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# A THEORY OF QUANTAL SETS*

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A quantale is a complete lattice provided with a (generally non commutative) binary multiplication which, in particular, distributes over arbitrary suprema in each variable. That notion is intended to replace locales in the spectral constructions involving non commutative algebraic structures. We propose a corresponding notion of quantale-valued set, which generalizes the classical notions of boolean or heyting-valued sets. We study the category obtained in this way and show that it can be seen as a fibration over the original quantale, whose fibres are toposes.

## Introduction

Locales have been recognized since many years as a fruitful algebraic generalization of topological spaces. They are those complete lattices where finite meets distribute over arbitrary joins. The canonical example of a locale is, of course, the lattice of open subsets of a topological space.

In classical algebra, it is common to construct the spectrum of a given algebraic gadget - and this spectrum is generally a topological space - and try to represent the given gadget via the global sections of some suitable sheaf on that space. Unfortunately, in non-commutative algebra it happens very often that topological spaces are inadequate for producing a relevant spectrum. For example, in the case of rings or algebras, the algebraic multiplication is often related, in one or the other way, with the intersection of open subsets of the spectrum. But that intersection is by nature commutative, a characteristic which is not shared in general by an algebraic multiplication.

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In 1983, Mulvey suggested replacing the intersection of open subsets by a not necessarily commutative multiplication of these open subsets (cf. [8]). In fact, he generalized directly the situation for locales by considering a quantale to be a complete lattice provided with a suitable multiplication (cf. [2]). Now, with a view to proving some representation theorems using quantales, we need some replacement for the notion of a sheaf on a locale; and below, we suggest one such.

A sheaf on a locale $\Omega$ is generally defined to be some functor $F$ from $\Omega$ to the category of sets. It is known from the work of Higgs and Walters (cf. [5] and [11]) that an equivalent approach consists in considering sets $A$ provided with an equality with values in $\Omega$. The link between the two definitions is given by the formulas (where $a, a' \in A$)

$$A = \coprod_{u \in \Omega} F(u), \qquad [a, a'] = \bigvee \{u \in \Omega \mid a|_u = a'|_u\}.$$

In his thesis, Nawaz, a student of Mulvey, generalized this approach to the case of a quantale (cf. [9]). We pursue his idea and show that an adequate generalization of Higgs's definitions produces a substitute for the notion of sheaf on a quantale, which is good enough to recapture the original quantale from the corresponding category of 'sheaves' alone.

Our technique is completely different from that adopted by Nawaz in [9]. We define first our category of $Q$-sets, which are sets provided with a suitable equality with values in a fixed quantale $Q$. Instead of generalizing to that new context the classical proofs concerning $\Omega$-sets (with $\Omega$ a locale), we construct from the category of $Q$-sets a fibration-cofibration over the quantale $Q$ itself, whose fibres are precisely the categories of $\Omega_u$-sets, for a family of locales $\Omega_u$, indexed by the elements $u$ of the quantale $Q$. All the results concerning $Q$-sets are then deduced from the corresponding results on $\Omega$-sets, via the techniques of fibered categories. Among other things, we prove the completeness, cocompleteness and regularity of the category of $Q$-sets. It should be mentioned that the stalk at 1 of our fibration is equivalent to the category of $Q$-sets as defined by Nawaz (cf. [9]).

An important property of our category of $Q$-sets is that it does characterize completely the quantale $Q$, with both its order and its multiplication. In fact, $Q$ appears as the lattice of subobjects of 1 in the category of $Q$-sets and its multiplication is recaptured using a Galois connection which is proved to exist between subobjects and regular subobjects of 1. This is a major difference between our construction and that given by Nawaz in [9], where the corresponding category of $Q$-sets does not characterize the quantale $Q$.

The canonical fibration associated with a category $\mathbb{E}$ is the fibration over $\mathbb{E}$ whose fibre at $A$ is precisely the slice category $\mathbb{E}/A$. In the case of $Q$-sets, we generalize the situation for $\Omega$-sets by restricting the slice construction to those morphisms which do have a (regular epi)–(regular mono) factorization (each morphism has a (regular epi)–mono and an epi–(regular mono) factorization). In that way we produce an interesting fibration over the category of $Q$-sets which is finitely complete and cocomplete, locally small and well powered, generated and cogenerated.

A version of this paper with fully detailed proofs appeared as a preprint in Louvain-la-Neuve.

## 1. The category of quantal sets

We recall (cf. [2]) that a quantale is a complete lattice $Q$ provided with an associative and idempotent binary multiplication & which admits 1 as a right unit (as usually, we write $1 = \bigvee Q$ and $0 = \bigwedge Q$) and distributes over $\vee$ in each variable. A first important example is the lattice of closed right ideals of a $\mathbb{C}^*$-algebra, provided with the multiplication $I \& J = \overline{IJ}$. Elementary properties of quantales can be found in [2], including the weak commutativity property $u \& v \& w = u \& w \& v$.

An element $u$ in a quantale $Q$ is 2-sided when it satisfies the equality $1 \& u = u$. For 2-sided elements of $Q$, the relation $u \& v = u \wedge v$ holds so that these elements constitute a locale Bil($Q$). Moreover, for each $u \in Q$, the down segment $\downarrow u$ is a quantale so that we obtain new locales Bil($\downarrow u$). Finally, each element $u \in Q$ has a 2-sided closure $\hat{u}$ in $Q$, namely $1 \& u$ (cf. [2]).

**Proposition 1.1.** *For each element u of the quantale Q, the mapping*

$$\Gamma : \mathrm{Bil}(\downarrow \hat{u}) \to \mathrm{Bil}(\downarrow u); \qquad x \mapsto u \& x$$

*is an isomorphism of locales whose inverse is given by* $\Gamma^{-1}(y) = \hat{y}$. □

**Definition 1.2.** For a quantale $Q$, a $Q$-set is a pair $(A, [\cdot = \cdot])$ *where A is a set and* $[\cdot = \cdot]$ *is a mapping*

$$[\cdot = \cdot] : A \times A \to Q$$

satisfying

(S1) $\quad [a = a'] \& [a' = a''] \leq [a = a'']$,

(S2) $\quad [a = a']^{\wedge} = [a' = a]^{\wedge}$

for all $a, a', a''$ in $A$.

Axioms (S1) and (S2) generalize those for $\Omega$-sets (cf. [5]); notice that equality is no longer symmetric, but (S2) expresses the symmetry of the 2-sided closure of the equality. Here are some easy consequences of the definition.

**Proposition 1.3.** *Let Q be a quantale and A a Q-set. For all* $a, a' \in A$,

(1) $\quad [a = a] \& [a = a'] = [a = a']$;

(2) $\quad [a = a'] \& [a' = a'] = [a = a']$;

(3) $\quad [a = a] \& [a' = a] = [a = a']$. □

**Definition 1.4.** If $Q$ is a quantale and $A, B$ are $Q$-sets, a morphism $f: A \to B$ of $Q$-sets is a pair of mappings

$$[f \cdot = \cdot]: A \times B \to Q, \qquad [\cdot = f \cdot]: B \times A \to A$$

satisfying

(M1) $\bigvee_{a \in A} [a = a] \leq \bigvee_{b \in B} [b = b]$;

(M2) $[a' = a] \,\&\, [fa = b] \leq [fa' = b]$, $\quad [b = fa] \,\&\, [a = a'] \leq [b = fa']$;

(M3) $[fa = b] \,\&\, [b = b'] \leq [fa = b']$, $\quad [b' = b] \,\&\, [b = fa] \leq [b' = fa]$;

(M4) $[b = fa] \,\&\, [fa = b'] \leq [b = b']$;

(M5) $\bigvee_{b \in B} [fa = b] \,\&\, [b = fa] = [a = a]$;

(M6) $[fa = b]^{\frown} = [b = fa]^{\frown}$

for all elements $a, a' \in A$ and $b, b' \in B$.

$[a = a]$ can be thought as the truth value of the formula "$a \in A$"; we shall write $\varepsilon(A)$ for the element $\bigvee_{a \in A} [a = a] \in Q$. $\varepsilon(A)$ is thus measuring the level at which $A$ is inhabited and this explains axiom (M1). On the other hand, axiom (M6) indicates again that, both 'truth values' $[fa = b]$ and $[b = fa]$ become equal when taking the 2-sided closures. Axioms (M1) and (M6) are consequences of (M2) to (M4) in the case of $\Omega$-sets (cf. [5]). It can be proved that dropping axiom (M1) produces a category equivalent to that studied by Nawaz (cf. [9]); in fact, Nawaz was using an asymmetric definition of morphism, as in our Definition 1.7, but this makes no difference.

**Proposition 1.5.** *Given a quantale $Q$ and a morphism $f: A \to B$ of $Q$-sets,*

(1) $[a = a] \,\&\, [fa = b] = [fa = b]$, $\quad [b = fa] \,\&\, [a = a] = [b = fa]$;

(2) $[fa = b] \,\&\, [b = b] = [fa = b]$, $\quad [b = b] \,\&\, [b = fa] = [b = fa]$

*for all $a \in A$, $b \in B$.* □

**Lemma 1.6.** *Given a quantale $Q$ and a morphism $f: A \to B$ of $Q$-sets,*

$$[b = fa] = [b = b] \,\&\, [fa = b]$$

*for all $a \in A$, $b \in B$.*

**Proof.**

$$[b = fa] = [b = b] \,\&\, 1 \,\&\, [b = fa] = [b = b] \,\&\, [fa = b]. \qquad \square$$

This lemma allows an asymmetric but shorter equivalent definition of a morphism of $Q$-sets.

**Definition 1.7.** If $Q$ is a quantale and $A, B$ are $Q$-sets, a morphism $f: A \to B$ of $Q$-sets is a mapping $[f\cdot = \cdot]: A \times B \to Q$ satisfying

(M1)′ $\bigvee_{a \in A} [a = a] \le \bigvee_{b \in B} [b = b]$;

(M2)′ $[a' = a] \,\&\, [fa = b] \le [fa' = b]$;

(M3)′ $[fa = b] \,\&\, [b = b'] \le [fa = b']$;

(M4)′ $[b = b] \,\&\, [fa = b] \,\&\, [fa = b'] \le [b = b']$;

(M5)′ $\bigvee_{b \in B} [fa = b] = [a = a]$;

(M6)′ $[fa = b] \le [b = b]^{\wedge}$

for all $a, a' \in A$ and $b, b' \in B$.

**Proposition 1.8.** *If $Q$ is a quantale, the $Q$-sets and their morphisms constitute a category for the following composition law: if $f: A \to B$ and $g: B \to C$ are morphisms of $Q$-sets, then*

$$[(g \circ f)(a) = c] = \bigvee_{b \in B} [fa = b] \,\&\, [gb = c].$$

*The identity on a $Q$-set $A$ turns out to be given exactly by its equality.* □

We shall write $Q$-**Sets** for the category of $Q$-sets. The following lemma is useful:

**Lemma 1.9.** *If $Q$ is a quantale and $f, g: A \rightrightarrows B$ are morphisms of $Q$-sets, the relation $[fa = b] \le [ga = b]$ for all $a \in A$, $b \in B$ implies the equality $f = g$.*

**Proof.**

$$\begin{aligned}[ga = b] = [a = a] \,\&\, [ga = b] &= \bigvee_{b' \in B} [fa = b'] \,\&\, [ga = b] \\ &= \bigvee_{b' \in B} [fa = b'] \,\&\, [b' = b'] \,\&\, [ga = b'] \,\&\, [ga = b] \\ &\le \bigvee_{b' \in B} [fa = b'] \,\&\, [b' = b] \le [fa = b].\end{aligned}$$ □

**Proposition 1.10.** *Given a quantale $Q$ and an element $u \in Q$, we produce an endofunctor $F_u$ on the category $Q$-***Sets** *by putting*

$$F_u(A, [\cdot = \cdot]) = (A, u \,\&\, [\cdot = \cdot]), \qquad F_u([f\cdot = \cdot]) = u \,\&\, [f\cdot = \cdot].$$ □

We shall abbreviate $F_u(A, [\cdot = \cdot])$ as $A_u$ and $F_u([f\cdot = \cdot])$ as $f_u$.

**Proposition 1.11.** *Let $Q$ be a quantale. For each $u \in Q$ and each $Q$-set $A$ such that $\varepsilon(A)$ is 2-sided in $u$, $A$ and $A_u$ are isomorphic $Q$-sets.*

**Proof.** The inverse isomorphisms $f:A\to A_u$ and $g:A_u\to A$ are defined by

$$[fa=a']=[a=a']\,\&\,u, \qquad [ga'=a]=u\,\&\,[a'=a]$$

for all $a,a'\in A$. $\square$

**Corollary 1.12.** *Given a quantale $Q$ and a $Q$-set $A$, $A$ is isomorphic to $A_{\varepsilon(A)}$.* $\square$

It should be noticed that the equality on $A_{\varepsilon(A)}$ takes in fact its values in the locale $\mathrm{Bil}(\downarrow\varepsilon(A))$.

**Proposition 1.13.** *Let $Q$ be a quantale. A morphism $f:A\to B$ of $Q$-sets is a monomorphism iff*

$$[fa=b]\,\&\,[fa'=b]\le[a=a']$$

*for all $a,a'\in A$ and $b\in B$.*

**Proof.** If $f$ is a monomorphism and $a,a'\in A$, we put

$$q=\bigvee_{b\in B}[fa=b]\,\&\,[fa'=b].$$

We consider the $Q$-set $(\downarrow q,\&)$ and the morphisms $f_a,f_{a'}:\downarrow q\rightrightarrows A$ defined by

$$[f_a x=\bar a]=x\,\&\,[a=\bar a], \qquad [f_{a'}x=\bar a]=x\,\&\,[a'=\bar a].$$

From the relation $f\circ f_a=f\circ f_{a'}$ we deduce $f_a=f_{a'}$. Choosing $x=q$ and $\bar a=a'$ we get $q\,\&\,[a=a']=q$ and therefore

$$[fa=b]\,\&\,[fa'=b]\le q=q\,\&\,[a=a']\le[a=a']\,\&\,[a=a']=[a=a'].$$

Conversely, if $g,h:c\rightrightarrows A$ are such that $f\circ g=f\circ h$, for all $a,a'\in A$, $b\in B$, $c\in C$,

$$\begin{aligned}
gc=a'&=[gc=a']\,\&\,[a'=a']\\
&=[gc=a]\,\&\bigvee_{b\in B}[fa'=b]\\
&\le\bigvee_{b\in B}\bigvee_{a\in A}[gc=a]\,\&\,[fa=b]\,\&\,[fa'=b]\\
&=\bigvee_{b\in B}\bigvee_{a\in A}[hc=a]\,\&\,[fa=b]\,\&\,[fa'=b]\\
&\le\bigvee_{a\in A}[hc=a]\,\&\,[a=a']\\
&=[hc=a']
\end{aligned}$$

and one concludes by Lemma 1.9. $\square$

**Proposition 1.14.** *Let $Q$ be a quantale. A morphism $f:A\to B$ of $Q$-sets is an epimorphism iff*

$$[b=b]=\bigvee_{a\in A}[b=b]\,\&\,[fa=b]$$

*or equivalently*

$$[b=b]=\bigvee_{a\in A}[b=fa]$$

*for all* $b\in B$. *This implies the density of* $\varepsilon(A)$ *in* $\downarrow\varepsilon(B)$, *i.e.*

$$\varepsilon(B)\,\&\,\varepsilon(A)=\varepsilon(B).$$

**Proof.** If $f$ is an epimorphism, let us consider the disjoint union $B\amalg B$. For every $b\in B$, we write $b_1, b_2$ for the two corresponding elements of $B\amalg B$. We provide $B\amalg B$ with the structure of a $Q$-set by putting

$$[b_1=b_1']=[b_2=b_2']=[b=b'],$$

$$[b_1=b_2']=[b_2=b_1']=\bigvee_{a\in A}[b=b]\,\&\,[fa=b]\,\&\,[b=b']$$

for all $b, b'\in B$. Now define morphisms $\varphi,\gamma: B\rightrightarrows B\amalg B$ by

$$[\varphi b=c]=[b_1=c],\qquad [\gamma b=c]=[b_2=c]$$

for all $b\in B$, $c\in B\amalg B$. From the relation $\varphi\circ f=\gamma\circ f$ we deduce $[\varphi b=b_1]=[\gamma b=b_1]$ for every $b\in B$, which is the required relation.

The converse implication is proved by arguments analogous to that developed in the case of monomorphisms. Finally, our characterization of an epimorphism implies immediately $\varepsilon(B)\le\varepsilon(B)\,\&\,\varepsilon(A)$, while the converse inequality is obvious. □

## 2. The technical fibration

The aim of this paragrah is to describe a technical tool – more precisely, a fibration – which will be used in the next paragraph to deduce the main properties of the category of quantal sets.

Given a quantale $Q$ and an element $u\in Q$, we write $\mathbb{E}_u$ for the full subcategory of $Q$-**Sets** whose objects are those $Q$-sets $A$ such that $\varepsilon(A)$ is 2-sided in $\downarrow u$, thus $u\,\&\,\varepsilon(A)=\varepsilon(A)$.

**Proposition 2.1.** *Given a quantale* $Q$ *and* $u\in Q$, *the category* $\mathbb{E}_u$ *is isomorphic to the topos of* Bil($\downarrow u$)*-sets.*

**Proof.** The category of Bil($\downarrow u$)-sets is clearly a subcategory of $\mathbb{E}_u$; let us prove it is a full subcategory. If $A, B$ are Bil($\downarrow u$)-sets and $f: A\to B$ is a morphism in $\mathbb{E}_u$,

$$\begin{aligned}u\,\&\,[fa=b]&=u\,\&\,[a=a]\,\&\,[fa=b]\\&=[a=a]\,\&\,[fa=b]=[fa=b]\end{aligned}$$

for all $a \in A$, $b \in B$; thus $[fa = b]$ is indeed 2-sided in $\downarrow u$. Now that inclusion is an equivalence by Corollary 1.12. □

**Proposition 2.2.** *Given a quantale $Q$ and elements $v \leq u$ in $Q$, the functor $F_v$ (cf. Proposition* 1.10) *induces a logical morphism of topoi*

$$F_v^u : \mathbb{E}_u \to \mathbb{E}_v$$

*which has both a left and a right adjoint.*

**Proof.** Via the isomorphisms described in Propositions 1.1 and 2.1, the problem reduces to the classical case of the locale Bil($Q$) and its elements $\hat{v} \leq \hat{u}$. □

If $Q$ is a quantale, with each $u \in Q$ we can associate the topos $\mathbb{E}_u$ and with each $v \leq u$, we can associate the function $F_v^u$. This produces a contravariant functor from $Q$ to the category of large categories, thus a fibration $p : \mathbb{E} \to q$. It should be noticed that the stalk of that fibration over the terminal object 1 is equivalent to the category of sheaves over the locale of 2-sided elements of $Q$ and is thus equivalent to the category of $Q$-sets as defined by Nawaz (cf. [9]).

**Proposition 2.3.** *Given a quantale $Q$, the corresponding fibration $p : \mathbb{E} \to Q$ can be described in the following way:*
- *the fibre at $u \in Q$ is $\mathbb{E}_u$*
- *if $v \leq u$, $A \in \mathbb{E}_v$, $B \in \mathbb{E}_u$, then $\mathbb{E}(A, B) = Q$-**Sets**$(A, B)$.*

*Each Cartesian morphism is a monomorphism in $Q$-**Sets**.*

**Proof.** With the notations of the statement, $F_v(B) = B_v$ (cf. Proposition 1.10) is in $\mathbb{E}_v$ and there exists a morphism of $Q$-sets $\alpha(B) : B_v \to B$ defined by

$$[\alpha(B)(b) = b'] = v \,\&\, [b = b']$$

for all $b, b' \in B$. It follows from Proposition 1.13 that $\alpha(B)$ is a monomorphism in $Q$-**Sets**. Every morphism $f : A \to B$, of $Q$-sets factorizes uniquely through $\alpha(B)$ via the morphism $g : A \to B_v$ defined by

$$[gc = b] = [fc = b] \,\&\, w$$

for all $b \in B$, $c \in C$. Therefore, $Q$-**Sets**$(A, B_v) \cong Q$-**Sets**$(A, B)$, which proves the statement; $\alpha(B)$ is a Cartesian morphism over $v \leq u$. □

**Proposition 2.4.** *Given a quantale $Q$, the corresponding fibration $p : \mathbb{E} \to Q$ is also a cofibration. Each cocartesian morphism is an epimorphism in $Q$-**Sets**.*

**Proof.** With the notations of Proposition 2.3, $F_u(A) = A_u$ (cf. Proposition 1.10) is in $\mathbb{E}_u$ and there exists a morphism of $Q$-sets $\beta(A) : A \to A_u$ defined by

$$[\beta(A)(a) = a'] = [a = a']$$

for all $a, a' \in A$; this is an epimorphism in $Q$-**Sets** by Proposition 1.14. $\beta(A)$ is cocartesian since, given $B \in \mathbb{E}_u$ and $f: A \to B$, the unique factorization $g: A_u \to B$ is defined by

$$[ga = b] = u \,\&\, [fa = b]$$

for all $a \in A$, $b \in B$. □

It should be noticed that the fibration–cofibration $p: \mathbb{E} \to Q$ is generally not a bifibration (cf. [1]): the Beck condition is not satisfied.

**Proposition 2.5.** *In the fibration $p: \mathbb{E} \to Q$ associated with a quantale $Q$, each cocartesian morphism is Cartesian as well.*

**Proof.** With the notations of Proposition 2.3, applying successively the cofibration and the fibration transports $A$ on $A_v$. One concludes using Proposition 1.11. □

## 3. The basic properties of the category of quantal sets

We refer to [5] and [11] for the basic properties of the category of $\Omega$-sets (with $\Omega$ a locale). We use them in conjunction with the results of Section 2 to deduce the corresponding properties for $Q$-sets (with $Q$ a quantale).

**Proposition 3.1.** *Let $Q$ be a quantale. The category of $Q$-**sets** is complete and cocomplete.*

**Proof.** We use the notations of Section 2. Given a diagram in $Q$-**Sets** with vertices $(A_i)_{i \in I}$, we view each $A_i$ as an object in $\mathbb{E}_{\varepsilon(A_i)}$ and reproduce in this way the same diagram in $\mathbb{E}$. Using the fibration $p: \mathbb{E} \to Q$, we obtain a diagram with the same shape in the fibre $\mathbb{E}_{\bigwedge \varepsilon(A_i)}$ whose limit $(L, (p_i)_{i \in I})$ does exist in that fibre. We shall prove that $(L, (\alpha(A_i) \circ p_i)_{i \in I})$ is the limit in $Q$-**Sets** of the original diagram. Indeed, if $(M, (q_i)_{i \in I})$ is another cone of the diagram $(A_i)_{i \in I}$ in $Q$-**Sets**, one has $\varepsilon(M) \leq \bigwedge_{i \in I} \varepsilon(A_i)$. Using the fibration, we obtain a cone $(M, (q_i')_{i \in I})$ on the diagram in $\mathbb{E}_{\bigwedge \varepsilon(A_i)}$. Next, we transport $M$ itself in the fibre $\mathbb{E}_{\bigwedge \varepsilon(A_i)}$, using the cofibration, and get a cone in this fibre, thus a unique factorization through $L$. The unicity of that factorization follows from Propositions 2.1 and 2.2.

The case of colimits is analogous, reversing the roles of Cartesian and cocartesian morphisms. □

**Proposition 3.2.** *Let $Q$ be a quantale and $f: A \to B$ a morphism of $Q$-sets. The following conditions are equivalent* (*notations of Section* 2):

(1) *$f$ is a regular monomorphism in $Q$-**Sets**;*

(2) *$f$ is a monomorphism in some fibre $\mathbb{E}_u$;*

(3) *$f$ is a monomorphism in $Q$-**Sets** and $\varepsilon(A)$ is 2-sided in $\downarrow\varepsilon(B)$.*

**Proof.** Follows from the way equalizers are constructed in $Q$-**Sets** and the fact that in $\mathbb{E}_{\varepsilon(B)}$ each monomorphism is an equalizer. □

**Proposition 3.3.** *Let $Q$ be a quantale and $f: A \to B$ be a morphism of $Q$-sets. The following conditions are equivalent* (*notations of Section* 2):
(1) *$f$ is a regular epimorphism in $Q$-***Sets***;*
(2) *$f$ is an epimorphism in some fibre $\mathbb{E}_u$;*
(3) *$f$ is an epimorphism in $Q$-***Sets*** and $\varepsilon(A) = \varepsilon(B)$.*

**Proof.** Analogous to that of Proposition 3.2. □

**Proposition 3.4.** *Let $Q$ be a quantale. The category of $Q$-sets is regular.*

**Proof.** Given a morphism $f: A \to B$ of $Q$-sets, we view it in $\mathbb{E}$ (notations of Section 2) with $A \in \mathbb{E}_{\varepsilon(A)}$ and $B \in \mathbb{E}_{\varepsilon(B)}$. The Cartesian morphism $\alpha(B): B_{\varepsilon(A)} \to B$ is a monomorphism in $Q$-**Sets** (cf. Proposition 2.3) and $f$ factors through it via a morphism $f': A \to B_{\varepsilon(A)}$ in $\mathbb{E}_{\varepsilon(A)}$. Factoring $f'$ through its image in $\mathbb{E}_{\varepsilon(A)}$ produces the required factorization of $f$ in $Q$-**Sets**.

If, moreover, $g: C \to B$ is a regular epimorphism of $Q$-sets, $g$ is an epimorphism in $\mathbb{E}_{\varepsilon(B)}$ and so $F_{\varepsilon(A)}^{\varepsilon(B)}(g)$ (notations of Section 2) is an epimorphism in $\mathbb{E}_{\varepsilon(A)}$ since $F_{\varepsilon(A)}^{\varepsilon(B)}$ has a right adjoint (cf. Proposition 2.2); the pullback of that morphism over $A$ is thus again an epimorphism in the topos $\mathbb{E}_{\varepsilon(A)}$, and this is precisely the pullback of $g$ along $f$ in $Q$-**Sets** (cf. Proposition 3.1). This completes the proof (cf. Proposition 3.3). □

**Proposition 3.5.** *Let $Q$ be a quantale. In the category of $Q$-sets, the equivalence relations are universal and effective.*

**Proof.** If $p_1, p_2: R \rightrightarrows A$ is an equivalence relation in $Q$-**Sets**, the reflexivity implies $\varepsilon(R) = \varepsilon(A)$ so that $R, A$ and the corresponding quotient (cf. Proposition 3.1) all live in the fibre $\mathbb{E}_{\varepsilon(A)}$ (notations of Section 2). The rest follows easily from Propositions 2.2 and 3.1 as well as the corresponding result for the fibres, which are topoi. □

The category of $Q$-sets is not coregular, but the following properties hold:

**Proposition 3.6.** *Let $Q$ be a quantale. In the category of $Q$-sets,*
(1) *each morphism factors uniquely as an epimorphism followed by a regular monomorphism.*
(2) *the pushout of a regular monomorphism is still a monomorphism and the corresponding square is a pullback.*

**Proof.** The first part of the proof is analogous to that of Proposition 3.4, using now the cofibration.

For the second part, if $f: A \to B$ is the morphism and $g: A \to C$ the regular epimorphism, the pushout is now computed in $\mathbb{E}_{\varepsilon(B)\vee\varepsilon(C)}$ (cf. Proposition 3.1). It follows from Propositions 1.13 and 2.4 that the direct image functors of the cofibration $p: \mathbb{E} \to Q$ preserve monomorphisms and we know that the cocartesian morphisms are monomorphisms (cf. Propositions 2.5 and 2.3). Thus the second statement follows from the corresponding result in the topos $\mathbb{E}_{\varepsilon(B)\vee\varepsilon(C)}$ and the way pullbacks and pushouts are constructed in $Q$-**Sets** (cf. Proposition 3.1). □

Some classical properties of coproducts of $\Omega$-sets ($\Omega$ a locale) carry over to the case of $Q$-sets ($Q$ a quantale).

**Proposition 3.7.** *Let $Q$ be a quantale. In the category of $Q$-sets,*
(1) *the initial object is strict;*
(2) *the canonical injections in a coproduct are monomorphisms;*
(3) *the coproducts are disjoint.*

**Proof.** First notice that the initial object of $Q$-**Sets** is also the initial object in each fibre $\mathbb{E}_u$, $u \in Q$ (notations of Section 2). So the statement follows from the corresponding statement in the fibres (cf. Proposition 2.1) and the fact for the cocartesian morphisms to be monomorphisms (cf. Propositions 2.5 and 2.3). □

## 4. The subobjects of quantal sets

We study the properties of subobjects and regular subobjects in $Q$-**Sets** and deduce from it that the knowledge of the category $Q$-**Sets** implies that of $Q$.

**Proposition 4.1.** *Let $Q$ be a quantale. Given a family $(A_i \rightarrowtail A)_{i\in I}$ of subobjects in* $Q$-**Sets**,
(1) *the intersection $\bigcap_{i\in I} A_i$ and the union $\bigcup_{i\in I} A_i$ of that family exist;*
(2) *when each $A_i$ is a regular subobject of $A$, so are the intersection and the union;*
(3) $\varepsilon(\bigcup_{i\in I} A_i) = \bigvee_{i\in I} \varepsilon(A_i)$.

**Proof.** The existence of intersections follows from that of limits (cf. Proposition 3.1) and the construction of unions follows from that of coproducts (cf. Proposition 3.1) and regular epi-mono factorizations (cf. Proposition 3.4). The last two statements follow directly from the various constructions and the fact that, when each $\varepsilon(A_i)$ is 2-sided in $\varepsilon(A)$, so are $\bigwedge_{i\in I} \varepsilon(A_i)$ and $\bigvee_{i\in I} \varepsilon(A_i)$ (cf. [2]; the inclusion of 2-sided elements in $\downarrow\varepsilon(A)$ has both a left and a right adjoint). □

**Proposition 4.2.** *Let $Q$ be a quantale and $A$ a $Q$-set. The regular subobjects of $A$ constitute a locale* Reg($A$) *which is both reflective and coreflective in the complete lattice* Sub($A$) *of subobjects of $A$.*

**Proof.** Clearly a $Q$-set $A$ has just a set of subobjects. From Proposition 4.1(2), Reg($A$) is a complete lattice which is both reflective and coreflective in Sub($A$). From Proposition 3.2 we deduce that Reg($A$) is the lattice of subobjects of $A$ in the fibre $\mathbb{E}_{\varepsilon(A)}$ (notations of Section 2); so it is a locale. □

**Theorem 4.3.** *Let $Q$ be a quantale and* 1 *the terminal object of the category of $Q$-sets. The quantale $Q$ is isomorphic to the lattice* Sub(1) *provided with the multiplication.*

$$S\&T = S \cap \hat{T}$$

*where* $(-)\hat{}$ *denotes the left adjoint to the inclusion of* Reg(1) *in* Sub(1).

**Proof.** The terminal object 1 of $Q$-**Sets** is also that of the fibre $\mathbb{E}_1$ (notations of Section 2). Each subobject $U$ of 1 in $Q$-**Sets** is also the terminal object of the fibre $\mathbb{E}_{\varepsilon(U)}$, from which it follows that $Q$ as a lattice is isomorphic to Sub(1). Moreover, Proposition 3.2(3) indicates that under this isomorphism, the locale Reg(1) is identified with the locale Bil($Q$) of 2-sided elements of $Q$. But, given two elements $u, v$ in $Q$, $u\&v = u \wedge \hat{v}$ where $(-)\hat{}$ is the 2-sided closure operation, thus the left adjoint to the inclusion of Bil($Q$) in $Q$ (cf. [2]). □

**Proposition 4.4.** *Let $Q$ be a quantale. The category of $Q$-sets has a classifier of regular subobjects.*

**Proof.** With the notations of Section 2, write $\Omega$ for the subobject classifier in the fibre $\mathbb{E}_1$. For each $Q$-set $A$, the morphisms of $Q$-sets from $A$ to $\Omega$ are in bijection with those from $A$ to $\Omega_{\varepsilon(A)}$ (cf. Proposition 2.1) and $\Omega_{\varepsilon(A)}$ is precisely the subobject classifier of $\mathbb{E}_{\varepsilon(A)}$ (cf. Proposition 2.2). On the other hand, the subobjects of $A$ in $\mathbb{E}_{\varepsilon(A)}$ are precisely the regular subobjects of $A$ in $Q$-**Sets** (cf. Proposition 3.2). This proves the required bijective correspondence; the existence of the corresponding pullback follows from the constructions in Proposition 3.1. □

## 5. The fibration of regular morphisms

If $Q$ is a quantale, we associate with the category of $Q$-sets a fibration which, in the case where $Q$ is a locale, is just the canonical fibration of the category $Q$-**Sets**. We prove it to satisfy, in the language of fibered categories, all the axioms defining a topos.

**Definition 5.1.** Let $Q$ be a quantale. A morphism of $Q$-sets is called regular when it factors as a regular epimorphism followed by a regular monomorphism.

**Proposition 5.2.** *Let $Q$ be a quantale.*

(1) *A morphism $f: A \to B$ is regular if and only if $\varepsilon(A)$ is 2-sided in $\downarrow\varepsilon(B)$.*

(2) *In $Q$-*Sets*, the pullback of a regular morphism along any morphism is again regular.*

(3) *In $Q$-*Sets*, the composite of two regular morphisms is again regular.*

(4) *In $Q$-*Sets*, if a composite $f \circ g$ is regular as well as the morphism $f$, then $g$ is also regular.*

**Proof.** The first statement follows from Proposition 3.2(3) and the second from Proposition 3.4. The last two statements are easy consequences of the first one. □

Let us denote by $\mathscr{R}$ the category whose objects are the regular morphisms in $Q$-**Sets** and whose arrows are arbitrary commutative squares in $Q$-**Sets**.

**Proposition 5.3.** *Let $Q$ be a quantale.*

(1) *The codomain functor $\varrho : \mathscr{R} \to Q$-*Sets* is a fibration.*

(2) *Each arrow in a fibre of $\mathscr{R}$ (thus over an identity in $Q$-*Sets*) is a regular morphism in $Q$-*Sets*.*

(3) *Given a $Q$-set $A$, the fibre $\mathscr{R}_A$ is isomorphic to the topos $\mathbb{E}_{\varepsilon(A)/A}$ (notations of Section 2).*

(4) *For every morphism $f : A \to B$ in $Q$-*Sets*, the inverse image functor $f^* : \mathscr{R}_B \to \mathscr{R}_A$ is a logical morphism of topoi with both a right and a left adjoint.*

**Proof.** Let $x : X \to B$ be a regular morphism of $Q$-sets and $f : A \to B$ an arbitary morphism of $Q$-sets. The pullback of $f$ and $x$ in $Q$-**Sets** produces a morphism in $\mathscr{R}$ (cf. Proposition 5.2) which is obviously Cartesian over $f$.

(1) follows easily from that remark.

(2) is a consequence of Proposition 5.2(4) and (3) results from Proposition 5.2(1) and the definition of $\mathbb{E}_{\varepsilon(A)}$ (cf. Section 2).

To prove (4), we use the description of $\mathscr{R}_A$ given in (3), and replace successively $\mathbb{E}_{\varepsilon(A)}$ by the topos of Bil($\downarrow\varepsilon(A)$)-sets (cf. Proposition 2.1) and finally that of Bil($\downarrow(\varepsilon(A)^{\frown}$)-sets (cf. Proposition 1.1); the same for $\mathscr{R}_B$. In that way the problem is entirely reduced to a corresponding problem about Bil($Q$)-sets. It suffices then to check that the various isomorphisms involved transform the inverse image functor $f^*$ of the fibration $\mathscr{R}$ in an inverse image functor of the canonical fibration associated with the category of Bil($Q$)-sets; this is long but straightforward. □

**Theorem 5.4.** *Consider a quantale $Q$; the corresponding fibration $\varrho : \mathscr{R} \to Q$-*Sets* of regular morphisms...*

(1) *is finitely complete and cocomplete;*

(2) *is locally small;*

(3) *is well-powered;*

(4) *has a generator and a cogenerator.*

**Proof.** The fibres are finitely complete and cocomplete (cf. Proposition 5.3(3)) and the inverse image functors are exact and coexact (cf. Proposition 5.3(4)), so (1) holds.

If $x: X \to A$ and $y: Y \to A$ are two objects of the topos $\mathscr{R}_A$, then the exponential $y^x$ is preserved by each inverse image functor of the fibration (cf. Proposition 5.3(4)), from which it follows easily that the fibration is locally small. (3) is proved in an analogous way, noticing that the inverse image functors preserve also the power objects (cf. Proposition 5.3(4)).

If 1 is the terminal object of $Q$-**Sets**, then the identity on 1 is in $\mathscr{R}_1$ and generates the fibration. Indeed each fibre $\mathscr{R}_A$ is a localic topos and so two distinct morphisms in $\mathscr{R}_A$ can be separated by a morphism with domain a subobject of $1 \in \mathscr{R}_A$, i.e. a regular monomorphism $S \rightarrowtail A$ of $Q$-**Sets**. Since the inverse image of $\mathrm{id}_1 \in \mathscr{R}_1$ in the fibre $\mathscr{R}_S$ is precisely $\mathrm{id}_S$, we can conclude.

If $\Omega$ is the subobject classifier of the topos $\mathbb{E}_1$ (cf. Section 2), the regular morphism $\omega: \Omega \to 1$ is the subobject classifier of the topos $\mathscr{R}_1$. Indeed, since each fibre $\mathscr{R}_A$ is a localic topos, two distinct morphisms in $\mathscr{R}_A$ can be separated by a morphism of that fibre with values in its subobject classifier $\omega_Q$. This concludes the proof since $\omega_A$ is the inverse image of $\omega$ in the fibration. □

It should be mentioned that the fibration $\varrho: \mathscr{R} \to Q$-**Sets** is generally not complete or cocomplete: the required Beck condition fails to be satisfied.

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