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A vanishing theorem for local cohomology modules

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Abstract

A conjecture of Huneke and Lyubeznik regarding the vanishing of certain local cohomology modules is proved in a special case. A topological counterpart of the vanishing result is also proved.

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All rings are assumed to be commutative, noetherian and with identity.

In [4, Theorem 3.8] Huneke and Lyubeznik prove, roughly speaking, that if k is a separably closed field, $R = k[[X_1,\dots,X_d]]$, and I is a nonzero and nonmaximal prime ideal of height c then $\mathrm{cd}(R,I) \leq d-1-[(d-2)/c]$, where $\mathrm{cd}(R,I)$ denotes the local cohomological dimension of R with respect to I, that is, the least integer n such that $H_I^j(R) = 0$ for all $j > n$. They also conjectured (cf. [3, Conjecture 2.4]) that the same result should hold if R is a regular local ring with separably closed residue field and I is an ideal such that the nilradical of I is the intersection of t primes of maximal height c. The main goal of this paper is to prove the conjecture if I is the intersection of 'not too many' minimal primes P_i, such that R/P_i is a normal domain for all i (cf. Theorem 2). We also prove a topological corollary of our main result, which, roughly speaking, gives the vanishing of singular homology groups of projective algebraic sets depending upon the number of irreducible components and the codimension.

We begin with some notations and definitions. For a local ring $(R,\mathfrak{m}_R)$, we recall that $\mathrm{embdim}(R) = \dim_{R/\mathfrak{m}_R}(\mathfrak{m}_R/\mathfrak{m}_R^2)$. We set $\widehat{R}$ to be the completion of R with respect to the $\mathfrak{m}_R$-adic topology, and R^{sh} the strict henselization. The bigheight of an ideal I, $\mathrm{bight} I$, is the maximum height of the minimal primes of I.

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Definition 1 (*Huneke and Lyubeznik [4, Definition 2.1]*). Let R be a local ring. We set

$$\operatorname{mdim}(R) = \min\{\dim(R/Q) \mid Q \text{ is a minimal prime of } R\}.$$

If I is an ideal of R, then we let $c(I) = \operatorname{embdim}(R) - \operatorname{mdim}(R/I)$.

If R is a regular local ring then $c(I) = \operatorname{bight} I$. We now state the main result of this paper:

Theorem 2. *Let R be an excellent d-dimensional local ring containing a field and let $I = P_1 \cap \cdots \cap P_t$, where $P_i \in \operatorname{Spec} R$ for all i. Assume that the rings R/P_i are normal. Let $c(I\widehat{R}) = c$ and assume that $0 < c \le (d-1)/t$. Then $H_I^q(R) = 0$ for all $q \ge d - [(d-2)/c]$. In particular, $\operatorname{cd}(R,I) \le d - 1 - [(d-2)/c]$.*

In the proof of Theorem 2, we use induction on t. For such purpose, we apply a theorem of Huneke and Lyubeznik [4], which they call 'an induction theorem':

Theorem 3 (Huneke and Lyubeznik [4, Theorem 2.5]). *Let R be a noetherian local ring containing a field and let I be an ideal of R. Let $B = ((\widehat{R})^{\text{sh}})\hat{}$ and set $c = c(IB)$. Let M be a finitely generated R-module and let $n > c$ be an integer. Assume that for all integers s, with $1 \le s \le c-1$, and for all $q \ge n-s$, the following hold:*
(i) $H^q_{IR_P}(M_P) = 0$ for all $P \in \operatorname{Spec} R$ such that $I \subseteq P$ and $\dim R/P > s+1$;
(ii) $H^q_{IB_L}(M \otimes_R B_L) = 0$ for all $L \in \operatorname{Spec} B$ such that $IB \subseteq L$, $\dim B/L = s+1$ and $L + Q$ is primary to the maximal ideal of B for some minimal prime ideal Q of IB. Then $H_I^q(M) = 0$ for all $q \ge n$.

We also make use of properties of c, which are summarized in the

Lemma 4 (Huneke and Lyubeznik [4, Lemmas 2.2 and 2.3]). *Let R be a universally catenary local ring containing a field, I an ideal of R and P a prime ideal. Then*
(1) $\operatorname{mdim}(R) = \operatorname{mdim}(\widehat{R})$ and $c(I) = c(I\widehat{R})$;
(2) If $B = ((\widehat{R})^{\text{sh}})\hat{}$ then $c(I) = c(IB)$;
(3) $\dim(R/P) + \dim R_P \ge \operatorname{mdim} R$;
(4) $c(I) \ge c(I_P)$ if $P \supseteq I$.

Proof of Theorem 2. The ring R^{sh} is excellent since R is (cf. [2] or [8]), therefore the rings B/P_iB are normal. The theorem holds if $t = 1$ by [4, Theorem 3.8]. By induction, we may assume that the result holds for any ideal $J = Q_1 \cap \cdots \cap Q_r$ $(1 \le r < t)$ where the rings R/Q_i are normal and $0 < c(J\widehat{R}) \le (d-1)/r$. Let $B = ((\widehat{R})^{\text{sh}})\hat{}$ and let $\mathfrak{m}_B$ be its maximal ideal. Let s be an integer such that $1 \le s \le c-1$, and $n = d - [(d-2)/c]$. To apply the 'induction theorem', we first need to check that $n > c$. The computation is carried out in the proof of [4, Theorem 3.8]. We will now check that condition (ii) of Theorem 3 holds. If $L \in \operatorname{Spec} B$, $\dim B/L = s+1$ and $IB \subseteq L$, $\sqrt{L + L'} = \mathfrak{m}_B$ for some

$L' \in \operatorname{Min}(IB)$, then we need to show that $H^q_{IB_L}(B_L) = 0$ for all $q \geq d - [(d-2)/c] - s$. Since B is R-flat, we have that

$$\sqrt{I}B = (P_1 \cap \cdots \cap P_t)B = P_1B \cap \cdots \cap P_tB.$$

The ideals P_iB are prime because R/P_i is normal for all i, hence they are the minimal primes of IB. As $IB \subseteq L$, we have that $P_iB \subseteq L$ for some i. If $P_iB \subseteq L$ for all i, then $\operatorname{ht} L = d$, which contradicts the assumption that $\operatorname{ht} L = d - s - 1 \leq d - 2$. Therefore we may assume that there exists an integer j, $2 \leq j < t$, such that $P_1B + \cdots + P_jB \subseteq L$ and $P_lB \nsubseteq L$ for all $l \geq j + 1$. We then have to show that $H^q_{(P_1B \cap \cdots \cap P_jB)B_L}(B_L) = H^q_{(P_1B_L \cap \cdots \cap P_jB_L)}(B_L) = 0$ for all $q \geq d - [(d-2)/c] - s$. To apply the inductive assumption we need to prove that

– B_L is an excellent local ring containing a field;
– the rings $((\widehat{B_L})^{\mathrm{sh}})\hat{}/P_i((\widehat{B_L})^{\mathrm{sh}})\hat{}$ are normal;
– $0 < c(P_1B_L \cap \cdots \cap P_jB_L) \leq (\dim B_L - 1)/j = (d - s - 2)/j$.

Since B is a complete local ring, it is excellent, hence its localization B_L is also excellent (see [7, p.260]). Moreover, it is clear that B_L contains a field, as R does. Since B_L/P_iB_L is normal and B_L is excellent, $\widehat{B_L}/P_i\widehat{B_L}$ is also normal. Since B_L contains a field, the map $\widehat{B_L}/P_i\widehat{B_L} \longrightarrow ((\widehat{B_L})^{\mathrm{sh}})\hat{}/P_i((\widehat{B_L})^{\mathrm{sh}})\hat{}$ is regular, so $((\widehat{B_L})^{\mathrm{sh}})\hat{}/P_i((\widehat{B_L})^{\mathrm{sh}})\hat{}$, is normal for all i (here we have again used the fact that the rings R/P_i are normal). If $c(P_1B_L \cap \cdots \cap P_jB_L) = 0$, then, as pointed out in [4, Remark 2.8], B_L is regular and $P_1B_L \cap \cdots \cap P_jB_L = 0$, so we may assume that $c(P_1B_L \cap \cdots \cap P_jB_L) > 0$. We now prove that $c(P_1B_L \cap \cdots \cap P_jB_L) \leq (\dim B_L - 1)/j = (d - s - 2)/j$. By Lemma 4,

$$\begin{aligned} c(P_1B_L \cap \cdots \cap P_jB_L) &= c(P_1 \cap \cdots \cap P_j)B_L \leq c(P_1 \cap \cdots \cap P_j)B = c(P_1 \cap \cdots \cap P_j) \\ &= \operatorname{embdim}(R) - \min\{R/P_i,\ 1 \leq i \leq j\} \\ &= \operatorname{embdim}(R) - \min\{d - \operatorname{ht} P_i,\ 1 \leq i \leq j\} \\ &= \operatorname{embdim}(R) - d + \max\{\operatorname{ht} P_i,\ 1 \leq i \leq j\} \\ &\leq \operatorname{embdim}(R) - d + \max\{\operatorname{ht} P_i,\ 1 \leq i \leq t\} = c. \end{aligned}$$

Since $(d - s - 2)/j \geq (d - c + 1 - 2)/j = (d - c - 1)/j$ for all s, it suffices to show that $(d - c - 1)/j \geq c$. Now

$$(d - c - 1)/j \geq c \iff d - c - 1 \geq jc \iff (d - 1)/(j + 1) \geq c,$$

and the last statement holds, as $j+1 \leq t$ and by assumption $c \leq (d-1)/t$. By induction,

$$H^q_{(P_1B_L \cap \cdots \cap P_jB_L)}(B_L) = 0 \text{ for all } q \geq d - s - 1 - [(d - s - 3)/c(P_1B_L \cap \cdots \cap P_jB_L)].$$

Since $c(P_1B_L \cap \cdots \cap P_jB_L) \leq c$, it suffices to show that

$$d - s - 1 - [(d - s - 3)/c] \leq d - s - [(d - 2)/c].$$

This is equivalent to showing that $[(d-2)/c]-1 \leq [(d-s-3)/c]$. Since $-s \geq -c+1$, we have that

$$[(d-s-3)/c] \geq [((d-2)/c)-1] = [(d-2)/c]-1,$$

which proves the assertion.

The proof that condition (i) of Theorem 3 holds for n as above is the same as the one given in the proof of [4, Theorem 3.8]. □

Remark. If R and I are as in Theorem 2, and M is a finitely generated faithful R-module, then $H_I^q(M) = 0$ for all $q \geq d - [(d-1)/c]$. The proof is the same as that of Theorem 2: one only needs to observe that, for L as in (ii) of the 'induction theorem', $d = \dim M \otimes_R B_L$ by [1, Proposition 19, p. 107].

The proof of the following corollary is similar to that of Theorem 6.2 in [4].

Corollary 5. *Let $Y \subseteq \mathbb{P}^e_{\mathbb{C}}$ be a closed algebraic set such that $Y = Y_1 \cup \cdots \cup Y_n$, where the Y_i's are irreducible of codimension $\leq b$ and $n \leq [e/(e-b)]$. Assume that each Y_i is projectively normal. Then $H_i(\mathbb{P}^e_{\mathbb{C}}, Y, \mathbb{C}) = 0$ if $i \leq [(e-1)/b]$.*

Proof. Let R be the homogeneous coordinate ring of $\mathbb{P}^e_{\mathbb{C}}$, localized at the homogeneous maximal ideal $(x_o, \ldots, x_e)$, and let $I(Y) \subset R$ be the defining ideal of Y. Then $I(Y) = I(Y_1) \cap \cdots \cap I(Y_n)$, where $I(Y_i) \subset R$ is the defining ideal of Y_i. Since each Y_i is projectively normal, $R/(I(Y_j))$ is a normal ring for all j. By [4, Proposition 6.1], $H_i(\mathbb{P}^e_{\mathbb{C}}, Y, \mathbb{C}) = 0$ if $i \leq \dim \mathbb{P}^e_{\mathbb{C}} - \mathrm{cd}(R,I)$. Applying Theorem 2 we get

$$\dim \mathbb{P}^e_{\mathbb{C}} - \mathrm{cd}(R,I) \geq e+1-e-1+[e/(e-b)],$$

which completes the proof. □

In [4] Huneke and Lyubeznik prove that, if $t=1$, the bound given in Theorem 2 is not the best possible in general. More precisely, they prove the following theorem:

Theorem 6 (Huneke and Lyubeznik [4, Theorem 3.9]). *Let R be an excellent local ring containing a field. Let I be a formally geometrically irreducible ideal of R such that $0 < c = c(I\widehat{R})$, and assume that $(R/I)_P$ is normal for all prime ideals $P \supseteq I$ with $\dim R/P \geq 3$. Let M be a finitely generated R module of dimension d. Then $H_I^j(M) = 0$ for all $q \geq d+1-[d/(n+1)]-[(d-1)/(c+1)]$.*

The following example shows that even under the assumptions of Theorem 2 the bound given in Theorem 6 does not hold if $t \geq 2$.

Example. Let k be a field and let $R = k[[X_1, \ldots, X_7]]$. Set

$$I_1 = (X_1, X_2), \qquad I_2 = (X_3, X_4), \qquad I_3 = (X_5, X_6), \qquad I = I_1 \cap I_2 \cap I_3.$$

Clearly the rings R/I_j are normal domains. Moreover, ht I = bight $I = 2 = (7-1)/3$, so all the assumptions of Theorem 2 are satisfied. It follows that $H_I^q(R) = 0$ for all $q \geq 7-[(7-2)/3] = 5$ (if k is an infinite field then the main theorem of [6] may be used to see that $\mathrm{ara}_A(I) \leq 6-[6/2]+1 = 4$, where $A = k[X_1,X_2,X_3,X_4,X_5,X_6]_{(X_1, X_2, X_3, X_4, X_5, X_6)}$, so $\mathrm{cd}(R,I) = \mathrm{cd}(A,I) \leq \mathrm{ara}_A(I) = 4$). The bound given in Theorem 6 would imply that $H_I^q(R) = 0$ if $q \geq 8 - [7/3] - [6/3] = 4$, so the point of this example is to show that $H_I^4(R) \neq 0$. There are at least two different ways of seeing this. We can make use of [5], as I is generated by monomials in the R-sequence $X_1,X_{2.},X_3,X_4,X_5,X_6$. We get that $\mathrm{cd}(R,I) = \mathrm{pd}_R(R/I) = 4$, where the last computation has been performed y means of the computer algebra program MACAULAY. Alternatively, the Mayer–Vietoris long exact sequence may be used to get a surjective map:

$$H_I^4(R) \longrightarrow H^5_{I_1 \cap I_2 + I_3}(R) \longrightarrow 0,$$

as $H^5_{I_1 \cap I_2}(R) = H^5_{I_3}(R) = 0$ by Theorem 2. Set $J_1 = (X_1,X_2,X_3,X_4)$, $J_2 = (X_3,X_4,X_5,X_6)$. Then $I_1 \cap I_2 + I_3 = J_1 \cap J_2$. The Mayer–Vietoris long exact sequence gives the surjective map

$$H^5_{I_1 \cap I_2 + I_3}(R) \longrightarrow H^6_{J_1+J_2}(R) \longrightarrow 0.$$

Since $J_1 + J_2 = (X_1,X_2,X_3,X_4,X_5,X_6)$, the ideal $J_1 + J_2$ has height 6, so $H^6_{J_1+J_2}(R) \neq 0$. It follows that $H^5_{I_1 \cap I_2 + I_3}(R) \neq 0$, so $H_I^4(R) \neq 0$.

One may try to examine the same example in the case $R = k[[X_1,\ldots,X_6]]$. However, in this case the ideals I_1, I_2 and I_3 add up to the maximal ideal of R, so $\mathrm{cd}(R,I) = 4$ by [4, Corollary 5.3].

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