\delta b))\\&= g^i(\gamma(0))\delta b = \delta(g^i(\gamma(0))b)\end{aligned}$$

and so by the Kock–Lawvere axiom,

$$\mu^i(\gamma(0),(d,b)) = g^i(\gamma(0))b.$$

Thus

$$\mu^i(\gamma,(d,b)) = \mu^i(\gamma,(0,0)) + v^i(\gamma)d + g^i(\gamma(0))b.$$

By (4.1), $0 = \mu^i(a,(0,1)) = g^i(a)$ for all $a \in A$. By (4.2), $0 = \mu^i(\gamma,(0,0))$. Thus

$$\mu^i(\gamma,(d,b)) = v^i(\gamma)d.$$

But consider the infinitesimal two-cube

$$\tau : (d_1,d_2) \mapsto (\gamma(d_1),d_2) \in A \times D.$$

By the definition of $d\mu^i$ [7, p. 136]

$$\begin{aligned}\int_{(\tau,h_1,h_2)} d\mu^i &= \int_{(\tau(\cdot,0),h_1)} \mu^i + \int_{(\tau(h_1,\cdot),h_2)} \mu^i - \int_{(\tau(\cdot,h_2),h_1)} \mu^i + \int_{(\tau(0,\cdot),h_2)} \mu^i\\&= h_1\mu^i(d \mapsto (\gamma(d),0)) + h_2\mu^i(d \mapsto (\gamma(h_1),d))\\&\quad - h_1\mu^i(d \mapsto (\gamma(d),h_2)) + h_2\mu^i(d \mapsto (\gamma(0),d)).\end{aligned}$$

By (4.1), the second and fourth terms vanish. By (4.2), the first term vanishes. Finally, the third term is $-h_1h_2v^i(\gamma)$.

On the other hand, by the integrability assumption,

$$d\mu^i = \psi^i_j \wedge \mu^j$$

for some forms ψ_j^i. Then

$$\int_{(\tau,h_1,h_2)} d\mu^i = h_1h_2\mu^i(\tau) = h_1h_2\psi_j^i \wedge \mu^j(\tau)$$
$$= h_1h_2(\psi_j^i(\Gamma)\mu^j(\Delta) - \psi_j^i(\Delta)\mu^j(\Gamma)),$$

where $\Gamma(d) = \tau(d,0), \Delta(d) = \tau(0,d)$. But (4.1) implies that $\mu^j(\Gamma) = 0$, and (4.2) implies that $\mu^j(\delta) = 0$, so the integral vanishes and we have $v^i(\gamma) = 0$.

Thus $\mu^i = 0$ and

$$\hom(A, F \times_M \mathscr{F}) \simeq \{H \in \hom(A \times D, M) : H^*\omega^i = 0\}$$
$$\simeq \hom(A, F^D). \qquad \Box$$

We now wish to investigate the linearity properties of F. Let $\mathscr{D}$ be a finite cocone of infinitesimal spaces $S_0, \ldots, S_k$ with vertex S_0, and suppose that $R^{\mathscr{D}}$ is a limit diagram. We wish to show that $F^{\mathscr{D}}$ is a limit diagram. So let A be an object and $f_j \in \hom(A, F^{S_j})$, $j = 1, \ldots, k$ be maps so that for all arrows $\varphi : S_i \to S_j, i,j \geq 1$ in $\mathscr{D}$, $F^{\varphi} \circ f_i = f_j$. We wish to show that there is a unique $f_0 \in \hom(A, F^{S_1})$ such that $F^{\psi_i} \circ f_0 = f_i, i \geq 1$, where $\psi_i : S_i \to S_0$ is the arrow to the vertex in $\mathscr{D}$. But

$$\hom(A, F^{S_j}) \simeq \hom(A \times S_j, F)$$
$$\simeq \{\psi \in \hom(A, M^{S_j}) : \omega^i(d \mapsto \psi(\alpha(d))(\sigma(d))) = 0$$
$$\forall i, \forall \alpha \in A^D, \forall \sigma \in S_j^D\}.$$

We are assuming that we have $f_j \in \hom(A, F^{S_j})$ that commute appropriately, therefore we have $\psi_j \in \hom(A, M^{S_j})$ that commute appropriately. Therefore, if $M^{\mathscr{D}}$ is a limit diagram, we have a unique $\psi_0 \in \hom(A, M^{S_0})$ commuting appropriately. We need to determine whether

$$\omega^i(d \mapsto \psi(\alpha(d))(\sigma(d))) = 0 \qquad \forall i, \forall \alpha \in A^D, \quad \forall \sigma \in S_j^D.$$

Lemma 4.2. *Let $H : V_1 \times V_2 \to W$ be a map of spaces such that*

(1) *V_1 and V_2 are acted on multiplicatively by R,*

(2) *W is a microlinear R-module satisfying the Kock–Lawvere axiom,*

(3) *H is homogeneous of degree one.*

Then $H(v_1, v_2) = H(v_1, 0) + H(0, v_2)$ for all $v_1 \in V_1, v_2 \in V_2$.

Proof. Fix $v_1 \in V_1, v_2 \in V_2$. Consider

$$\varphi(d_1, d_2) = H(d_1v_1, d_2v_2),$$

$$\psi(d_1, d_2) = H(d_1v_1, 0) + H(0, d_2v_2).$$

Both $\varphi, \psi : D(2) \to W$. Since $\varphi(d,0) = \psi(0,d)$ and $\varphi(0,d) = \psi(0,d)$, by the microlinearity of W, $\varphi = \psi$. Therefore, for all $d \in D$,

$$\begin{aligned} dH(v_1, v_2) &= H(dv_1, dv_2) \\ &= \varphi(d,d) = \psi(d,d) \\ &= H(dv_1, 0) + H(0, dv_2) \\ &= d(H(v_1, 0) + H(0, v_2)). \end{aligned}$$

We therefore have the result by applying the Kock–Lawvere axiom on W. □

Since $V_1 = A^D, V_2 = S_j^D, W = R$ and $H : V_1 \times V_2 \to R$ given by

$$H(\alpha, \sigma) = \omega^i(d \mapsto \psi_0(\alpha(d))(\sigma(d)))$$

satisfy the hypotheses of the lemma, we have

$$\omega^i(d \mapsto \psi(\alpha(d))(\sigma(d))) = \omega^i(d \mapsto \psi(\alpha(d))(\sigma(0))) + \omega^i(d \mapsto \psi(\alpha(d))(\sigma(d))).$$

Let us first consider $\omega^i(d \mapsto \psi(\alpha(d))(\sigma(0)))$. If we fix α, we obtain a map $S_0 \to R$

$$s \mapsto \omega^i(d \mapsto \psi(\alpha(d))(s)).$$

We have corresponding maps $S_j \to R$, all commuting nicely. Since $\mathscr{D}$ is an R-colimit, and the maps $S_j \to R$ are all zero, the map $S_0 \to R$ must also be zero.

We have left to consider the one-forms

$$\sigma \mapsto \omega^i(d \mapsto \psi(a)(\sigma(d))).$$

What we would like to show is that for any R-colimit diagram $\mathscr{D}$, if a one-form μ on S_0 pulls back to the zero one-form on each of the S_j, then $\mu = 0$. This of course is not true in this generality. (Notice that we have not mentioned the Frobenius integrability condition.) It is useful to look at the following.

4.1. Example 1

Consider the R-colimit

$$\begin{array}{ccc} D(2) & \xleftarrow{(d,0)} & D \\ \uparrow{\scriptstyle (0,d)} & & \uparrow{\scriptstyle 0} \\ D & \xleftarrow{0} & 0 \end{array}$$

Using the generalized Kock–Lawvere axiom it is easy to see that the one-forms on these spaces have the following form:

(1) On D, $a\,dx$, for some $a \in R$;

(2) On $D(2)$, $\mu = a\,dx + b\,dy + c(x\,dy - y\,dx)$ for some $a, b, c \in R$.

The form μ will pullback to zero if and only if $a=b=0$. So, clearly, there are non-zero forms which get pulled back to zero. However, notice that, on $D(2)$, $d\mu = c\,dx\wedge dy$, so that if $d\mu=\eta\wedge\mu$ for some form η, then $c=0$ (just equate the forms at $(x,y)=(0,0)$).

Indeed, for any finite number of forms μ^i on $D(2)$ satisfying the Frobenius integrability conditions, if they all pull back to zero, they must themselves vanish. A similar calculation works for the n-dimensional generalization of this diagram, and as a result we have the following.

Theorem 4.3. *If M is an infinitesimally linear space, and there are one-forms $\omega^1,\dots,\omega^n$ on M such that the ideal $(\omega^1,\dots,\omega^n)$ satisfies the Frobenius condition*

$$d\omega^i \equiv 0 \mod(\omega^1,\dots,\omega^n),$$

then the subobject $F=\{x\in M:\hat{\omega}^1(x)=\cdots=\hat{\omega}^n(x)=\hat{0}\}$ is infinitesimally linear.

However, even with the Frobenius integrability condition thrown in, the result is not true in general.

4.2. Example 2

Let

$$S=\{(x,y)\in D_5(2): 5x^4+4x^3y=x^4+4y^3=xy^4=y^5=0\}$$

and define $\varphi: S\to D$ by

$$\varphi(x,y)=x^5+x^4y+y^4.$$

The polynomials defining S were picked here so that $\varphi^*(dt)=d\varphi=0$. On the other hand, φ will be the zero function if and only if it lies in the ideal generated by the four polynomials defining S and the monomials of degree 6. An elementary calculation shows that this is not so. We now let

$$S' = \{(x,y)\in D_5(2): 5x^4+4x^3y=x^4+4y^3=xy^4 = y^5=x^5+x^4y+y^4=0\}$$

and let $i:S'\to S$ be the inclusion mapping. The reader may check that

$$S'\times S' \underset{i\circ\pi_2}{\overset{i\circ\pi_1}{\rightrightarrows}} S \xrightarrow{\varphi} D$$

is an R-coequalizer. (Perhaps the most difficult thing to check here is that the map $R^D\to R^S$ is injective. That, however, is just the statement that $\varphi\neq 0$.)

We now note that, provided that not all of the forms $\omega^1,\dots,\omega^n$ vanish, F is not microlinear. Indeed, consider the putative equalizer

$$F^D\to F^S \rightrightarrows F^{S'\times S'}.$$

Find a map $X: D \to M$ so that some $X^*(\omega^i) \neq 0$. (This can be accomplished by simply taking a point $X \in M^D$ at which $\omega^i(X) \neq 0$. Then $X^*(\omega^i)$ will be non-zero on $T_0 D$.) Then $i^*X : S \to F$, and if M is microlinear, i^*X has a unique lifting to a map $D \to M$, i.e., the map X. However, X does not map into F, hence F is not microlinear.

However, the arguments above do show the following.

Theorem 4.4. *Let M be a Frobenius microlinear space. Suppose that there are one-forms $\omega^1, \dots, \omega^n$ on M such that the ideal $(\omega^1, \dots, \omega^n)$ satisfies the Frobenius condition*

$$d\omega^i \equiv 0 \quad \mathrm{mod}(\omega^1, \dots, \omega^n),$$

then the subobject $F = \{x \in M : \hat{\omega}^1(x) = \cdots = \hat{\omega}^n(x) = \hat{0}\}$ is Frobenius microlinear.

It is worth noting that all of the diagrams used by Moerdjik and Reyes in discussing connections [7, Ch. V] are Frobenius diagrams. I certainly would be surprised to find that an example as bizarre as that of Example 2 above has geometric interest.

5. Possible applications of synthetic differential geometry to partial differential relations

My original motivation for studying synthetic differential geometry was a desire to use it to study Gromov's h-principle. In this Section I wish to describe some of the possibilities and difficulties involved. Even more so than in Section 1, there will be little that is precise. For details on the h-principle, see [2]. For details on the contact ideal and the differential forms approach to partial differential equations, see [1].

Let M and X be manifolds, and $\pi: X \to M$ a smooth map making X into a fiber bundle over M. Let $J^r(M,X)$ be the bundle of r-jets of sections of X. Heuristically, the fiber of $J^r(M,X)$ over a point $m \in M$ consists of all possible values of all derivatives up to order r of a section of X at m. Thus, one way of getting a point of $J^r(M,X)$ is to take a (local) section f of X and evaluate it and all its derivatives up to order r at m. The point $J^r f$ obtained thereby is called the jet of f at m. A partial differential relation is a subset $\mathscr{R}$ of $J^r(M,X)$. A strong solution of the partial differential relation $\mathscr{R}$ is a section σ of $\mathscr{R}$ which is the jet of a section of X, i.e., there should be a section f of X so that for all $m \in M$, $J^r f(m) = \sigma(m)$. (This is probably what most people would think of when thinking of what a "solution"of the partial differential relation ought to be.) A weak solution of $\mathscr{R}$ is a section of $\mathscr{R}$. (This has little and probably nothing to do with the analytical notion of "weak solution" that has to do with distributions.) The partial differential relation $\mathscr{R}$ is said to satisfy the h-principle if every weak solution is homotopic to a strong solution (through sections of $\mathscr{R}$). Thus a partial differential relation which satisfies the h-principle has a solution if and only if the space $\mathscr{R}$ over M has a section, reducing the solving of a partial differential relation to "mere topology". The problem is to identify those partial differential relations which satisfy the h-principle and those that do not, and, more generally, to find some measure of the extent to which certain partial differential relations (or classes of partial differential relations) do or do not satisfy the h-principle.

To study the h-principle, one needs a way of telling when a section of $\mathscr{R}$ is a strong solution, i.e., when it is the jet of a section of X. For smooth sections of $\mathscr{R}$ this can be done by considering the contact ideal. The contact ideal $\mathscr{I}$ is an ideal of differential forms defined on the jet bundle. It has the property that a section σ of the jet bundle is the jet of a section of X if and only if $\sigma^*\mathscr{I}$ is the zero ideal. The contact ideal is generated (as an ideal closed under exterior differentiation) by finitely many one-forms. Restricting these to $\mathscr{R}$, and using the amazing right adjoint, we get a map $\mathscr{R} \to \bigoplus \Lambda^1$. The zeroes of this map give us a subobject $\mathscr{S} \subset \mathscr{R}$ with the property that a section of $\mathscr{R}$ is a strong solution if and only if it factors through the inclusion $\mathscr{S} \subset \mathscr{R}$. The h-principle is satisfied if and only if every section of $\mathscr{R}$ is homotopic to a section of $\mathscr{S}$. One can then hope to apply homotopy theoretic methods directly to $\mathscr{R}$ and its "subset" $\mathscr{S}$ in order to study the h-principle.

One difficulty is that $\mathscr{S} \xrightarrow{\pi} M$ is, in general, not a fibration. Consider the following example. Let $M = \mathbb{R}$, $X = \mathbb{R}^2$ and $\pi =$ the standard projection. Then the jet bundle $J^1(M,X)$ is naturally identified with $\mathbb{R}^3$ with coordinates (x, y, p). (x being the coordinate on $M = \mathbb{R}$, (x, y) the coordinates on $X = \mathbb{R}^2$, and p being the value of dy/dx.) The contact ideal is then generated by the one-form $\omega = dy - p\,dx$. We let $\mathscr{R}$ be all of $J^1(M,X)$. Then the map $f : \xi \mapsto (0, 0, \xi)$ maps $\mathbb{R}$ to $\mathscr{S}$, since $f^*\omega = 0$. The map $\pi \circ f$ is the constant map to $0 \in \mathbb{R}$. The homotopy of this constant map which at time t is the constant map to $t \in \mathbb{R}$ does not lift to a homotopy H of f in $\mathscr{S}$, since any such lift would have $H^*(d\omega) \neq 0$, since $d\omega = dp \wedge dx$ and the image of the lift must have $dp \equiv d\xi \mod (dt)$ and $dx = dt$ at time zero. (Note that ω vanishes on a section of $\mathscr{S}$, and therefore so does $d\omega$.) Thus, even in this simple case, $\mathscr{S}$ is not a fibration. Thus the usual obstruction theory does not apply directly to the question of the existence of sections of $\mathscr{S}$.

Indeed, the usual methods for analyzing such questions run into another problem. Most such analysis seems to be based on cellular complexes. One fact that gets used in such analysis is that if $A, B \subset X$ are closed subsets, and $f : A \to Y$ and $g : B \to Y$ are continuous maps which agree on $A \cap B$, then there is a unique continuous map $h : A \cup B \to Y$ whose restriction to A is f and whose restriction to B is g. This enables one to examine questions of existence of continuous maps cell by cell. However, if we replace "continuous" by "smooth", the result is not true. The difficulty is that though there will be a continuous h, there is no guarantee that h will be smooth – its derivatives may not match up on the overlap.

This problem we may be able to get around. Instead of building a complex out of cells, we build a complex out of "germ neighborhoods" of cells. Fix the dimension n of the total complex (the dimension of the base M), and consider the space which is not the k-cell, but rather corresponds to the C^∞-ring $C^\infty(\mathbb{R}^n)/\mathscr{I}$, where $\mathscr{I}$ is the ideal of functions vanishing in some neighborhood of the k-cell $\Sigma^k \subset \mathbb{R}^n$. This germ-neighborhood will have "boundary" a germ-neighborhood of the $(k-1)$-sphere. The point is that smooth functions which agree on a germ-neighborhood of the intersection will patch together to form a smooth function. One can then work with "germ CW-complexes" made up by patching such fuzzy cells together. Much of the analysis should go through, replacing the usual fundamental groups by corresponding objects defined via these germ-neighborhoods of cells.

Thus, although the most naive of approaches fails, there is still some hope that synthetic methods may be used to analyze the global behavior of solutions of partial differential relations.

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