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A UNIQUENESS DECOMPOSITION THEOREM FOR ACTIONS OF FINITE GROUPS ON FREE GROUPS

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Let $G \to \operatorname{Aut} F$ be an action of a finite group G on a free group F. The main result of the paper is that the maximal free product decomposition $F = F_1 * F_2 * \cdots * F_n$, with factors $F_1, F_2, \ldots, F_n$ invariant under the action of G, is practically unique. As an application, a classification is obtained of all periodic automorphisms of free groups of rank ≤ 5.

1. Introduction

For both the intrinsic interest and a close connection with some important groups arising in topology, the automorphism groups of free groups have been among the most studied particular groups in combinatorial group theory. A considerable amount of information about them has accumulated, including highly non-trivial and deep results, c.f. [7, 8, 9, 10]. Yet some basic questions have so far been only partially answered. Problem 5 from Lyndon's list [7] reflects the present situation:

> "Determine the structure of Aut F, of its subgroups, especially its finite subgroups, and its quotient groups, as well as the structure of invidual automorphisms."

The present article offers a modest contribution to the understanding of the ways in which a finite group can act on a finitely generated free group. Our main result is that such an action uniquely decomposes with respect to the free product.

We shall consider actions $\theta : G \to \operatorname{Aut} F$ of a finite group G on a finitely generated free group F and say that θ *decomposes* if there exist non-trivial invariant subgroups F' and F'' of F such that $F = F' * F''$. In this situation we write $\theta = \theta' * \theta''$, where the factors θ' and θ'' are restrictions of θ on F' and F''.

Theorem 1. *If* $\theta = \theta_1 * \theta_2 * \cdots * \theta_r$ *and* $\theta = \theta'_1 * \theta'_2 * \cdots * \theta'_s$ *are two decompositions of* $\theta : G \to \operatorname{Aut} F$ *into indecomposable factors, then* $r = s$ *and, up to a reordering of factors,* θ_i *is equivalent to* θ'_i, $1 \leq i \leq r$.

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The paper is almost entirely devoted to proving this theorem.

The majority of known results about finite subgroups of Aut F [2, 3, 5, 11] is obtained by using the structure theorem for finite extensions of free groups [1, 4, 13, 14]. The essence of the method is contained in the realization theorem of Culler [2]: Every finite subgroup of Aut F is induced by a group of automorphisms of a graph. Our arguments will be based on Culler's theorem and accompanying it uniqueness theorem of [6]. These preliminaries are discussed in the next section.

The last three short sections contain some applications related to (the classification of) periodic automorphisms of free groups. In [11] McCool classified elements of finite order in Aut F_3 (F_3 = the free group of rank 3). Proofs that the automorphisms given on the list were pairwise non-conjugate were ad hoc and not supplied. For each of these automorphisms one can easily draw a graph realizing it. The question "Can't we tell non-conjugate automorphisms by looking at the differences in the graphs realizing them?" was the origin of the present work.

2. *G*-graphs

As in [6] we shall work with purely combinatorial *graphs* [8]. The vertex set and the edge set of the graph Γ are denoted by $V(\Gamma)$ and $E(\Gamma)$; $\iota(e)$, $\tau(e)$ and $e^{-1}=\bar{e}$ are respectively the initial vertex, the terminal vertex and the inverse edge of the edge e.

By a *G-graph* we shall understand a pointed graph together with a base-point preserving action of G on it. The finite group G will be fixed throughout. It is convenient and usual to assume that G does not invert any edge on graphs on which it acts. Every G-graph Γ induces an action θ_Γ of G on $\pi_1(\Gamma, *)$. Two G-graphs Γ and Γ' will be called *equivalent* here if the actions θ_Γ and $\theta_{\Gamma'}$ are equivalent. For example, every G-graph is equivalent with a *reduced* G-graph, where 'reduced' means that the orbit of every geometric edge is not simply connected [6].

An action θ of G on a free group F is said to be *realized* by the G-graph Γ if θ is equivalent with θ_Γ. The following is Culler's Realization Theorem:

Theorem A [2]. *Every action of G on a finitely generated free group is realized by a finite G-graph.* □

If e and e' are edges of a G-graph Γ such that $e^{\pm 1}\notin Ge'$, $\tau(e)=\tau(e')$ and $\mathrm{Stab}(e)\subseteq \mathrm{Stab}(e')$, then the graph Γ' obtained from Γ by keeping the same vertex set and the same edge set but changing the incidence functions to the amount that the terminal vertex of xe becomes the initial vertex of xe' (for every $x\in G$) is a G-graph equivalent with Γ. We say that Γ' is obtained from Γ by the *Nielsen transformation* $\langle e, e'\rangle$. Of course, $\langle xe, xe'\rangle$ is the same as $\langle e, e'\rangle$ (for every $x\in G$). For more details about G-graphs and Nielsen transformations the reader is referred to [6].

We shall use the term *transformation* to mean a finite product of Nielsen transformations. In particular, if $\sigma = e_n e_{n-1}\cdots e_1$ is a path in Γ such that $\tau(\sigma)=\tau(e)$,

$e_i \notin Ge^{\pm 1}$ and $\mathrm{Stab}(e) \subseteq \mathrm{Stab}(\sigma) = \mathrm{Stab}(e_1) \cap \cdots \cap \mathrm{Stab}(e_n)$ then we write $\langle e, \sigma \rangle$ for the product $\langle e, e_n \rangle \langle e, e_{n-1} \rangle \cdots \langle e, e_1 \rangle$. The transformation $\langle e, \sigma \rangle$ can be described as 'moving e along $\bar{\sigma}$' and we shall often say that σ as above is an *e-trail*. Furthermore, if $\iota(\sigma) = *$ we shall say that σ is a *basic e-trail*.

Transformations preserve the property of being reduced [6, Lemma 3]. For every G-graph Γ we denote by $\mathbf{V}(\Gamma)$ the set of all graphs which can be obtained by applying a transformation to Γ and call it the *Nielsen class* of Γ. The following theorem states that the classes of equivalent G-graphs are essentially the Nielsen classes:

Theorem B [6]. *If Γ and Γ' are equivalent reduced G-graphs, then there exists a transformation T applicable to Γ such that $T\Gamma$ and Γ' are isomorphic G-graphs.* □

Our arguments will proceed within a fixed Nielsen class $\mathbf{V}$. So fixed sets E and V will be the edge and the vertex sets of all members of $\mathbf{V}$. The incidence functions in a particular $\Gamma \in \mathbf{V}$ will be denoted by ι_Γ and τ_Γ.

3. Restating Theorem 1 in terms of graphs

It is clear how to define decompositions of G-graphs: Γ *decomposes* if there are G-subgraphs Γ' and Γ'' whose union is Γ and which have only base-point in common; i.e. $\Gamma = \Gamma' \vee \Gamma''$ (the wedge product). Γ' and Γ'' are *factors* of Γ. If $\Gamma = \Gamma' \vee \Gamma''$, then $\theta_\Gamma = \theta_{\Gamma'} * \theta_{\Gamma''}$, but the converse is not true. More precisely, if Γ is indecomposable, then θ_Γ need not be indecomposable. For example, take the $\mathbb{Z}_2$-graph Γ depicted in Fig. 1, the action being given by $a \leftrightarrow b$ and $c \leftrightarrow d$. This Γ is obviously indecomposable, but $\langle \bar{c}, a \rangle$ transforms it into decomposable graph Γ'. So the action θ_Γ on F_2, being equivalent with $\theta_{\Gamma'}$, decomposes. A G-graph Γ such that θ_Γ is indecomposable will be called *simple*. This is clearly a stronger condition than being an indecomposable graph. Observe also that every graph equivalent with a simple graph is simple itself. Finally, we define a G-graph to be *semisimple* if it is a wedge product of simple graphs. Every graph is equivalent with a semisimple graph or, in other words, every Nielsen class contains a semisimple graph. Indeed, given a G-graph Γ, there is a maximal decomposition $\theta_\Gamma = \theta_1 * \cdots * \theta_k$; if $\Gamma_1, \ldots, \Gamma_k$ are G-graphs realizing $\theta_1, \ldots, \theta_k$ (their existence is guaranteed by Theorem A), then Γ is equivalent with the semisimple graph $\Gamma_1 \vee \cdots \vee \Gamma_k$.

Every G-graph has a unique (up to the order of indecomposable factors) decom-

Fig. 1.

position $\Gamma = \Gamma_1 \vee \cdots \vee \Gamma_k$. If Γ is semisimple, then $\theta_\Gamma = \theta_{\Gamma_1} * \cdots * \theta_{\Gamma_k}$ is a maximal decomposition of θ_Γ. We shall say that two G-graphs are *strongly equivalent* ('factorwise' equivalent) if there is a bijection between their indecomposable factors such that every two corresponding factors are equivalent G-graphs. Now we can restate Theorem 1: Every two equivalent semisimple G-graphs are strongly equivalent. In view of Theorem B this is the same as

Theorem 1′. *Every two semisimple G-graphs belonging to the same Nielsen class are strongly equivalent.*

Remark. Let Γ and Γ' be semisimple graphs from the same Nielsen class. Theorem 1′ asserts that there are maximal decompositions $\Gamma = \Gamma_1 \vee \cdots \vee \Gamma_k$ and $\Gamma' = \Gamma_1' \vee \cdots \vee \Gamma_k'$ with Γ_i equivalent with Γ_i'. But it is not true in general that Γ_i' should be obtainable from Γ_i by a transformation; passing from Γ to Γ' by a sequence of Nielsen

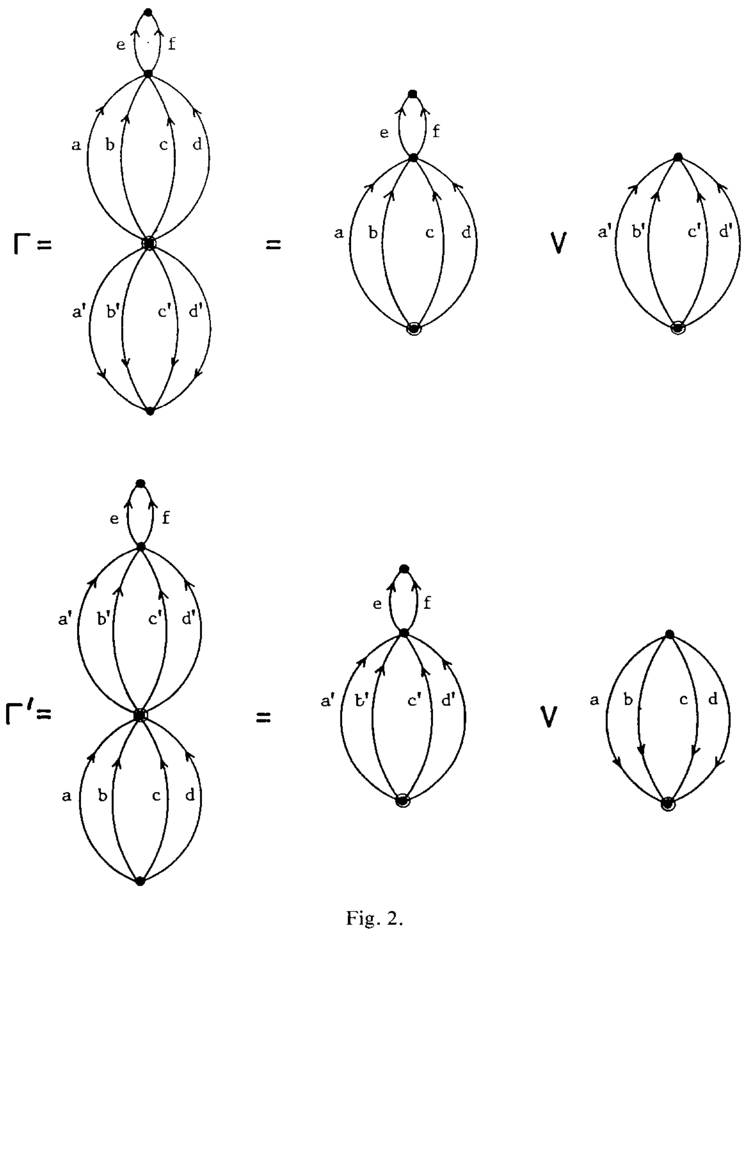


Fig. 2.

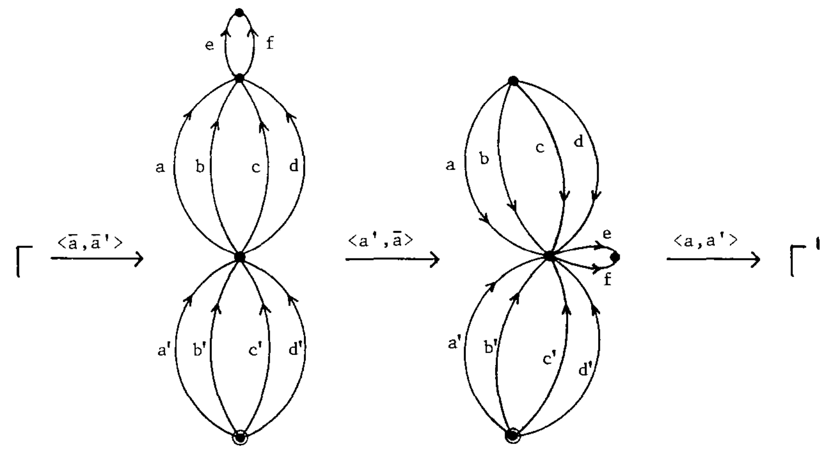


Fig. 3.

transformations may destroy the original decomposition into k factors and restore it differently. For example, take the $\mathbb{Z}_4$-graphs Γ and Γ' in Fig. 2, the action on both being the same: $a \to b \to c \to d \to a$, $a' \to b' \to c' \to d' \to a'$, $e \leftrightarrow f$. Their factors obviously do not belong to the same Nielsen class (the edge sets are different) but Γ and Γ' do belong to the same Nielsen class (Fig. 3).

In order to prove Theorem 1′ we need to have a more detailed knowledge of what is going on in a Nielsen class and, in particular, to be able to distinguish semisimple graphs in it. Fortunately, there is a class of graphs which is easily geometrically characterized and almost coincides with semisimple graphs. These graphs we shall call *subsided* and define them by the condition that no transformation of the form $\langle e, \sigma \rangle$ increases the degree of the basepoint. Thus, a graph is subsided if and only if there are no basic trails for any edge. In the next section we shall prove (Corollary 2) that every semisimple graph is strongly equivalent with a subsided one. So our final restatement of Theorem 1 is the following:

Theorem 2. *All subsided G-graphs belonging to the same Nielsen class are strongly equivalent.*

Since every graph strongly equivalent with a semisimple graph is semisimple itself, it immediately follows that subsided graphs are semisimple. Thus Theorem 2 gives an algorithm for obtaining a maximal decomposition of the action θ_Γ: perform transformations of the form $\langle e, \sigma \rangle$ which increase the degree of $*$ as long as possible; eventually the process will stop at a subsided graph whose maximal decomposition $\Gamma_1 \vee \cdots \vee \Gamma_k$ then gives a maximal decomposition $\theta_{\Gamma_1} * \cdots * \theta_{\Gamma_k}$ of θ_Γ.

4. The main lemma

A Nielsen class $\mathbf{V}$ of G-graphs together with the edge and vertex sets E and V of its members will be fixed throughout Sections 4–6. Let W be the set of all edges $e \in E$ which can be incident with $*$, i.e. such that for some $\Gamma \in \mathbf{V}$ one has $\iota_\Gamma(e) = *$ or $\tau_\Gamma(e) = *$. Elements of W will be called *weak edges*; all others will be called *strong*. The partition $E = W \cup S$ $(S = E - W)$ is central for our treatment; so is the following:

Lemma 1. *If e is a weak edge, then every $\Gamma \in \mathbf{V}$ contains a basic e-trail or a basic $\bar{e}$-trail.*

Proof. We prove the following statement from which the lemma immediately follows: If $\Gamma \in \mathbf{V}$ contains a basic e-trail and if $\langle e_1, e_2 \rangle : \Gamma \to \Gamma'$, then Γ' contains a basic e-trail or a basic $\bar{e}$-trail.

Let σ be a basic e-trail in Γ. If $\langle e_1, e_2 \rangle$ moves neither e nor any edge of σ, then σ remains an e-trail in Γ'. If $\langle e_1, e_2 \rangle$ moves e then, without a loss of generality, $e_1 = e$ or $e_1 = \bar{e}$ and in both cases no edge of σ is moved by $\langle e_1, e_2 \rangle$. If $e_1 = \bar{e}$, then σ remains a basic e-trail in Γ' and if $e_1 = e$, then $\sigma\bar{e}_2$ is a basic e-trail in Γ'.

So we are left with the case when some edge $xe_1^{\pm 1}$ occurs in σ (and, as a consequence, $e \notin Ge_1^{\pm 1}$). Write

$$\sigma = \sigma_0 e_{11}^{\varepsilon_1} \sigma_1 e_{12}^{\varepsilon_2} \cdots e_{1k}^{\varepsilon_k} \sigma_k,$$

where $\varepsilon_i = \pm 1$, $e_{1i} = x_i e_1$ and no $xe_1^{\pm 1}$ occurs in σ_i, $i = 0, 1, \ldots, k$.

If in this situation $e_2 \notin Ge^{\pm 1}$, then

$$\sigma' = \sigma_0 (e_{11} e_{21})^{\varepsilon_1} \sigma_1 (e_{12} e_{22})^{\varepsilon_2} \cdots (e_{1k} e_{2k})^{\varepsilon_k} \sigma_k,$$

with $e_{2i} = x_i e_2$, is a basic e-trail in Γ'. (See Fig. 4 for an example.)

The final case to be considered is when, with σ as above, e_2 or $\bar{e}_2$ is in the orbit Ge. If $\varepsilon_1 = -1$, then σ_0 is a basic e_{21}-trail in both Γ and Γ' (Fig. 5(a)). If $\varepsilon_1 = 1$, then $\sigma_0 e_{11}$ is a basic $\bar{e}_{21}$-trail in Γ' (Fig. 5(b)). Since $e \in Ge_{21}^{\pm 1}$, it follows that Γ' contains a basic e-trail or a basic $\bar{e}$-trail. □

Corollary 1. *In a subsided graph, every weak edge is incident with* $*$. □

Corollary 2. *Every semisimple G-graph is strongly equivalent with a subsided graph.*

Fig. 4.

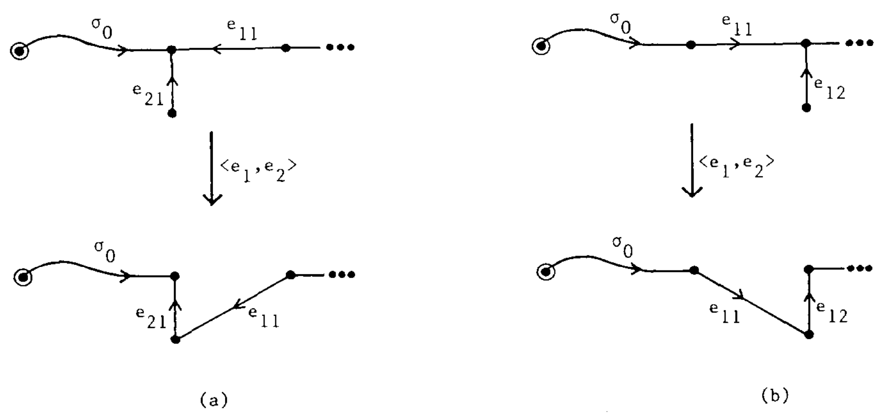


Fig. 5.

Proof. Let Γ be semisimple and $\Gamma=\Gamma_1\vee\cdots\vee\Gamma_k$ its maximal factorization. Suppose Γ is not subsided, i.e. there is an edge e not incident with $*$ in Γ and a basic e-trail σ in Γ. We may assume that σ is without self-intersections, so e and σ entirely belong to a Γ_i, say to Γ_1. Now $\langle e,\sigma\rangle\Gamma=(\langle e,\sigma\rangle\Gamma_1)\vee\Gamma_2\vee\cdots\vee\Gamma_k$ is strongly equivalent with Γ (because $\langle e,\sigma\rangle\Gamma_1$ is simple). Continuing the process we eventually arrive at a subsided graph strongly equivalent with Γ. □

Corollary 3. *No Nielsen transformation is of the form* $\langle e_1,e_2\rangle$ *with* e_1 *strong and* e_2 *weak. (Strong edges cannot move along weak edges.)*

Proof. Assume the contrary. There are two cases to be considered separately.

(i) $\mathrm{Stab}(e_1)=\mathrm{Stab}(e_2)$. Consider Fig. 6. It follows by symmetry that $e_1\in W\Leftrightarrow e_2\in W$ – a contradiction. (To put this in more detail, let $\Gamma'=\langle\bar e_1\bar e_2\rangle\langle\bar e_2,e_1\rangle\langle e_1,e_2\rangle\Gamma$. There is an equivariant isomorphism $f:\Gamma\to\Gamma'$ which fixes V and all edges not belonging to the orbit of $e_1^{\pm 1}$ or $e_2^{\pm 1}$ and such that $f(xe_1)=xe_2$ and $f(xe_2)=x\bar e_1$ for all $x\in G$. So σ is a basic $e_2^{\pm 1}$-trail in Γ if and only if $f(\sigma)$ is a basic $e_1^{\mp 1}$-trail in Γ'.)

(ii) $\mathrm{Stab}(e_1)\subsetneqq\mathrm{Stab}(e_2)$. Here $\mathrm{Stab}(e_1)\subsetneqq\mathrm{Stab}(\sigma)$ for every basic e_2-trail σ and so no edge of $Ge_1^{\pm 1}$ occurs in σ. Thus basic e_2-trails are in this case basic e_1-trails too. Similarly, if σ is a basic $\bar e_2$-trail, then σe_2 is a basic e_1-trail. So e_1 must be a weak edge – a contradiction. □

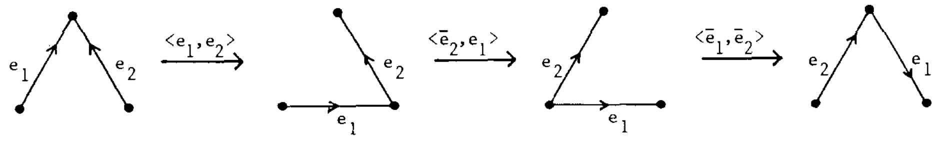


Fig. 6.

Corollary 4. *Two vertices which can be joined by a strong path (that is, a path consisting entirely of strong edges) in some $\Gamma \in \mathbf{V}$ can be joined by a strong path in every $\Gamma \in \mathbf{V}$.*

Proof. Suppose that vertices a and b can be joined by a strong path in Γ and let $\Gamma' = \langle e, e' \rangle \Gamma$. It suffices to show that a and b can be joined by a strong path in Γ'. If e is a weak edge, this is true because in this case the strong paths in Γ' are the same as the strong paths in Γ. It is also true if both e and e' are strong: every occurrence of $xe^{\pm 1}$ in a path joining a with b in Γ should be replaced by $x(ee')^{\pm 1}$. The remaining case when e is strong and e' weak does not occur by Corollary 3. □

Every G-graph admits an obvious decomposition $\Gamma = R(\Gamma) \vee S(\Gamma)$, where $R(\Gamma)$ -the rose of Γ - consists of all loops at $*$. Now we are in position to define a much finer natural decomposition of subsided graphs. Observe first that Corollary 4 defines an invariant equivalence relation on $V - \{*\}$, $a \sim b$ if there is a strong path connecting a with b in some (and hence in all) $\Gamma \in \mathbf{V}$. Let $\approx$ be the smallest equivalence relation on $V - \{*\}$ which contains $\sim$ and has invariant classes and let $V - \{*\} = V_1 + V_2 + \cdots + V_k$ be the partition into equivalence classes of $\approx$.

Corollary 5. *Every subsided graph $\Gamma \in \mathbf{V}$ has a decomposition*

$$\Gamma = R(\Gamma) \vee S_1(\Gamma) \vee \cdots \vee S_k(\Gamma),$$

where the vertex set of $S_r(\Gamma)$ is $V_r \cup \{*\}$ $(1 \le r \le k)$.

Proof. Follows from Corollaries 1 and 4. □

Our strategy for proving Theorem 2 will be to show that for every two subsided graphs $\Gamma, \Gamma' \in \mathbf{V}$, $R(\Gamma)$ and $R(\Gamma')$ are strongly equivalent and that $S_r(\Gamma)$ and $S_r(\Gamma')$ are simple and equivalent.

Since strong edges can move only along strong edges (Corollary 3) it easily follows that for every strong edge e there is r $(1 \le r \le k)$ such that for every $\Gamma \in \mathbf{V}$ both $\iota_\Gamma(e)$ and $\tau_\Gamma(e)$ belong to V_r. Thus we obtain the invariant partition $S = S_1 + \cdots + S_k$ of the set of strong edges with the property that $e \in S_r$ if and only if $\iota_\Gamma(e), \tau_\Gamma(e) \in V_r$ for every $\Gamma \in \mathbf{V}$.

Observe that if $\langle e, e' \rangle$ is a Nielsen transformation applicable to some $\Gamma \in \mathbf{V}$ and $e \in S_r$, then $e' \in S_r$ too. It follows that if $\Gamma' = \langle e_r, e_r' \rangle \langle e_s, e_s' \rangle \Gamma$ and $e_r \in S_r$, $e_s \in S_s$, then $\Gamma' = \langle e_s, e_s' \rangle \langle e_r, e_r' \rangle \Gamma$. Assume now $\Gamma' = \langle e_r, e_r' \rangle \langle e, e' \rangle \Gamma$ with $e_r \in S_r$ and $e \in W$. If $e' \notin Ge_r^{\pm 1}$, then we have $\Gamma' = \langle e, e' \rangle \langle e_r, e_r' \rangle \Gamma$. If $e' \in Ge_r^{\pm 1}$, it is no loss of generality to assume $e' = e_r$ or $e' = \bar{e}_r$. It is easy to check that in the first case we have $\Gamma' = \langle e, e_r e_r' \rangle \langle e_r, e_r' \rangle \Gamma$ and in the second case $\Gamma' = \langle e, \bar{e}_r' \bar{e}_r \rangle \langle e_r, e_r' \rangle \Gamma$. As a consequence we obtain

Lemma 2. *Let $\Gamma, \Gamma' \in \mathbf{V}$. Then $\Gamma' = T_w T_k T_{k-1} \cdots T_1 \Gamma$ where T_w moves only weak edges and T_r moves only edges belonging to S_r $(1 \le r \le k)$.* □

5. A characterization of subsided graphs

From the previous section we see that it is weak edges which make complications in a Nielsen class $\mathbf{V}$. For, according to Corollary 3, the strong parts of graphs of $\mathbf{V}$ are rather independent and, since strong edges always stay away from $*$, their impact on decompositions of graphs of $\mathbf{V}$ is clear – they just keep all vertices of each V_r in the same factor for every decomposition. On the other hand, weak edges behave rather wildly; for instance, they can move from $S_r(\Gamma)$ to $S_{r'}(\Gamma')$ as is demonstrated by the example following the statement of Theorem 1′. The aim of this section is to understand their behaviour; after that, the proof of Theorem 2 will be straightforward.

First we need to generalize the relation $\sim$ on vertices. Let H be a subgroup of G. For every $\Gamma \in \mathbf{V}$ we define $\sim_{\Gamma,H}$ on $V - \{*\}$ by $a \sim_{\Gamma,H} b$ if a and b can be connected in Γ by a strong path fixed by H. (For $H = 1$ we have $\sim_{\Gamma,H}$ is just $\sim$.) In the same way as we did in Corollary 4, we can now prove that $\sim_{\Gamma,H}$ does not depend on Γ. So we denote this equivalence relation by $\sim_H$. Its classes will be called *H-classes* and $C_H(a)$ will denote the H-class containing the vertex a. Clearly $H \subseteq H'$ implies $C_{H'}(a) \subseteq C_H(a)$; in particular, every H-class is contained in some V_r.

An edge e will be called an *H-edge* if $\mathrm{Stab}(e) = H$. We shall say that an H-class C is *occupied in Γ* if C contains an endpoint of a weak H-edge. Notice that it is possible that a subset C of $V - \{*\}$ be at the same time an H-class and an H'-class for $H' \ne H$ but its being occupied as an H-class is not the same as being occupied as an H'-class. Finally, we define an H-class to be an *essential H-class* if it is occupied (as an H-class) in every $\Gamma \in \mathbf{V}$.

Lemma 3. *Let C be an H-class occupied in some $\Gamma \in \mathbf{V}$. Then, for every $\Gamma' \in \mathbf{V}$ there is $H' \supseteq H$ and an H'-class C' which is contained in C and occupied in Γ'.*

Proof. We can assume $\Gamma' = \langle e, e' \rangle \Gamma$. The lemma is true with $H' = H$ and $C' = C$ unless e is a weak H-edge with $\tau_\Gamma(e) \in C$ and e' a weak edge. In the latter case we can take $H' = \mathrm{Stab}(e')$ and $C' = C_{H'}(a)$. □

Corollary 6. *Every H-class occupied in some $\Gamma \in \mathbf{V}$ contains an essential H'-class for some $H' \supseteq H$.*

Proof. Let C be an H-class occupied in Γ. If it is not essential, there is $\Gamma' \in \mathbf{V}$ in which C is not occupied. But, by Lemma 3, there must be an H'-class $C' \subseteq C$ which is occupied in Γ'. Since $H' \supsetneq H$, iterating the argument we shall eventually end up with an essential class contained in C. □

The last corollary guarantees the existence of essential classes. Observe that xC $(x\in G)$ is an essential H^x-class whenever C is an essential H-class $(H^x=xHx^{-1})$. Thus G acts on the set of all essential classes. (Again we want to remark that the same subset of V may be an essential H-class for different H and that we should distinguish the H-class C and the H'-class C when $H\neq H'$. Of course, we could have put this in a formally correct way by defining classes as pairs (C,H) etc.)

Define now $\mathbf{O}_1,\dots,\mathbf{O}_m$ to be all orbits of essential classes. For every $\mathbf{O}_i$ there is $H\leq G$ such that every element of $\mathbf{O}_i$ is an essential H'-class, where H' is a conjugate of H. So we may define the *index* of $\mathbf{O}_i$ by $\mathrm{ind}(\mathbf{O}_i)=|G:H|$. Now we state the main result of this section.

Lemma 4. *Let* $d=\#W-\sum_{i=1}^m \mathrm{ind}(\mathbf{O}_i)$. *For every* $\Gamma\in\mathbf{V}$ *and every* i, $1\leq i\leq m$, *the number of weak edges having in* Γ *an endpoint in* $\bigcup_{C\in\mathbf{O}_i} C$ *is* $\geq\mathrm{ind}(\mathbf{O}_i)$. *For every* $\Gamma\in\mathbf{V}$ *one has* $\deg_\Gamma(*)\leq d$ *with the equality if and only if* Γ *is subsided.*

Proof. Let C be an H-class belonging to $\mathbf{O}_i$. So there is a weak H-edge e such that $\tau_\Gamma(e)\in C$. Then for every $x\in G$, $\tau_\Gamma(xe)\in xC\in\mathbf{O}_i$. Since $\#Ge=|G:H|=\mathrm{ind}(\mathbf{O}_i)$, the first statement of the lemma follows.

Now we prove $\deg_\Gamma(*)\leq d$ for every $\Gamma\in\mathbf{V}$. In view of Lemma 1 we may assume that $*$ is an endpoint in Γ of every weak edge. By the first paragraph of the proof, there exist weak edges $e_1,\dots,e_m$ such that $\tau_\Gamma(xe_i)\in\bigcup\mathbf{O}_i$ for every $x\in G$ and $1\leq i\leq m$. Since $\#Ge_i=\mathrm{ind}(\mathbf{O}_i)$, it suffices to prove that $e_i\notin Ge_j$ when $i\neq j$. Assume the contrary, i.e. $e_j=xe_i$ for some $x\in G$ and $i\neq j$. Let $H=\mathrm{Stab}(e_i)$. So there is an essential H-class $C\in\mathbf{O}_i$ and an essential H^x-class $C'\in\mathbf{O}_j$ such that $\tau_\Gamma(e_i)\in C$ and $\tau_\Gamma(e_j)\in C'$. Since xC is an essential H^x-class which contains $\tau_\Gamma(xe_i)$ and $e_j=xe_i$, it follows that $xC=C'$ - a contradiction.

So we have proved $\deg(*)\leq d$ for every $\Gamma\in\mathbf{V}$. A consequence of this is that every Γ with $\deg_\Gamma(*)=d$ is subsided. It only remains to prove that a graph $\Gamma\in\mathbf{V}$ with $\deg_\Gamma(*)<d$ cannot be subsided.

Assume $\deg_\Gamma(*)<d$ and that in Γ every weak edge had $*$ as an endpoint. (If the last assumption is not met, Γ is not subsided by Lemma 1.) Let as above $e_1,\dots,e_m$ be weak edges emanating from $*$ and terminating at vertices which respectively belong to essential classes $C_1,\dots,C_m$ such that $C_i\in\mathbf{O}_i$. From $\deg_\Gamma(*)<d$ we get that there exists a weak edge e with $\iota_\Gamma(e)=*\neq\tau_\Gamma(e)$ and $e\notin Ge_1\cup\cdots\cup Ge_m$. Let $H=\mathrm{Stab}(e)$ and let C be the H-class containing τe. From Lemma 2 we obtain an essential H'-class $C'\subseteq C$ with $H'\supseteq H$. Then $C'=xC_j$ for some $x\in G$ and $1\leq j\leq m$. Let σ be a strong path fixed by H which connects $\tau_\Gamma(xe_j)\in C'$ with $\tau_\Gamma(e)$. Then $(xe_j)\sigma$ is a basic e-trail in Γ and therefore Γ is not subsided. □

6. Proof of Theorem 2

Assuming that $\Gamma,\Gamma'\in\mathbf{V}$ are subsided we shall prove the following three facts:

(a) $R(\Gamma)$ is strongly equivalent with $R(\Gamma')$;
(b) each $S_r(\Gamma)$ is simple;
(c) $S_r(\Gamma)$ is equivalent with $S_r(\Gamma')$, $1 \le r \le k$.
In view of Corollary 5, this will prove Theorem 2.

(a) We have two partitions $W = W_0 + W_1$ and $W = W_0' + W_1'$ where W_0 and W_0' are the sets of all loops at $*$ in Γ and Γ' respectively. Both partitions are invariant. Furthermore, W_0 and W_0' are respectively the edge sets of $R(\Gamma)$ and $R(\Gamma')$. Proving the equivalence of $R(\Gamma)$ and $R(\Gamma')$ is clearly the same as proving that W_0 and W_0' are equivalent G-sets. We shall prove that W_1 and W_1' are equivalent G-sets and this will clearly suffice.

Let $e_1, \dots, e_m$ be weak edges such that ($\iota_\Gamma(e_i) = *$ and) $\tau_\Gamma(e_i) \in \bigcup \mathbf{O}_i$. We have seen in the proof of Lemma 4 that $Ge_i \cap Ge_j = \emptyset$ for $i \ne j$ and that $W_1 = Ge_1 + \cdots + Ge_m + G\bar{e}_1 + \cdots + G\bar{e}_m$. Similarly, $W_1' = Ge_1' + \cdots + Ge_m' + G\bar{e}_1' + \cdots + G\bar{e}_m'$, where $\iota_{\Gamma'}(e_i') = *$ and $\tau_{\Gamma'}(e_i') \in \bigcup \mathbf{O}_i$. Since e_i and e_i' have conjugate stabilizers, the orbits Ge_i and Ge_i' are equivalent G-sets. Hence W_1 and W_1' are equivalent G-sets.

(b) Suppose $S_1(\Gamma)$ is not simple. Then, for some transformation T, $TS_1(\Gamma) = \Delta' \vee \Delta''$, where Δ' and Δ'' are non-trivial. Then $T\Gamma = R(\Gamma) \vee \Delta' \vee \Delta'' \vee S_2(\Gamma) \vee \cdots \vee S_k(\Gamma) \in \mathbf{V}$ and in view of Corollary 4 and the definition of the partition $V - \{*\} = V_1 + \cdots + V_k$ it follows that one of Δ', Δ'' (say Δ') is a rose. Thus $R(T\Gamma) \supseteq R(\Gamma) \vee \Delta'$. But this contradicts the fact (which immediately follows from Lemma 4) that the number of edges in the rose of any graph in $\mathbf{V}$ is $\le \# W - 2\sum \operatorname{ind}(\mathbf{O}_i)$ with the equality if and only if the graph is subsided.

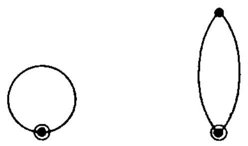
(c) Write $\Gamma' = T_w T_k T_{k-1} \cdots T_1 \Gamma$ where, as in Lemma 2, T_w moves only weak edges and T_r $(1 \le r \le k)$ moves only edges belonging to $S_r(\Gamma)$.

Let $\Gamma'' = T_k T_{k-1} \cdots T_1 \Gamma$. Then Γ'' is subsided, $R(\Gamma'') = R(\Gamma)$ and $S_r(\Gamma'') = T_r S_r(\Gamma)$ for every r. In particular, $S_r(\Gamma'')$ is equivalent with $S_r(\Gamma)$. Therefore, in order to prove that $S_r(\Gamma)$ is equivalent with $S_r(\Gamma')$ we may assume that $\Gamma = \Gamma''$, i.e. that all transformations $T_1, \dots, T_k$ are trivial.

So let $\Gamma' = T_w \Gamma$. Let $\mathbf{O}_1', \dots, \mathbf{O}_p' \in \{\mathbf{O}_1, \dots, \mathbf{O}_m\}$ be all orbits of essential classes contained in V_r. Choose a representative C_i in each $\mathbf{O}_i'$ and let $H_i \subseteq G$ be such that C_i is an essential H_i-class $(1 \le i \le p)$. Let e_i and e_i' be weak H_i-edges such that $\tau_\Gamma(e_i) \in C_i$ and $\tau_{\Gamma'}(e_i') \in C_i$. Since Γ and Γ' are subsided, $S_r(\Gamma)$ and $S_r(\Gamma')$ are subsided too and hence $W_r = (Ge_1 + \cdots + Ge_p)^{\pm 1}$ and $W_r' = (Ge_1' + \cdots + Ge_p')^{\pm 1}$ are sets of weak edges of $S_r(\Gamma)$ and $S_r(\Gamma')$ respectively.

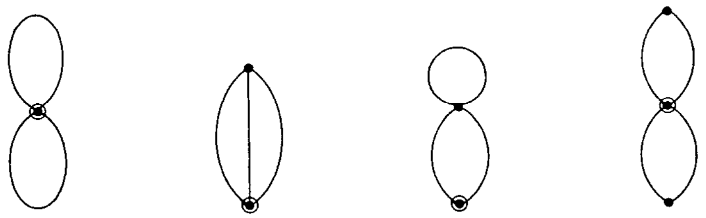
Let σ_i $(1 \le i \le p)$ be a strong path in Γ connecting $\tau_{\Gamma'}(e_i')$ with $\tau_\Gamma(e_i)$. Then $T = \langle e_1, \sigma_1 \rangle \cdots \langle e_p, \sigma_p \rangle$ is a transformation applicable to Γ. Let $\Gamma'' = T\Gamma$; we have got $\tau_{\Gamma''}(e_i) = \tau_{\Gamma'}(e_i')$. Also $S_r(\Gamma'') = TS_r(\Gamma)$, so $S_r(\Gamma'')$ and $S_r(\Gamma)$ are equivalent.

It remains to see that $S_r(\Gamma'')$ and $S_r(\Gamma')$ are equivariantly isomorphic. Both graphs have the vertex set $V_r \cup \{*\}$ and their edge sets are $W_r \cup S_r$ and $W_r' \cup S_r$ respectively. It is easy to see that the graph map $\phi : S_r(\Gamma'') \to S_r(\Gamma')$ defined to be the identity map on vertices and on S_r, and on W_r given by $\phi : xe_i^{\pm 1} \mapsto (xe_i')^{\pm 1}$, is an equivariant isomorphism.



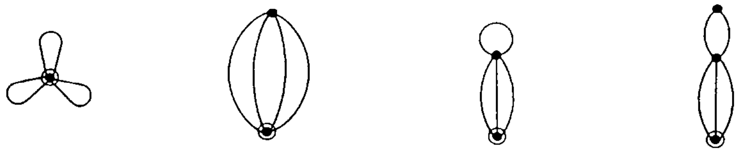
1.1 (1,1) 1.2 (2,1)

Fig. 7. Rank one.

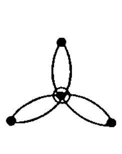


2.1 (2,1) 2.2 (3,1) 2.3 (2,2) 2.4 (4,1)

Fig. 8. Rank two.



3.1 (3,1) 3.2 (4,1) 3.3 (3,2) 3.4 (6,2)



3.5 (6,2) 3.6 (2,3) 3.7 (6,1)

Fig. 9. Rank three.

7. A list of indecomposable cyclic actions on F_n, $n \leq 5$

This section contains empirical material only. We just list all non-equivalent reduced indecomposable subsided $\mathbb{Z}_k$-graphs of rank ≤ 5 and all possible k. Making the list did not require any particular effort.

The base vertex is in each case distinguished by encircling.

Under each graph there is a pair of numbers (k, m) where k is the order of the

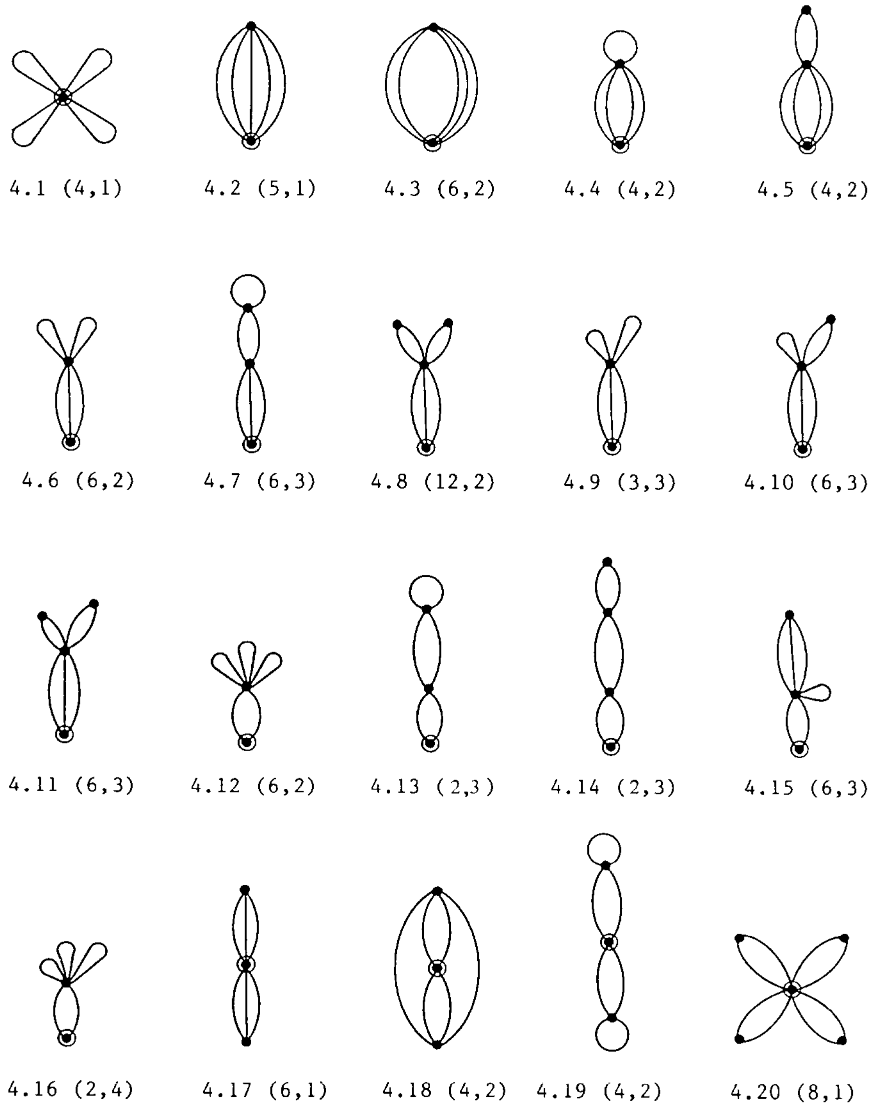


Fig. 10. Rank four.

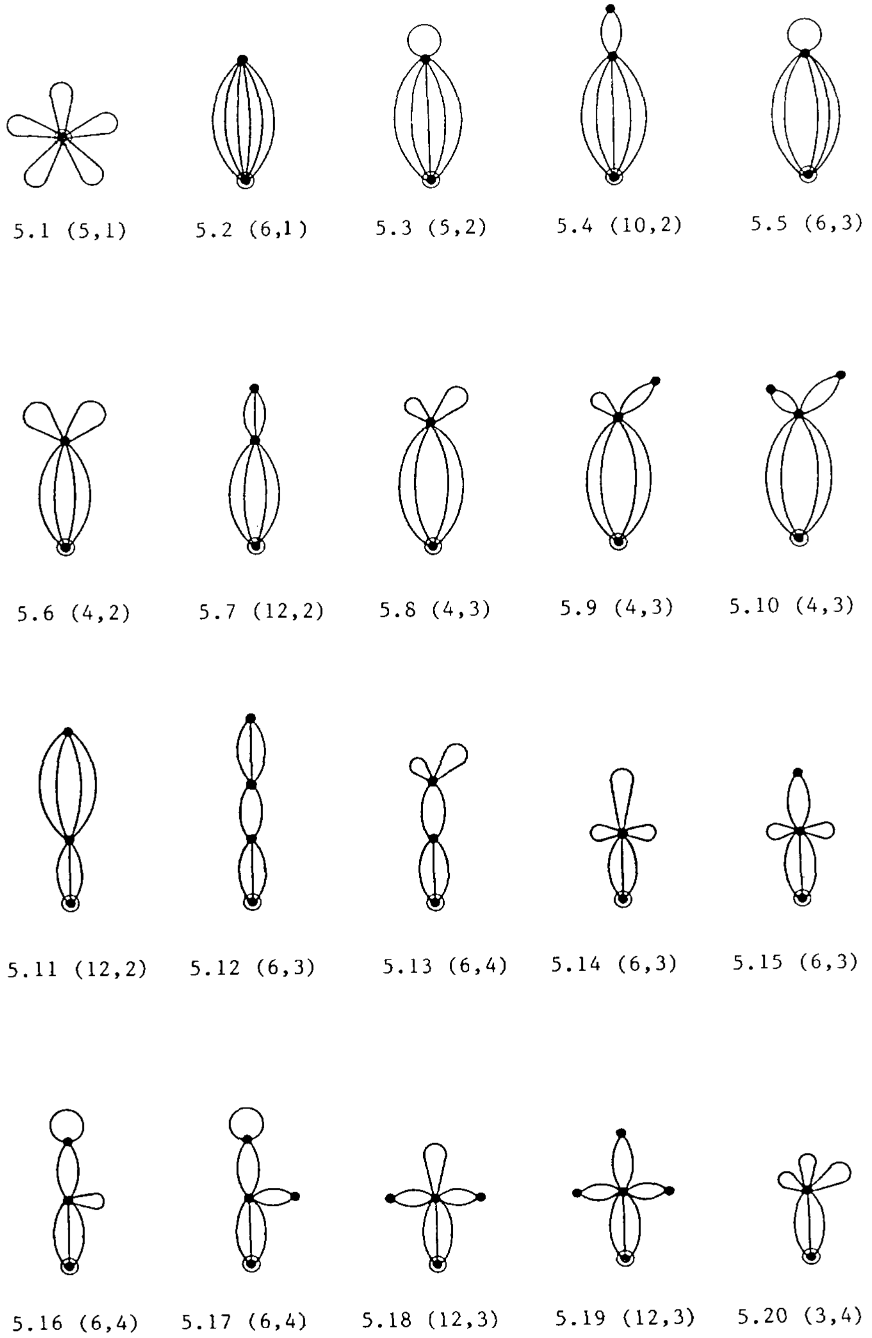
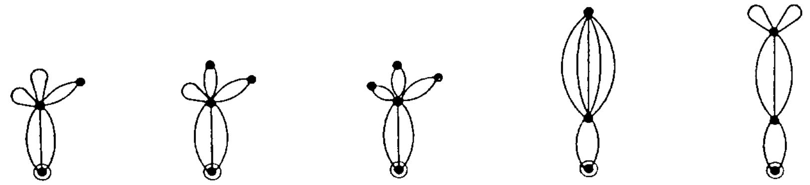
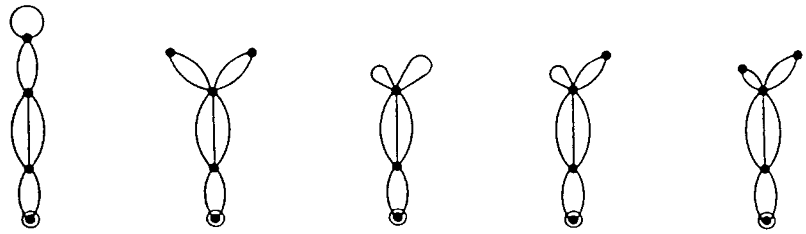


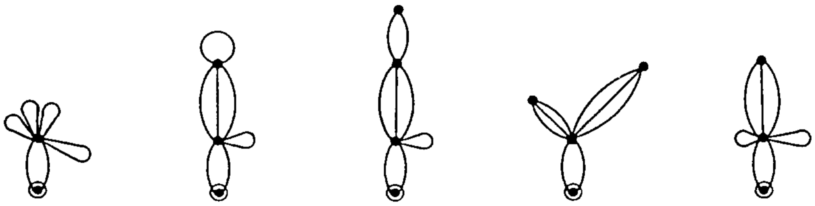
Fig. 11. Rank five.



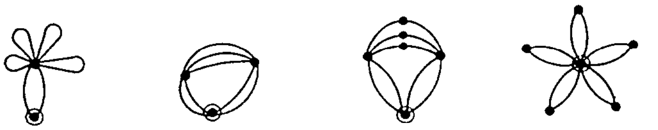
5.21 (6,4) 5.22 (6,4) 5.23 (6,4) 5.24 (10,2) 5.25 (12,3)



5.26 (6,4) 5.27 (12,3) 5.28 (6,4) 5.29 (6,4) 5.30 (6,4)



5.31 (6,3) 5.32 (6,4) 5.33 (6,4) 5.34 (6,3) 5.35 (6,4)



5.36 (2,5) 5.37 (6,3) 5.38 (12,2) 5.39 (10,2)

Fig. 11. (Contd.)

cyclic group acting and m is the number of orbits of geometrical edges. The graph and the pair (k, m) uniquely (and in the way which should be visible) determine the action. So there is no need to write down the actions explicitly.

8. Periodic automorphisms of free groups of rank ≤5

The classification of periodic automorphisms of F_n is for $n=2$ due to Meskin [12] and for $n=3$ to McCool [11]. Using the list of the preceding section, here we reprove their results and extend to obtaining all periodic automorphisms of F_n for $n=4$ and $n=5$.

There is a slight difference between the task of classifying (up to conjugacy) automorphisms of F_n of order k and that of classifying (up to equivalence) $\mathbb{Z}_k$-graphs of rank n. For the latter is in fact the task of classifying (up to conjugacy) cyclic subgroups of order k in $\operatorname{Aut} F_n$. And there may well be non-conjugate elements of the same order in $\operatorname{Aut} F_n$ such that the cyclic subgroups of $\operatorname{Aut} F_n$ they generate are conjugate. A minimal example of this (and incidentally a counter-example to [5, Corollary 2]) follows.

Example. Consider graphs Γ_1 and Γ_2 depicted in Fig. 12. Let a_i be the (essentially unique) automorphism of order 10 of Γ_i which permutes vertices so that $1 \to 2 \to 3 \to 4 \to 5 \to 1$. Fix a generator a of $\mathbb{Z}_{10}$ and let $\theta_i : \mathbb{Z}_{10} \to \operatorname{Aut} \Gamma_i$, $\theta_i(a) = a_i$ define the structure of a $\mathbb{Z}_{10}$-graph on Γ_1 and Γ_2. Let α_1 and α_2 be automorphisms of F_{10} realized by a_1 and a_2 respectively. Of course, α_1 and α_2 are determined up to conjugacy. For both Γ_1 and Γ_2 there are two Nielsen transformations applicable ('rotation of radial edges in $\pm 2\pi/5$' for Γ_1 and 'rotation of radial edges in $\pm 4\pi/5$' for Γ_2). As is easily seen, both Nielsen tranformations applicable to Γ_i give rise to a $\mathbb{Z}_{10}$-graph equivariantly isomorphic with Γ_i. Since Γ_1 and Γ_2 are reduced and since, as is also easily seen, they are not equivariantly isomorphic, it follows by Theorem

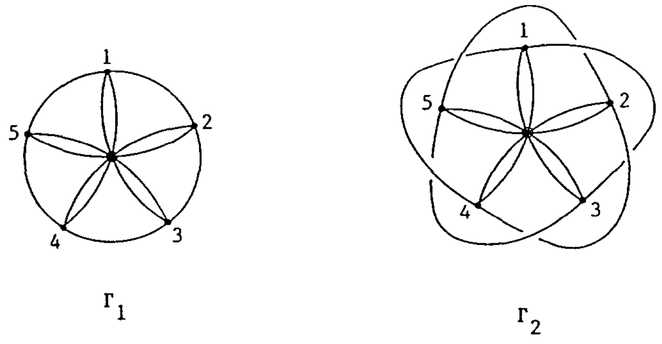


Fig. 12.

B that Γ_1 and Γ_2 are non-equivalent $\mathbb{Z}_{10}$-graphs. In other words, α_1 and α_2 are not conjugate.

On the other hand, if we define the action of $\mathbb{Z}_{10}$ on Γ_1 by $\theta_1' : \mathbb{Z}_{10} \to \operatorname{Aut} \Gamma_1$, $\theta_1'(a) = a_1^3$, then Γ_1 and Γ_2 become equivariantly isomorphic. That is, α_1^3 is conjugate in Aut F_{10} with α_2.

Remark. In the same manner one can, for every $k \geq 10$ which is not a prime, get an automorphism α of some F_n which has order k and is not conjugate to some of its powers of the same order. Khramtsov (private communications) has constructed an automorphism of F_{215} of order 60 which is not conjugate to any of its powers. The number 10 above is minimal because, if $\alpha \in \operatorname{Aut} F_n$ is of order $k < 10$ and if $\gcd(t,k) = 1$, then α is conjugate in Aut F_n with α^t. The number 10 is minimal in another sense too: If $\alpha \in \operatorname{Aut} F_n$, $n < 10$, and if α and α^t are of the same order, then they are conjugate. Both these minimality statements have straightforward proofs which we do not include.

So, in general, a given $\mathbb{Z}_k$-graph may realize several non-conjugate automorphisms. However, this is not the case for any of the graphs in Figs 7–11. For, as we have already remarked and is readily seen, there is essentially (i.e. up to an equivariant isomorphism) only one $\mathbb{Z}_k$-structure on each of them, k being the first component of the pair of numbers associated with each of these graphs.

We can conclude that irreducible automorphisms of F_n, $n \leq 5$ are all in fact given in Figs 7–11. In general, for every automorphism α of Aut F_n there is a maximal decomposition $F_n = F_{n_1} * \cdots * F_{n_r}$ invariant under α and if α_j $(j = 1, \ldots, r)$ is the restriction of α on F_{n_j}, then α_j are all irreducible. Moreover, by Theorem 1, the

Table 1. Rank two (Meskin)

type	order				
	1	2	3	4	
(2)		2	1	1	4
(1, 1)	1	2			3
	1	4	1	1	7

Table 2. Rank three (McCool)

type	order					
	1	2	3	4	6	
(3)		1	2	1	3	7
(1, 2)		4	1	2		8
(1, 1, 1)	1	3				4
	1	8	3	3	4	19

Table 3. Rank four

type	order								
	1	2	3	4	5	6	8	12	
(4)		1	3	5	1	8	1	1	20
(1,3)		2	2	2		8			14
(2,2)		3	1	3		2		1	10
(1,1,2)		6	1	3		2			12
(1,1,1,1)	1	4							5
	1	16	7	13	1	20	1	2	61

Table 4. Rank five

type	order									
	1	2	3	4	5	6	8	10	12	
(5)		1	1	4	2	21		3	7	39
(1,4)		6	1	10	1	17	2	1	2	40
(2,3)		2	2	4		14			6	28
(1,1,3)		3	2	3		13				21
(1,2,2)		6	1	6		5			2	20
(1,1,1,2)		8	1	4		3				16
(1,1,1,1,1)	1	5								6
	1	31	8	31	3	73	2	4	17	170

numbers r and $n_1, \ldots, n_r$ such that $1 \leq n_1 \leq \cdots \leq n_r$ and automorphisms $\alpha_1, \ldots, \alpha_r$ are all uniquely determined by α. We call the r-tuple $(n_1, \ldots, n_r)$ the *type* of α. Clearly, the order of α is the least common multiple of orders of α_j.

Therefore, to find all automorphisms of F_n, $n \leq 5$, we consider all partitions $n = n_1 + \cdots + n_r$ and all possible r-tuples $(\alpha_1, \ldots, \alpha_r)$ where α_j is an irreducible automorphism of F_{n_j}. An account of the numbers of automorphisms obtained this way is given in Tables 1–4.

9. Elements of prime order in Aut F_n

Elements of prime order in Aut F_n were described first by Dyer and Scott [3] (cf. also [2]). Here we derive their result and the corresponding uniqueness result. The latter is that every element of prime order p is, up to conjugacy, determined by the free product decomposition given in [3, Theorem 3].

So let p be a prime and Γ a reduced subsided $\mathbb{Z}_p$-graph. If e is an edge of Γ such that $\mathrm{Stab}(e) = 1$, then e can move along any edge incident with it which in turn implies that e is weak edge and so e is incident with the basepoint. Assume that $\iota(e) = *$.

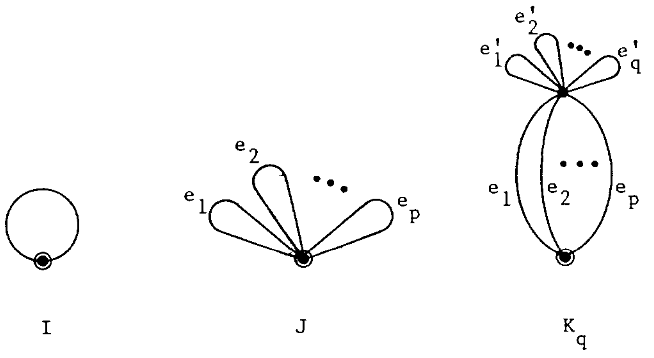


Fig. 13.

Then $\tau(xe)=\tau(e)$, for otherwise Γ would not be reduced. Now it follows that the stabilizer of every strong edge is the whole $\mathbb{Z}_p$, i.e. that strong edges are all loops. From these observations one easily concludes that the only indecomposable reduced subsided $\mathbb{Z}_p$-graphs are the ones depicted in Fig. 13.

The edges $e_1',\ldots,e_q'$ in K_q are fixed and the edges $e_1,\ldots,e_p$ are in both J and K_q, permuted by the action of $\mathbb{Z}_p$.

Clearly, J realizes a unique (up to conjugacy) element of order p in Aut F_p and similarly K_q realizes a unique element of order p in Aut F_{p+q-1}.

Every $\mathbb{Z}_p$-graph is then equivalent to a graph of the form

$$\Gamma(s,t,q_1,\ldots,q_r)=I^s * J^t * K_{q_1} * \cdots * K_{q_r},$$

where I^s denotes the wedge product of s copies of I, and similarly for J^t. The rank of this graph is $s+tp+r(p-1)+q_1+\cdots+q_r$. By Theorem 1, $\Gamma(s,t,q_1,\ldots,q_r)$ is equivalent with $\Gamma(s',t',q_1',\ldots,q_{r'}')$ if and only if the tuples $(s,t,q_1,\ldots,q_r)$ and $(s',t',q_1',\ldots,q_{r'}')$ are equal, up to a permutation of q's.

If follows that the tuples $(s,t,q_1,\ldots,q_r)$ with properties

$$s,t,r\geq 0,\qquad 0\leq q_1\leq\cdots\leq q_r,\qquad s+tp+r(p-1)+q_1+\cdots+q_r=n$$

classify automorphisms of order p of F_n.

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