Journal of Pure and Applied Algebra 28 (1983) 189–196
North-Holland Publishing Company

A TEST FOR THE FUNDAMENTAL GROUP OF A 3-MANIFOLD

Akio KAWAUCHI
Department of Mathem, *Osaka City University, Sugimoto-cho, Osaka, Japan*

Communicated by H. Bass
Received 8 November 1982

It is well known that a finitely presented group is necessarily isomorphic to the fundamental group of a closed orientable n-manifold for each $n \geq 4$. On the contrary, it is not necessarily isomorphic to the fundamental group of a compact 3-manifold. It is a difficult problem to determine when a given finitely presented group is isomorphic to the fundamental group of a compact 3-manifold. For example, it is known that there is no algorithm which decides whether or not finitely presented groups are isomorphic to the fundamental groups of compact 3-manifolds (See Lyndon–Schupp [15, p. 192]). (This fact was suggested by González–Acuña to the author.) The purpose of this paper is to present a method of testing that a finitely presented group with an element of infinite order is *not* isomorphic to the fundamental group of any compact 3-manifold. Similar questions were considered by Heil in [6] and Jaco in [8] and [9], using other methods. Our method is based upon the theory of infinite cyclic covering spaces. This method has its applications in knot theory (cf. Blanchfield [2], Milnor [17], Farber [4] and [11], [12], [13]). We summarize our main idea here. If G is the group in question, the criterion in the case of an orientable 3-manifold is that any Alexander module produced in G is self-reciprocal, and the criterion in the case of a non-orientable 3-manifold is that there is an index 2 subgroup G' of G such that any Alexander module produced in G' is self-reciprocal. This is a generalized revised version of the author's earlier arguments [13, Application 1] where an incorrect theorem, 'Theorem A', was claimed. ["We may consider..." (p. 194, line 32) is false.] The example following after 'Theorem A' can be proved by the present method, but we omit the proof since it is a well-known fact by Heil [6] and Jaco [8, 10].

In Section 1 we define and study a module induced from a group with an infinite cyclic quotient group. In Section 2 we discuss two special kinds of modules, called Alexander modules and self-reciprocal modules. In Section 3 we discuss some properties of the fundamental groups of 3-manifolds which are useful for our purpose. In Section 4 we state and prove our main theorem giving a necessary condition for a group to be the fundamental group of a 3-manifold. Also, we give there a plan for our test and one example.

0022-4049/83/$03.00

1. A module induced from a group with an infinite cyclic quotient group

Let $\langle t \rangle$ be the infinite cyclic group generated by a letter t. Let $Z\langle t \rangle$ be its integral group ring. We consider a group K with an epimorphism $\gamma : K \to \langle t \rangle$. Let $\tilde{K}$ be the kernel of γ. The lemma which follows shows how the integral homology group $H_1(\tilde{K}; Z)$ has a natural $Z\langle t \rangle$-module structure, called the *module induced from K by γ*.

Consider a presentation $(x_0, x_1, \ldots, x_n \mid r_1, \ldots, r_m)$ $(n \leq +\infty, m \leq +\infty)$ of K such that $\gamma(x_0) = t$ and $\gamma(x_j) = 1$ for all $j \geq 1$. Let $\{r_1^*, \ldots, r_m^*\}$ freely generate $\bigoplus_m Z\langle t \rangle$ and let $\{x_0^*, x_1^*, \ldots, x_n^*\}$ freely generate $\bigoplus_{n+1} Z\langle t \rangle$. Define a $Z\langle t \rangle$-sequence

$$\bigoplus_m Z\langle t \rangle \xrightarrow{d_2} \bigoplus_{n+1} Z\langle t \rangle \xrightarrow{d_1} Z\langle t \rangle$$

by $d_2(r_i^*) = \sum_{j=0}^n (\partial r_i / \partial x_j)^\gamma x_j^*$ and $d_1(x_j^*) = \gamma(x_j) - 1$. Then $d_1 d_2 = 0$ because $r_i - 1 = \sum_{j=0}^n (\partial r_i / \partial x_j)(x_j - 1)$ in the integral group ring of the free group generated by $\{x_0, x_1, \ldots, x_n\}$ (cf. Fox [5]). By our choice of a presentation of K, we have $d_1(x_0^*) = t - 1$ and $d_1(x_j^*) = 0$ for all $j \geq 1$. Since $d_1 d_2 = 0$, it follows that $(\partial r_i / \partial x_0)^\gamma = 0$ and $\ker d_1 = \bigoplus_n Z\langle t \rangle$, where x_i^* generates the i-th free factor, $1 \leq i \leq n$. In particular, d_2 determines a map

$$d_2' : \bigoplus_m Z\langle t \rangle \to \bigoplus_n Z\langle t \rangle,$$

where $d_2'(r_i^*) = \sum_{j=1}^n (\partial r_i / \partial x_j)^\gamma x_j^*$. By Crowell [3, p. 39], $H_1(\tilde{K}; Z)$ is $Z\langle t \rangle$-isomorphic to $\operatorname{Ker} d_1 / \operatorname{Im} d_2 = \bigoplus_n Z\langle t \rangle / \operatorname{Im} d_2'$. Let J be the $m \times n$ matrix whose $(i-j)$-th entry is $(\partial r_i / \partial x_j)^\gamma$, where $1 \leq i \leq m$, $1 \leq j \leq n$. Then we have proved the following:

Lemma 1.1. *The matrix J is a $Z\langle t \rangle$-presentation matrix of the $Z\langle t \rangle$-module $H_1(\tilde{K}; Z)$. In other words, there is a $Z\langle t \rangle$-exact sequence*

$$\bigoplus_m Z\langle t \rangle \xrightarrow{J} \bigoplus_n Z\langle t \rangle \to H_1(\tilde{K}; Z) \to 0.$$

Corollary 1.2. *If K is finitely generated, then $H_1(\tilde{K}; Z)$ is finitely generated as a $Z\langle t \rangle$-module.*

Lemma 1.3. *If K is finitely generated and $H_1(K; Q) \cong Q$, then $H_1(\tilde{K}; Q)$ is a finitely generated torsion $Q\langle t \rangle$-module.*

Proof. We see from Corollary 1.2 that $H_1(\tilde{K}; Q)$ is finitely generated over the principal ideal domain $Q\langle t \rangle$. Then if $H_1(K; Q) \cong Q$, Milnor [17, the proof of Assertion 5] shows that $H_1(\tilde{K}; Q)$ is a torsion $Q\langle t \rangle$-module. This completes the proof.

2. Alexander modules and self-reciprocal modules

For a $Z\langle t\rangle$-module T, we denote the $Q\langle t\rangle$-module $T\otimes_Z Q$ and the $Z_p\langle t\rangle$-module $T\otimes_Z Z_p$ by T_Q and T_p, respectively, where Z_p is the field of prime order p. The integral torsion product $\mathrm{Tor}_Z(T, Z_p)=\{x\in T \mid px=0\}$ is denoted by $T^{(p)}$. Then $T^{(p)}$ forms a $Z_p\langle t\rangle$-module.

The following is easily proved:

Lemma 2.1. *A $Z\langle t\rangle$-module T is a torsion $Z\langle t\rangle$-module if and only if T_Q is a torsion $Q\langle t\rangle$-module.*

Definition 2.2. A finitely generated torsion $Z\langle t\rangle$-module is called an *Alexander module.*

Combining Lemma 2.1 with Corollary 1.2 and Lemma 1.3, we obtain the following:

Lemma 2.3. *For a finitely generated group K with $H_1(K; Q)\cong Q$, the $Z\langle t\rangle$-module $H_1(\tilde{K}; Z)$ is an Alexander module.*

For example, consider the knot group $K=\pi_1(S^3-k)$ of a tame knot k in a 3-sphere S^3. Since K is finitely presented and $H_1(K; Z)\cong Z$, we see from Lemma 2.3 that the knot module $H_1(\tilde{K}; Z)$ is an Alexander module. The Alexander module was named after J.W. Alexander, who introduced these ideas in [1].

Let R be a commutative ring with a unit. Let T be an $R\langle t\rangle$-module. If $f(t)\in R\langle t\rangle$, $x\in T$, define $f(t)\cdot x=f(t^{-1})x$. This gives T the structure of an $R\langle t\rangle$-module in a second way. Denote it by T^*.

Definition 2.4. A finitely generated $Z\langle t\rangle$-module T is said to be *self-reciprocal* if:
(i) $T_Q\cong T_Q^*$ as $Q\langle t\rangle$-modules,
(ii) $T^{(p)}\cong \mathrm{Hom}_{Z_p\langle t\rangle}[T_p, Z_p\langle t\rangle]^*$ as $Z_p\langle t\rangle$-modules for all p.

For an Alexander module T, let $A(t)\in Q\langle t\rangle$ be the characteristic polynomial of $t: T_Q\to T_Q$. We call $A(t)$ up to unit multiples of $Q\langle t\rangle$ the *Alexander polynomial* of the Alexander module T. Since $Q\langle t\rangle$ is a principal ideal domain, $A(t)$ is a generator of the order ideal of a cyclic $Q\langle t\rangle$-splitting of T_Q (cf. Lang [14, p. 401]).

The following lemma is easily proved.

Lemma 2.5. *For an Alexander module T with property* (i), *the Alexander polynomial $A(t)$ is self-reciprocal, i.e., $A(t)=uA(t^{-1})$ for some unit $u\in Q\langle t\rangle$.*

Using that $Z_p\langle t\rangle$ is a principal ideal domain and T_p is finitely generated over $Z_p\langle t\rangle$, we see the following:

Lemma 2.6. *For an Alexander module T with property* (ii), *$T^{(p)}$ is a free $Z_p\langle t\rangle$-module having the same finite $Z_p\langle t\rangle$-rank as T_p. In particular, $T^{(p)}$ is infinite or trivial as an abelian group.*

Corollary 2.7. *For a finitely generated group K with $H_1(K;Z)\cong Z$, assume that $H_1(\tilde{K};Z)$ has the property* (ii). *Then $H_1(\tilde{K};Z)$ is a torsion-free abelian group.*

Proof. By Lemma 2.3, $H_1(\tilde{K};Z)$ is an Alexander module. Since $H_1(K;Z_p)\cong Z_p$, Milnor [17, Assertion 5] shows that $H_1(\tilde{K};Z_p)=H_1(\tilde{K};Z)_p$ is a torsion $Z_p\langle t\rangle$-module. By Lemma 2.6, $H_1(\tilde{K};Z)^{(p)}=0$ for all prime p. This implies that $H_1(\tilde{K};Z)$ is a torsion-free abelian group. This completes the proof.

3. Some properties of the fundamental groups of 3-manifolds

Hempel's book [7] and Jaco's book [9] are useful for general references in this section. Unless otherwise stated, 3-manifolds will be assumed to be connected piecewise-linear 3-manifolds with or without boundary.

3.1. Any subgroup G of the fundamental group $\pi_1(M)$ of a 3-manifold M is the fundamental group of a 3-manifold M', namely the covering space of M belonging to G. Moreover, if G is finitely generated, then M' is compact and G is finitely presented. Also, if M is orientable, then M' is also orientable. (See Hempel [7, Chapter 8], Jaco [9, Chapter V].)

Lemma 3.2. *If $G=\pi_1(M)$ is a finitely generated infinite group, then G has an element of infinite order.*

Proof. Assume G is a finitely generated torsion group. Then it suffices to show that G is finite. By 3.1 we may assume M is compact. By considering, if necessary, an index 2 subgroup in place of G, we may further assume that M is orientable. Then since $H_1(M;Z)=H_1(G;Z)$ is finite, we may assume that M is a closed orientable 3-manifold. Now by the sphere theorem, we have $\pi_2(M)=0$, because any non-trivial free product has an element of infinite order. To show that G is finite, suppose G is infinite. Then M is a $K(G,1)$-space and hence G is torsion-free (cf. [7, Chapter 9]), which is a contradiction. This completes the proof.

Let M be a compact oriented 3-manifold with an epimorphism $\gamma:\pi_1(M)\to\langle t\rangle$. Let $\tilde{M}$ be the covering space of M belonging to Ker γ, called the *infinite cyclic covering space* associated with γ. The covering transformation group of $\tilde{M}$ is identified with $\langle t\rangle$. The homology group $H_1(\tilde{M};Z)$ has the structure of a finitely generated $Z\langle t\rangle$-module.

Theorem 3.3. *Assume* $\dim_Q H_1(\tilde{M}; Q) < +\infty$. *Then the* $Z\langle t\rangle$*-module* $H_1(\tilde{M}; Z)$ *is self-reciprocal.*

Proof. After easy modifications of M, we can assume that ∂M is non-empty and contains no 2-sphere as a component. By [11], $H_2(\tilde{M}, \partial\tilde{M}; Q) \cong H^0(\tilde{M}; Q) = Q$, since $\dim_Q H_1(\tilde{M}; Q) < +\infty$. So, $\dim_Q H_1(\partial\tilde{M}; Q) < +\infty$. This implies that $\partial\tilde{M}$ is the disjoint union of finite copies of $S^1 \times R^1$ (cf. Milnor [17, Assertion 6]). Let $I = \mathrm{Im}[i_*: H_1(\partial\tilde{M}; Q) \to H_1(\tilde{M}; Q)]$ and $H = H_1(\tilde{M}; Q)/I$. Consider the $Q\langle t\rangle$-primary splittings $I = \bigoplus_q C_q$ and $H = \bigoplus_q H_q$, where $q = q(t)$ ranges over all irreducible polynomials of $Q\langle t\rangle$ up to unit multiples. For any q with $I_q \neq 0$ we see that q is a self-reciprocal polynomial, since q must be a factor of some $t^n - 1$. Then the natural epimorphism $H_1(\tilde{M}; Q) \to H$ induces a $Q\langle t\rangle$-isomorphism $C_q \cong H_q$ for all non-self-reciprocal polynomials q, for $I \cap C_q = 0$. Using the cohomology exact sequence of $(\tilde{M}, \partial\tilde{M})$, we obtain the following composite $Q\langle t\rangle$-isomorphism:

$$H \cong \mathrm{Hom}_Q[H, Q] \cong \mathrm{Ker}[i^*: H^1(\tilde{M}; Q) \to H^1(\partial\tilde{M}; Q)]$$
$$\cong H^1(\tilde{M}, \partial\tilde{M}; Q)/\mathrm{Im}[\delta: H^0(\partial\tilde{M}; Q) \to H^1(\tilde{M}, \partial\tilde{M}; Q)].$$

Then from [13, Corollary 3.5] we can see that $H \cong H^*$ (cf. Blanchfield [2], [12, 2.8]). This implies that for any non-self-reciprocal $q, C_{q(t)} \cong (C_{q(t^{-1})})^*$. For any self-reciprocal $q, C_{q(t)} = C_{q(t^{-1})} \cong (C_{q(t)})^*$. Therefore, we have a $Q\langle t\rangle$-isomorphism $H_1(\tilde{M}; Q) \cong H_1(\tilde{M}; Q)^*$. Next, since $\dim_Q H_1(\tilde{M}, \partial\tilde{M}; Q) < +\infty$, we see from [11] that $H_2(\tilde{M}; Z) \cong H^0(\tilde{M}, \partial\tilde{M}; Z) = 0$. Then by the universal coefficient theorem, $H_2(\tilde{M}; Z_p) = H_1(\tilde{M}; Z)^{(p)}$ for all prime p. By [13, Duality Theorem (II)], $\mathrm{Tor}_{Z_p\langle t\rangle} H_2(\tilde{M}; Z_p) \cong \mathrm{Tor}_{Z_p\langle t\rangle} H_0(\tilde{M}, \partial\tilde{M}; Z_p)^* = 0$, for $H_0(\tilde{M}, \partial\tilde{M}; Z_p) = 0$. That is, $H_2(\tilde{M}; Z_p)$ is a free $Z_p\langle t\rangle$-module. Using that $H_*(\partial\tilde{M}; Z_p)$ is a torsion $Z_p\langle t\rangle$-module, we see that

$$H_1(\tilde{M}; Z)^{(p)} = H_2(\tilde{M}; Z_p) \cong H_2(\tilde{M}, \partial\tilde{M}; Z_p)/Z_p\langle t\rangle\text{-torsion}.$$

By [13, the proof of Duality Theorem (I)],

$$\mathrm{Hom}_{Z_p\langle t\rangle}[H_1(\tilde{M}; Z_p), Z_p\langle t\rangle]^* \cong \mathrm{Hom}_{Z_p\langle t\rangle}[H_1(\tilde{M}; Z_p)/Z_p\langle t\rangle\text{-torsion}, Z_p\langle t\rangle]^*$$
$$\cong H_2(\tilde{M}, \partial\tilde{M}; Z_p)/Z_p\langle t\rangle\text{-torsion}.$$

Thus, we have a $Z_p\langle t\rangle$-isomorphism $H_1(\tilde{M}; Z)^{(p)} \cong \mathrm{Hom}_{Z_p\langle t\rangle}[H_1(\tilde{M}; Z_p), Z_p\langle t\rangle]^*$. This completes the proof.

Combining Theorem 3.3 with Lemma 2.6, we rediscover Farber's result [4, Theorem 6].

4. The main theorem

Let G be a group which contains a finitely generated subgroup K with $H_1(K; Z)$

infinite. Assume an Alexander module T is induced from the group K by an epimorphism $\gamma: K \to \langle t \rangle$.

Definition 4.1. We say that the Alexander module T *is produced in* the group G.

The following is our main theorem:

Theorem 4.2. *Let G be a group with an element of infinite order.*

(1) Assume G is isomorphic to the fundamental group of an orientable 3-manifold. Then any Alexander module produced in G is self-reciprocal.

(2) Assume G is isomorphic to the fundamental group of a non-orientable 3-manifold. Then there exists an index 2 subgroup G' of G such that any Alexander module produced in G' is self-reciprocal.

Proof. Let K be a finitely generated subgroup of G which induces an Alexander module $H_1(\tilde{K}; Z)$ by an epimorphism $\gamma: K \to \langle t \rangle$. In the case (1), by 3.1 $K \cong \pi_1(M)$ for a compact orientable 3-manifold M. Let $\tilde{M}$ be the infinite cyclic covering space of M associated with γ. Since there is a $Z\langle t \rangle$-isomorphism $H_1(\tilde{K}; Z) \cong H_1(\tilde{M}; Z)$ and $\dim_Q H_1(\tilde{K}; Q) = \dim_Q H_1(\tilde{M}; Q) < +\infty$, we see from Theorem 3.3 that $H_1(\tilde{K}; Z)$ is self-reciprocal, proving (1). In the case (2), G must have an index 2 subgroup G' which is isomorphic to the fundamental group of an orientable 3-manifold, namely a unique double covering space of the original 3-manifold. Apply the case (1) to G'. This completes the proof.

4.3. *A plan for test.* Assume we are given a group G and a (possibly infinite) presentation P_K for a finitely generated subgroup K (possibly $K = G$) and an epimorphism $\gamma: K \to \langle t \rangle$. If $K \cong Z$, then the test fails, since $Z = \pi_1(S^1 \times S^2)$, so assume $K \neq Z$. Then check whether or not the induced $Z\langle t \rangle$-module $H_1(\tilde{K}; Z)$ is an Alexander module by using the presentation P_K and Lemma 1.1. For example, if $H_1(K; Q) \cong Q$, then by Lemma 2.3 $H_1(\tilde{K}; Z)$ is an Alexander module. In the case of an Alexander module, check whether or not $H_1(\tilde{K}; Z)$ is self-reciprocal. If it is not self-reciprocal, then by Theorem 4.2 (1) G is not isomorphic to the fundamental group of any orientable 3-manifold. Next, for the non-orientable case, assume $H^1(G; Z_2) \neq 0$ and we are given all of the index 2 subgroups G_i ($i \in I$) of G. (If $H^1(G; Z_2) = 0$, then by Theorem 4.2 (2) G is not isomorphic to the fundamental group of any non-orientable 3-manifold.) Futher, assume, for each i, we are given a (possibly infinite) presentation P_{K_i} for a finitely generated subgroup K_i of G_i (possibly $K_i = G_i$) and an epimorphism $\gamma_i: K_i \to \langle t \rangle$. If for each i, (K_i, γ_i) induces a non-self-reciprocal Alexander module, then by Theorem 4.2 (2) G is not isomorphic to the fundamental group of any non-orientable 3-manifold. In this case, G is of course not isomorphic to the fundamental group of any orientable 3-manifold by Theorem 4.2 (1).

Example 4.4. *For non-zero integers l, m and a prime $p \geq 2$, the group $G = G(l, m; p) = (a, b \mid a^{-1}b^l a = b^m, b^p = 1)$ is the fundamental group of a 3-manifold if and only if p divides $2lm$.*

Proof. If p divides lm, then G is isomorphic to Z or the free product $Z * Z_p$. So G is realized as the fundamental group of a 3-manifold. Assume p does not divide lm. Then if $p=2$, $G \cong (a, b \mid a^{-1}ba = b, b^2 = 1) \cong Z \times Z_2$. This is the fundamental group of $S^1 \times P^2$. Now assume p does not divide $2lm$. Then we show that G is not isomorphic to the fundamental group of any 3-manifold. Since $H^1(G; Z_2) = Z_2$, G has just one subgroup G' of index 2. By the Reidemeister–Schreier method (cf. [16]), G' has the presentation

$$(a', b_1, b_2 \mid a'^{-1} b_1^l a' = b_2^m, b_1^m = b_2^l, b_1^p = b_2^p = 1).$$

Let $\gamma : G' \to \langle t \rangle$ be the epimorphism sending a' to t^{-1} and b_1, b_2 to 1. By Lemma 1.1, $H_1(\tilde{G}'; Z) \cong Z_p\langle t \rangle / (l^2 t - m^2)$ ($\cong Z_p$ as an abelian group), which is a non-self-reciprocal Alexander module by Lemma 2.6. By Theorem 4.2, G is not isomorphic to the fundamental group of any 3-manifold. The proof is completed.

Acknowledgements

The author would like to thank the referee of this paper and F. González-Acuña for their helpful suggestions.

References

[1] J.W. Alexander, Topological invariants of knots and links, Trans. Amer. Math. Soc. 30 (1928) 275–306.

[2] R.C. Blanchfield, Intersection theory of manifolds with operators with applications to knot theory, Ann. of Math. 65 (1957) 340–356.

[3] R.H. Crowell, Corresponding group and module sequences, Nagoya Math. J. 19 (1961) 27–40.

[4] M.Š. Farber, Linking coefficients and two-dimensional knots, Soviet Math. Dokl. 16 (1975) 647–650.

[5] R.H. Fox, Free differential calculus I, Ann. of Math. 57 (1953) 547–560.

[6] W. Heil, Some finitely presented non-3-manifold groups, Proc. Amer. Math. Soc. 53 (1975) 497–500.

[7] J. Hempel, 3-Manifolds, Ann. of Math. Studies 86 (Princeton University Press, Princeton, NJ, 1976).

[8] W. Jaco, Roots, relations and centralizers in 3-manifold groups, in: Geometric Topology, Lecture Notes in Mathematics No. 438 (Springer, Berlin–New York, 1975) 283–309.

[9] W. Jaco, Lectures on Three-Manifold Topology, Regional Conference Series in Mathematics No. 43 (Amer. Math. Soc., Providence, RI, 1980).

[10] W. Jaco and P. Shalen, Seifert Fibered Spaces in 3-Manifolds, Memoirs Vol. 21, No. 220 (Amer. Math. Soc., Providence, RI, 1979).

[11] A. Kawauchi, A partial Poincaré duality theorem for infinite cyclic coverings, Quart. J. Math. 26 (1975) 437–458.
[12] A. Kawauchi, Three dimensional homology handles and circles, Osaka J. Math. 12 (1975) 565–581.
[13] A. Kawauchi, On quadratic forms of 3-manifolds, Invent. Math. 43 (1977) 177–198.
[14] S. Lang, Algebra (Addison-Wesley, Reading, MA, 1969).
[15] R.C. Lyndon and P.E. Schupp, Combinatorial Group Theory, Ergebnisse der Mathematik und ihrer Grenzgebiete 89 (Springer, Berlin–New York, 1977).
[16] W. Magnus, A. Karrass and D. Solitar, Combinatorial Group Theory (Interscience, New York, 1966).
[17] J.W. Milnor, Infinite cyclic coverings, in: Conference on the Topology of Manifolds, Complementary Series in Mathematics (Prindle, Weber & Schmidt, Boston, MA, 1968) 115–133.