Acceleration by Aggregation of Successive Approximation Methods

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ABSTRACT

Methods of successive approximation for solving linear systems or minimization problems are accelerated by aggregation-disaggregation processes. These processes, which modify the iterates being produced, are characterized by a two directional flow of information between the original higher dimensional problem and a lower dimensional aggregated version. This technique is characterized by means of Galerkin approximations, and this in turn permits analysis of the method. A deterministic as well as probabilistic analysis is given of a number of specific aggregation-disaggregation examples. Numerical experiments have been performed, and these confirm the analysis and demonstrate the acceleration.

1. INTRODUCTION

The $n \times n$ linear system

$$x = Ax + b \tag{1.1}$$

may be solved by a variety of methods. For large values of n, iterative methods such as the method of successive approximations

$$x^{k+1} = Ax^k + b \tag{1.2}$$

are frequently used. Such a method requires that the spectral radius $\rho(A) < 1$.

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When $\rho(A)$ is close to unity, the convergence of the process (1.2) may be very slow. We propose to speed it up by interrupting it at the kth step and replacing $\tilde{x} := x^k$ by an improved approximation $\tilde{x}$ to the solution $x^* = (I - A)^{-1}b$. Then the iterative process is restarted with $\tilde{x}$. This accelerative process may be repeated as required.

Variants of this idea have been proposed in a number of settings. Most such variants having been designed to speed up the resolution of systems arising through the discretization of differential equations such as multigrid methods [2, 5], coarse-mesh corrections [3, 10], and additive corrections [11]. For large systems not specifically associated with discretization, other types of methods, called aggregation methods, have been recently proposed [9, 13] (see [8] also). Systems of this latter type occur in the social sciences, e.g. input-output economics.

Our view of aggregation methods corresponds to the two way flow of information in a multilevel hierarchical management system. Indeed, it is this two way flow of information which we regard as the key feature of the aggregation step for accelerating the process (1.2). A simple prototype of this feature is illustrated as follows in the case that the eigenelements of $I - A$ are (λ_i, v^i), $i = 1, \ldots, n$, and

$$0 < \lambda_n < \lambda_{n-1} < \cdots < \lambda_1 < 2.$$

When λ_1 is very close to two and/or when λ_n is very close to zero, the error $\epsilon^k = x^* - x^k = (I - A)^k \epsilon^0$, which diminishes very slowly, tends toward the one dimensional subspace spanned by v^1. The residual $r(x^k) = b - Ax^k = A\epsilon^k$ likewise tends to this subspace, and so a correction to x^k in the form $\tilde{x} = x^k + cr^k$ is suggested. This correction (i.e. the constant c) may be found by solving the least squares problem

$$\min_c \|A(x^k + cr^k) - b\|. \tag{1.3}$$

We observe a two directional flow of information: The iterative process (1.2) in n dimensions defines a one dimensional manifold (and corresponding problem). The one dimensional problem (1.3) is solved, and its solution defines an n-vector $\tilde{x}$ with which to restart (1.2).

We begin in Section 2 with the introduction of notation and two aggregation methods for forming p-dimensional corrections ($p < n$): the multiplicative and the additive. Then we show that these methods may be characterized by means of Galerkin approximations. Examples, including some from input-output economics, are then presented. In Section 3 we review several minimization techniques for system solving, and show their connection to the aggregation processes developed in Section 2. In Section 4 we analyze the

processes introduced in Sections 2 and 3. The minimization methods are treated by simple observation, but the aggregation methods are supplied with a derivation of an error propagation formula which exploits their characterization as Galerkin approximations. From this formula we are able to make two types of deductions concerning the efficacy of aggregation as a convergence accelerating process. The first depends on the quality with which a set is described by its aggregated form; the second is a stochastic analysis of the error propagation formula. Finally, in Section 5 we give the results of numerical experiments performed on a set of matrices (a matrix from input-output economics, a randomly selected matrix, and a matrix arising in finite differences) and with several of the aggregation methodologies derived here. The computations show the methods to be effective.

2. METHODS OF AGGREGATION

Our study of aggregation methods employs a special notation and formalism, and we begin with a review of them.

2.1. *Notation*

As before A is an $n \times n$ matrix and b is an n-vector. $\bar{x}$ denotes an approximation to $x^* := (I - A)^{-1}b$, the solution of (1.1), $(I - A)x = b$. The residual at x is $r(x) := b - (I - A)x = (I - A)(x^* - x)$. For the error $\epsilon := x^* - \bar{x}$ at $\bar{x}$, we have $r := r(\bar{x}) = (I - A)\epsilon$.

Let $H = \{\mathcal{H}_i\}_1^p$ and $G = \{\mathcal{G}_i\}_1^p$ each be a partition of the integers $\{1, \ldots, n\}$ into p disjoint subsets. Let $\{e^i\}_1^n$ be the canonical basis of $\mathbb{R}^n$. For each $i = 1, \ldots, p$, we define H_i (respectively G_i) to be the matrix representing the orthogonal projection onto $\{e^i,\ j \in \mathcal{H}_i\}$ (respectively $\{e^i, j \in \mathcal{G}_i\}$). We introduce two mappings R and P which will in general depend on these partitions and projections:

$R: \mathbb{R}^n \to \mathbb{R}^p$ is a linear mapping which we view as a restriction or aggregation.

$P: \mathbb{R}^p \to \mathbb{R}^n$ is a linear mapping which we view as prolongation or disaggregation.

Finally, $e := (1, \ldots, 1)^t$, $\|\cdot\|$ denotes the Euclidean norm, and $(x, y) = x^t y$, each in $\mathbb{R}^n$ or $\mathbb{R}^p$ as the case may be.

2.2. *The Corrected Approximation* $\tilde{x}$

Given $\bar{x}$, a corrected approximation $\tilde{x}$ is computed by means of the solution of a $p \times p$ linear system of equations. With the linear mappings R and

P given, we define

$$C := R(I - A)P, \qquad d := Rb, \quad \text{and} \quad \rho := Rr.$$

We suppose that C is regular, and we specify two ways of computing $\tilde{x}$ as a correction to $\bar{x}$:

(i) multiplicative correction,

$$Cy = d, \qquad \tilde{x} = Py; \tag{2.1}$$

(ii) additive correction,

$$C\delta = \rho, \qquad \tilde{x} = \bar{x} + P\delta := \bar{x} + \tilde{\epsilon}. \tag{2.2}$$

The point here is that R and P are to be chosen so that $\|\tilde{x} - x^*\|$ is smaller than $\|\bar{x} - x^*\|$. [In fact, this is not strictly required, since it is also acceptable if we are able to increase the rate of decay of errors arising from the iterations restarted at $\tilde{x}$. This increase of decay rate, typically temporary in nature, may be so substantial—e.g., if the correction $\tilde{x}$ results in the annihilation of the component of $\epsilon(\tilde{x})$ along the principle eigenvector—that some degradation in $\|x^* - \tilde{x}\|$ compared to $\|x^* - \bar{x}\|$ may be tolerated. We return to this point in Section 4.2.4.]

This restriction-prolongation process will be combined with simple relaxation by interrupting the latter occasionally for a step of the former. For brevity we shall refer to such a step as an a/d (aggregation/disaggregation) step.

2.3. *Aggregation as a Galerkin Process*

Let $\Pi = PR$. If $RP = I_p$, the p-dimensional identity matrix, then $\Pi^2 = \Pi$. Thus Π is a projection (in general nonorthogonal) onto the p-dimensional subspace, $M = \Pi(\mathbb{R}^n)$, along $N = \operatorname{Ker} \Pi$. Thus the processes described in Section 2.2 amounts to a projection method on the subspace M, and we may write C as

$$C = I - B \qquad \text{with} \quad B := RAP.$$

(i) In the case of the multiplicative correction, $Cy = d$ implies $PR(I - A)Py = PRb$. Since, moreover, $y = RPy = R\tilde{x}$, then

$$\Pi(I - A)\Pi\tilde{x} = \Pi b. \tag{2.3}$$

Thus $\tilde{x} \in M$ is the Galerkin solution in M of $(I - A)x = b$. (See [7] for a discussion of the concept of such Galerkin solutions.) Indeed, this equation is solved in the sense that $\Pi r(\tilde{x}) = 0$. We shall see in Section 4 that the error in the corrected approximation $\tilde{x} - x^*$ depends on $(I - \Pi)x^*$.

(ii) In the additive correction case, $C\delta = \rho$ implies $PR(I-A)P\delta = PRr$. Since moreover $\delta = RP\delta = R\tilde{\epsilon}$, then

$$\Pi(I-A)\Pi\tilde{\epsilon} = \Pi r. \tag{2.4}$$

Thus $\tilde{\epsilon} \in M$ is the Galerkin solution in M of $(I-A)\epsilon = r$. Indeed, this equation is solved in the sense that $\Pi[r-(I-A)\tilde{\epsilon}] = 0$. We shall see in Section 4 that $\tilde{\epsilon} - \epsilon$ depends on $(I-\Pi)\epsilon$. Note that $\tilde{\epsilon} - \epsilon = \tilde{x} - \bar{x} - (x^* - \bar{x}) = \tilde{x} - x^*$.

While the multiplicative and additive aggregation methods result in different corrected values, we may seek conditions under which the respective values of $\tilde{x}$ are the same.

Suppose that $\Pi\bar{x} = \bar{x}$. Let $\tilde{x}$ be the solution of (2.3), and set $\tilde{x} = \bar{x} + \tilde{\epsilon}$. Then $\tilde{\epsilon}$ is the solution of (2.4). Conversely, if (2.3) and (2.4) furnish the same $\tilde{x}$, then it follows that $\Pi\bar{x} = \bar{x}$.

In Figure 1, we give schematics illustrating the two aggregation methods.

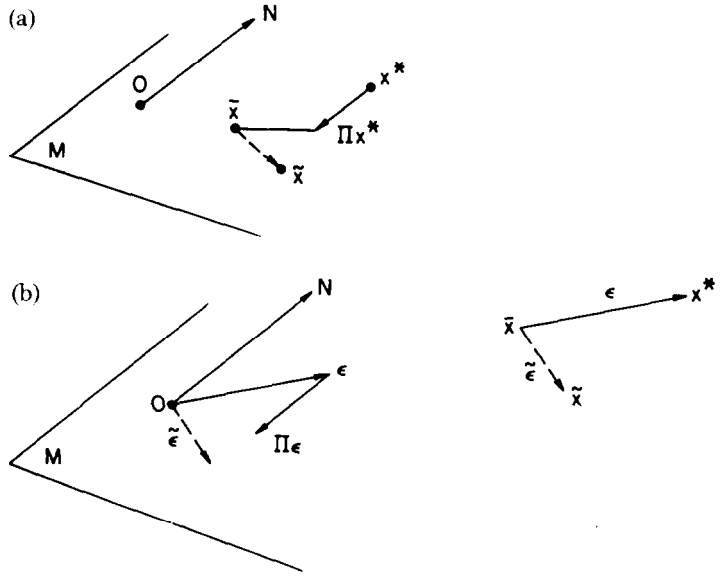


FIG. 1. (a) Multiplicative Aggregation: Solution x^* of $(I - A)x = b$ is approximated in M by $\tilde{x}$.
(b) Additive Aggregation: Solution ϵ of $(I - A)\epsilon = r$ is approximated in M by $\tilde{\epsilon}$.

2.4. Choices of R and P

There are many possible choices for R and P. We shall consider the following family of them in which R depends on a given partition H and a given vector $z \in \mathbb{R}^n$, while P depends on a given partition G and a given vector $t \in \mathbb{R}^n$.

$$\begin{aligned} R&: (x_i)_1^n \to (y_i)_1^p = \left(\sum_{i \in \mathcal{K}_i} z_i x_i \right), \\ P&: y \to x = \sum_{j=1}^{p} y_j G_j t. \end{aligned} \tag{2.5}$$

Upon setting $d=(d_i)_1^p$, $\rho=(\rho_i)_1^p$, and $C=(c_{ij})$, $i,j=1,\ldots,p$, it follows from this choice of R and P that

$$d_i = (H_i z, b) = (z, H_i b), \tag{2.6i}$$

$$\rho_i = (H_i z, r) = (z, H_i r), \tag{2.6ii}$$

$$c_{ij} = (H_i z, (I-A)G_j t) = (z, H_i(I-A)G_j t). \tag{2.6iii}$$

These formulas show that only the block $H_i(I-A)G_j t$ (respectively $H_i b$) plays a role in the definition of c_{ij} (respectively d_i).

Note that the R and P defined here become the ones specified by Miranker and Pan [7] when the restriction $t=\bar{x}$ is imposed.

2.5. Examples Where $RP=I_p$

Using (2.5), the condition $RP=I_p$ developed in Section 2.3 may be written as follows:

$$(RPy)_i = \left(H_i z, \sum_{j=1}^{p} y_j G_j t \right) = y_i, \qquad i=1,\ldots,p.$$

To achieve this it suffices to set $H=G$ and $\Sigma_{k \in \mathcal{G}_i} z_k t_k = 1$, $i=1,\ldots,p$.

2.5.1. An Additive Correction Example. A simple example of this sufficient condition is supplied by

$$t=e \quad \text{and} \quad z=\sum_{j=1}^{p} \frac{1}{\sigma_j} G_j e,$$

where σ_j is the cardinality of G_j. To stress the dependence of this example on

the vector e, we will call it example (e). [This will also distinguish it from the examples (a), (b) and (c) to be introduced in section 2.5.2.] For this example (e), (2.5) yields

$$R: (x_i)_1^n \to (y_j)_1^p, \quad \text{with} \quad y_j = \frac{1}{\sigma_j} \sum_{l \in \mathcal{G}_j} x_l,$$

$$P: y \to \tilde{x} = \sum_{j=1}^{p} y_j G_j e.$$

Then

$$\Pi: x \to \sum_{j=1}^{p} \frac{1}{\sigma_j} \left(\sum_{l \in \mathcal{G}_j} x_l \right) G_j e,$$

which displays the dependence of the projection Π on e. Setting $B=(b_{ij})$, a $p \times p$ matrix, the aggregation formula (2.6iii) for a matrix becomes

$$b_{ij} = (z, G_i A G_j e) = \frac{1}{\sigma_i} \sum_{k \in \mathcal{G}_i} \sum_{l \in \mathcal{G}_j} a_{kl}.$$

The aggregation formulas (2.6i) or (ii) for a vector yield

$$d_i = \frac{1}{\sigma_i} \sum_{k \in \mathcal{G}_i} b_k.$$

The property $\Pi e = e$ (that is, $e \in M$) is easily checked. Note that B and d are invariant with respect to the iterates. Additive aggregation is used with this choice of vectors t and z.

A collection of such mappings is considered by Settari and Aziz [11], corresponding to various weightings for $\mathcal{G}_j$. Application to matrices arising in finite difference approximations for partial differential equations is also studied in [11]. The two level multigrid method [2] is also a projection method wherein R and P depend on the geometry of the grids.

2.5.2. Examples of Multiplicative Correction. Mappings of the type Π also occur in models developed in the social sciences. In these cases, Π often depends on $\bar{x} = x^k$, the current iteration vector, and typically $\bar{x} \in M$. $\Pi(\bar{x})$ varies over the iteration. Multiplicative aggregation is used.

For the typical example where A is the interproduct input-output matrix, the aggregation process and the aggregated system $y = By + d$ as well may be interpreted in terms of economics, as we shall see:

(a) Consider the linear Leontiev model for a productive economy, which consists of n industrial sectors each of which produces one output. Let the

partition G of $\{1,\dots,n\}$ correspond to $p<n$ aggregated sectors. Let $\bar{x}$ be the production of the economy. $X_i=\Sigma_{k\in\mathcal{G}_i}\bar{x}_k$ is the production of the ith aggregated sector $\mathcal{G}_i$, and $X_{ij}=\Sigma_{k\in\mathcal{G}_i}\Sigma_{l\in\mathcal{G}_j}a_{kl}\bar{x}_l$ is the output of $\mathcal{G}_i$ required for the production of $\mathcal{G}_j$. Then $b_{ij}=X_{ij}/X_j$, $i,j=1,\dots,p$, are the input-output coefficients of the Leontiev matrix corresponding to the aggregated level. The aggregated balance equation is the p-dimensional system $y=By+d$ where $d_i=\Sigma_{k\in\mathcal{G}_i}b_k$. Here d is called the aggregated demand and y is called the vector of aggregated production.

The disaggregation formula is taken to be

$$y\to\tilde{x}=\sum_{j=1}^{p}\left(\frac{y_j}{\sum_{l\in\mathcal{G}_j}\bar{x}_l}\right)G_j\bar{x},$$

where it is assumed that $\Sigma_{l\in\mathcal{G}_i}\bar{x}_l\neq 0$, $j=1,\dots,p$. This assumption means that each sector $\mathcal{G}_j$ has a nonzero output. The production of the sector $\mathcal{G}_j$ is then modified by the factor $y_j/\Sigma_{l\in\mathcal{G}_j}\bar{x}_l$. Then for the first iterate

$$\hat{x}:=A\tilde{x}+b \tag{2.7}$$

obtained from $\tilde{x}$ by employing (1.2), we have

$$\hat{x}_k=\sum_{j=1}^{p}\frac{y_j}{\sum_{l\in\mathcal{G}_j}\bar{x}_l}\sum_{l\in\mathcal{G}_j}a_{kl}\tilde{x}_l+b_k,\qquad k=1,\dots,n.$$

$\bar{x}\to y$ represents a flow of information from $\mathbb{R}^n$ to $\mathbb{R}^p$, while $y\to\tilde{x}$ represents the flow back. $\hat{x}$ is the new production (i.e., the production after correction by y).

This process is the iterative aggregation method presented in [12]. It corresponds to

$$z=e,\qquad t=\sum_{i=1}^{p}\frac{1}{\sum_{k\in\mathcal{G}_i}\bar{x}_k}\mathcal{G}_i\bar{x}\quad\left(\text{if }\sum_{k\in\mathcal{G}_i}\bar{x}_k\neq 0,\ i=1,\dots,p\right),$$

$$R:x\to(y_i)_1^p,\qquad y_i=\sum_{l\in\mathcal{G}_i}x_l=(x,\mathcal{G}_i e),$$

$$P:y\to\tilde{x}=\sum_{j=1}^{p}\left(\frac{y_j}{\sum_{l\in\mathcal{G}_j}\bar{x}_l}\right)G_j\bar{x},$$

and

$$\Pi: x \to \tilde{x} = \sum_{j=1}^{p} \left(\frac{\sum_{l \in \mathcal{G}_j} x_l}{\sum_{l \in \mathcal{G}_j} \bar{x}_l} \right) G_j \bar{x}.$$

Note that $\Pi \bar{x} = \bar{x}$.

(b) Our second example, which has a similar input-output economics character, is considered in [9]. If $\bar{x}_k \neq 0$, $k = 1, \ldots, n$, we set

$$v = \left(\frac{1}{\bar{x}_k} \right)_1^n, \qquad z = \sum_{j=1}^{p} \frac{1}{\sigma_j} G_j v, \quad \text{and} \quad t = \bar{x}.$$

Here σ_j is the cardinality of $\mathcal{G}_j$. Then

$$R: x \to (y_j)_1^p, \qquad \text{where} \quad y_j = \frac{1}{\sigma_j} \sum_{l \in \mathcal{G}_j} \frac{x_l}{\bar{x}_l},$$

$$P: y \to \tilde{x} = \sum_{j=1}^{p} y_j G_j \bar{x},$$

$$\Pi: x \to \tilde{x} = \sum_{j=1}^{p} \frac{1}{\sigma_j} \left(\sum_{l \in \mathcal{G}_j} \frac{x_l}{\bar{x}_l} \right) G_j \bar{x}.$$

Note that $\Pi \bar{x} = \bar{x}$. We also have

$$b_{ij} = \frac{1}{\sigma_i} \sum_{k \in \mathcal{G}_i} \frac{1}{\bar{x}_k} \sum_{l \in \mathcal{G}_j} a_{kl} \bar{x}_l, \qquad d_i = \frac{1}{\sigma_i} \sum_{k \in \mathcal{G}_i} \frac{b_k}{\bar{x}_k}.$$

(c) We give a third example. Let i^* be a distinguished index in $\mathcal{G}_i$, $i = 1, \ldots, p$, and let K_{i*} be the orthogonal projection onto e_{i*}. Let

$$v = \left(\frac{1}{\bar{x}_i} \right)_1^n, \qquad z = \sum_{i=1}^{p} K_{i*} v, \quad \text{and} \quad t = \bar{x}.$$

Then

$$R: x \to (y_j)_1^p, \qquad y_j = \frac{x_{j*}}{\bar{x}_{j*}},$$

$$P: y \to \tilde{x} = \sum_{j=1}^{p} y_j G_j \bar{x},$$

$$\Pi: x \to \tilde{x} = \sum_{j=1}^{p} \frac{x_{j*}}{\bar{x}_{j*}}.$$

Note that $\Pi\bar{x} = \bar{x}$ and that

$$b_{ij} = \frac{1}{\bar{x}_{i*}} \sum_{l \in \mathcal{G}_j} a_{i*l} \bar{x}_l, \qquad d_i = \frac{b_{i*}}{\bar{x}_{i*}}.$$

$\Pi\bar{x} = \bar{x}$ in these three examples. This is equivalent to the existence of a vector η in $\mathbb{R}^p$ such that $\bar{x} = P\eta$:

$$\eta = e, \qquad \text{example (a)},$$

$$\eta = \left(\sum_{l \in \mathcal{G}_j} \bar{x}_l \right), \qquad \text{example (b)},$$

$$\eta = (\bar{x}_{i*}), \qquad \text{example (c)}.$$

Then the multiplicative method is equivalent to the additive one (compare the concluding comments of Section 2.3). Set y, the solution of (2.1), equal to $\eta + \delta$. Then $\tilde{x} = Py = \bar{x} + P\delta = \bar{x} + \tilde{\epsilon}$, where δ is the solution of (2.2): indeed, $C\eta = d + Rr = d + \rho$ implies $C\delta = \rho$.

3. MINIMIZATION METHODS IN NUMERICAL ANALYSIS

There are methods for accelerating the iteration (1.2) which are based on the minimization of the residual $r(x)$ or of the error $x^* - x$. The a/d step is replaced by the solution of a least squares problem of size $p < n$ which is applied at each iteration step.

In these minimization methods, the correction $\tilde{x}$ to $\bar{x}$ is sought in the form

$$x=\bar{x}+\sum_{i=1}^{p}\xi_i u^i,$$

where the $\{u^i\}_1^p$ are p independent vectors in $\mathbb{R}^n$. The p-vector of coefficients ξ is determined through minimization of the residual or of the error over all $\xi\in\mathbb{R}^p$ (cf. Householder and Bauer [6]). For convenience we introduce

$$A'=I-A.$$

3.1. Minimization of $\|r(x)\|$

Let U be the $n\times p$ matrix with the columns u^i, $i=1,\ldots,p$. Then $x=\bar{x}+U\xi$ and

$$r(x)=b-A'x=r(\bar{x})-\sum_{i=1}^{p}\xi_i A'u^i=r-A'U\xi.$$

To minimize $\|A'U\xi-r\|$ is equivalent to finding the least squares solution of $A'U\xi=r$—that is, to solving

$$U^tA'^tA'U\xi=U^tA'^tr,$$

or equivalently

$$\sum_{i=1}^{p}(A'u^i, A'u^j)\xi_i=(r, A'u^j), \qquad j=1,\ldots,p. \tag{3.1}$$

$A'\tilde{x}$ is the orthogonal projection of r onto the manifold $\mathcal{V}$ generated by $\{A'u^i\}_1^p$ and passing through $A'\bar{x}$. Clearly $\|r(\tilde{x})\|\leq\|r\|$ (cf. Figure 2).

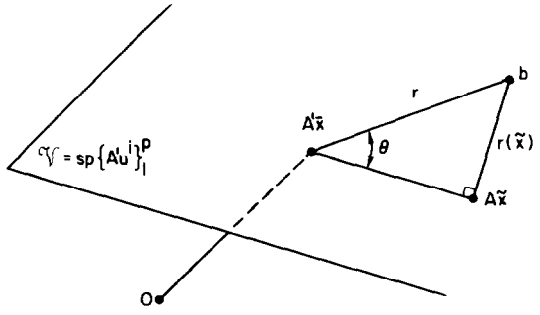


FIG. 2. Minimization of $\|r(x)\|$.

We now give several choices of the vectors u^i which have been proposed elsewhere. Some of these choices arise in connection with the linear systems associated with discretized partial differential equations.

(1) Taking $e \in \mathbb{R}^n$ and given a partition G, set $u^i = G_i e$, $i = 1, \ldots p$ (cf. Beneu [1]). In this case, solving (3.1) corresponds to computing the additive correction $U\xi$ such that $\tilde{x} = \bar{x} + U\xi$.

(2) In the case that $\bar{x}_k \neq 0$, $k = 1, \ldots, n$, set $u^i = G_i \bar{x}$, $i = 1, \ldots, p$ (cf. Froehlich and Nakamura in Beneu [1]). Note that $\bar{x} = \sum_{i=1}^{p} u^i$, so that (3.1) is equivalent to

$$\sum_{i=1}^{p} (A'u^i, A'u^j)(1+\xi_i) = (r, A'u^j) + (A'\bar{x}, A'u^j) = (b, A'u^j). \quad (3.2)$$

This may be interpreted as a multiplicative correction method:

$$\tilde{x} = U(e+\xi) \qquad \text{with} \quad e \in \mathbb{R}^p,$$

or alternatively as an additive correction method:

$$\tilde{x} = \bar{x} + U\xi,$$

since

$$Ue = \sum_i u^i = \bar{x}.$$

(3) In case $p = 1$, the previous example (2) corresponds to an aggregation method with $u^1 = t = \bar{x}$ and $z = A'\bar{x}$. This is procedure 2 in [9].

3.2. *Minimization of* $\|x^* - x\|$

$\|x^* - \bar{x} - \sum_{i=1}^{p} \xi_i U^i\|$ is minimized for $\xi = (U^t U)^{-1} U^t \epsilon$, where $\epsilon = x^* - \bar{x}$. Here ϵ is unknown but $r = A'\epsilon$ is known. Then suppose that the matrix U (that is, the set of vectors u^i) is given in terms of a known matrix W as $U = A'^t W$. Then $U^t = W^t A'$ and $\xi = (W^t A' A'^t W)^{-1} W^t r$.

When $p = 1$, W is a well-defined vector, which we call z. In this case

$$\tilde{x} = \bar{x} - \frac{(z, r)}{\|A'^t z\|^2} A'^t z.$$

$\tilde{x}$ is the orthogonal projection of x^* onto the one dimensional manifold $\mathcal{U}$ parallel to $u = A'^t z$ and passing through $\bar{x}$. Thus $\|x^* - \tilde{x}\| \leq \|x^* - \bar{x}\|$ (cf. Figure 3).

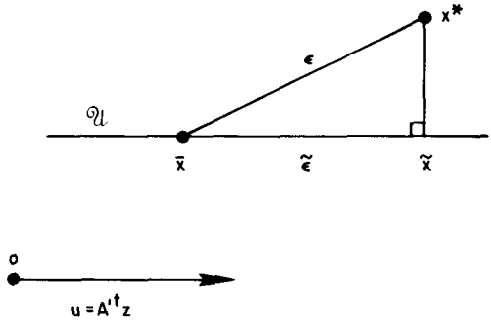


FIG. 3. Minimization of $\|\epsilon\|$ when $p=1$.

Gastinel [4] proposes a choice for z which is computed from r by means of norm decomposition and which ensures that $\|x^*-\tilde{x}\| \leq \alpha \|x^*-\bar{x}\|$ with $\alpha<1$. For the Euclidean norm, Gastinel's choice yields the method of steepest descent.

We conclude this section with the following two observations.

REMARK 1. If $p=1$, the Miranker-Pan method (cf. Section 2.4) yields $\tilde{z}=\bar{x}y=[(z,b)/(x,A'\bar{x})]\bar{x}$. If $A'^t z=\bar{x}$ (which is procedure 1 in [9]), this formula is identical to Gastinel's.

REMARK 2. We may note that the minimization methods of Sections 3.1 and 3.2 correspond to an orthogonal projection onto a p-dimensional manifold (passing through $A'\bar{x}$ in the former section and through $\bar{x}$ in the latter). On the other hand, the Galerkin method corresponds to a projection Π onto a p-dimensional subspace of $\mathbb{R}^n$. We have restricted the term projection method to this latter case.

4. ANALYSIS

We begin with the minimization methods, for which some simple observations and references are made. Then we deal with aggregation methods which are Galerkin approximations. For these we derive error equations for the a/d step, from which we characterize the associated error reduction. One characterization is given in terms of the quality with which the aggregated variables describe the unaggregated; the second characterization is a stochastic one.

4.1. Minimization Methods

The minimization property is exploitable for analyzing the minimization methods. In fact, we confine our study of these methods to a simple observation for the minimization of $\|r(x)\|$ or $\|x^*-x\|$.

4.1.1. Minimization of $\|r(x)\|$. $\|r(\tilde{x})\| < \|r\|$ if the acute angle θ between r and the manifold $\mathcal{V}$ is less than $\pi/2$. Set $\hat{r} = r/\|r\|$, and set $v^i = A'u^i/\|A'u^i\|$ and $\alpha_i = (\hat{r}, v^i)$, $i = 1,\ldots,p$. We refer to the following result of Beneu [1]: If for a given $\delta > 0$ there exists an index i such that $|\alpha_i| \geqslant \delta > 0$, then $\cos\theta \geqslant \delta$. Then corresponding to $\delta > 0$ and a given vector t with nonzero components, there exists a partition G such that for the corresponding vectors $u^i = G_i t$, $i = 1,\ldots,p$, there exists an index i for which $|\alpha_i| \geqslant \delta > 0$.

This result establishes this minimization method as well as the particular aggregation methods for $p = 1$ which are defined by $t = \bar{x}$ (if $\bar{x}_i \neq 0$) and $z = A't$ (procedure 2 in [9]).

4.1.2. Minimization of $\|x^*-x\|$. We refer to Gastinel [4] for these results.

4.2. Aggregation Methods

Our analysis of aggregation methods is confined to the case where $RP = I_p$; that is, to the case in which the a/d step is a Galerkin approximation in $M = \Pi(\mathbb{R}^n)$.

4.2.1. The a/d Step. Using (2.1) and its form (2.3) as a Galerkin approximation, we find

$$\begin{aligned}(I-\Pi A)(\tilde{x}-x^*) &= (\Pi-\Pi A)\tilde{x}-\Pi(I-A)x^*+(\Pi-I)x^* \\ &= \Pi\big[b-(I-A)x^*\big]+(\Pi-I)x^* \\ &= (\Pi-I)x^*.\end{aligned}$$

C is regular by assumption. Then the equivalence of (2.1) and (2.3) in M implies that $I-\Pi A$ is regular. Thus

$$\tilde{x}-x^* = (I-\Pi A)^{-1}(\Pi-I)x^*. \tag{4.1}$$

Applying (4.1) to $(I-A)\epsilon = r$, we get

$$\tilde{\epsilon}-\epsilon = (I-\Pi A)^{-1}(\Pi-I)\epsilon. \tag{4.2}$$

Now making use of $\epsilon = x^* - \bar{x}$ and $\tilde{\epsilon} - \epsilon = \tilde{x} - \bar{x} - (x^* - \bar{x}) = \tilde{x} - x^*$, we get

$$\tilde{x} - x^* = (I - \Pi A)^{-1}(\Pi - I)(x^* - \bar{x}), \tag{4.3}$$

for additive aggregation, such as for the example (e) in Section 2.5.1.

We also obtain the identity (4.3) directly from (4.1) for multiplicative aggregation for which the relation $\Pi\bar{x} = \bar{x}$ also holds [such as the examples (a), (b), and (c) in Section 2.5.2].

Let $\|(I - \Pi A)^{-1}\| \leq \tilde{K}$. Then (4.3) may be written as

$$\|\tilde{x} - x^*\| \leq \tilde{K}\|(\Pi - I)(x^* - \bar{x})\|. \tag{4.4}$$

We assume that $\tilde{K}$ is a moderate constant (a property verified by computation in Section 5 below), and we proceed to estimate $\|(\Pi - I)(x^* - \bar{x})\|$.

4.2.2. Estimate of $\|(\Pi - I)\epsilon\|$ *with* $\epsilon = x^* - \bar{x}$. Corresponding to each of the four examples presented in Section 2.5, we derive bounds for the expression

$$z := \frac{\|(I - \Pi)\epsilon\|}{\|\epsilon\|}, \tag{4.5}$$

which, as (4.4) shows, characterizes the gain in error reduction, i.e., the acceleration provided by an a/d step.

For example (e) of Section 2.5.1,

$$\begin{aligned}(I - \Pi)\epsilon &= \epsilon - \sum_{j=1}^{p} \frac{1}{\sigma_j}\left(\sum_{l \in \mathcal{G}_j} \epsilon_l\right) G_j e \\ &= \sum_{j=1}^{p} \sum_{l \in \mathcal{G}_j} \epsilon_l \left[1 - \frac{1}{\epsilon_l}\left(\frac{1}{\sigma_j} \sum_{i \in \mathcal{G}_j} \epsilon_i\right)\right] e^l. \end{aligned} \tag{4.6}$$

Then by employing the Cauchy-Schwartz inequality, we find

$$\|(I - \Pi)\epsilon\| \leq C(e)\|\epsilon\|, \tag{4.7}$$

where

$$C(e) = \max_{\substack{l \in \mathcal{G}_j \\ j = 1, \ldots, p}} \left|1 - \frac{1}{\epsilon_l}\left(\frac{1}{\sigma_j} \sum_{i \in \mathcal{G}_j} \epsilon_i\right)\right|.$$

Now consider the examples (a), (b), and (c) of Section 2.5.2. For (a) we have

$$(I-\Pi)\epsilon=\epsilon-\sum_{j=1}^{p}\left(\frac{\sum_{l\in\mathcal{G}_j}\epsilon_l}{\sum_{l\in\mathcal{G}_j}\bar{x}_l}\right)G_j\bar{x}$$

$$=\sum_{j=1}^{p}\sum_{l\in\mathcal{G}_j}\epsilon_l\left[1-\left(\frac{\sum_{i\in\mathcal{G}_j}\epsilon_i}{\sum_{i\in\mathcal{G}_j}\bar{x}_i}\right)\frac{\bar{x}_l}{\epsilon_l}\right]e^l. \tag{4.8}$$

Then using the Cauchy-Schwartz inequality, we find

$$\|(I-\Pi)\epsilon\|\leqslant C_a(\bar{x})\|\epsilon\|, \tag{4.9}$$

where

$$C_a(\bar{x})=\max_{\substack{l\in\mathcal{G}_j\\ j=1,\ldots,p}}\left|1-\left(\frac{\sum_{i\in\mathcal{G}_j}\epsilon_i}{\sum_{i\in\mathcal{G}_j}\bar{x}_i}\right)\frac{\bar{x}_l}{\epsilon_l}\right|.$$

The estimate (4.9) with C_a replaced by C_b [C_c] is valid in the case of example (b) [example (c)]. In particular

$$C_b(\bar{x})=\max_{\substack{l\in\mathcal{G}_j\\ j=1,\ldots,p}}\left|1-\frac{1}{\sigma_j}\left(\sum_{i\in\mathcal{G}_j}\frac{\epsilon_i}{\bar{x}_i}\right)\frac{\bar{x}_l}{\epsilon_l}\right|,$$

and

$$C_c(\bar{x})=\max_{\substack{l\in\mathcal{G}_j\\ j=1,\ldots,p}}\left|1-\frac{\epsilon_{l^*}}{\bar{x}_{l^*}}\frac{\bar{x}_l}{\epsilon_l}\right|.$$

Using (4.4) with (4.8) and (4.9), we write

$$\tilde{\alpha} := \frac{\|x^* - \tilde{x}\|}{\|x^* - \bar{x}\|} \leqslant \tilde{K} C(\cdot), \tag{4.10}$$

$C(\cdot)$ being $C(\bar{x})$ or $C(e)$ as the case may be. We also suppress the subscript of $C(\cdot)$. Thus $C(\cdot)$ is a measure of the improvement produced by the a/d step.

We see that $C(\bar{x})$ is small if in each set of aggregation $\mathcal{G}_j$, the ratio $\epsilon_l/\bar{x}_l$ is well represented by

the ratio of the sums:	$\dfrac{\sum_{i\in\mathcal{G}_j} \epsilon_i}{\sum_{i\in\mathcal{G}_j} \bar{x}_i}$,	example (a),
the average ratio:	$\dfrac{1}{\sigma_j} \sum_{i\in\mathcal{G}_j} \dfrac{\epsilon_i}{\bar{x}_i}$,	example (b),
the ratio of distinguished components:	$\epsilon_{j*}/\bar{x}_{j*}$,	example (c).

Similarly $C(e)$ is small if in each set of aggregation $\mathcal{G}_j$, ϵ_l is well represented by the average value $(1/\sigma_j)\Sigma_{i\in\mathcal{G}_j}\epsilon_i$ [cf. (4.7)].

4.2.3. A Stochastic Estimate of $(I-\Pi)\epsilon$. A figure of merit which characterizes the improvement provided by an a/d step may be obtained from a stochastic point of view. Consider the totality of real matrices A and vectors b for which the problem (1.1) is well defined. The set of such problems corresponds to a subset of $\mathbb{R}^{n(n+1)}$. Suppose that some probability measure is defined on that subset. This measure induces a corresponding measure on the corresponding class of expressions $(I-\Pi)\epsilon$ whose norm characterizes the improvement provided by an a/d step.

Consider in particular the expected values of z^2 [cf. (4.5)]:

$$E(z^2) := E\left\{ \frac{\|(I-\Pi)\epsilon\|^2}{\|\epsilon\|^2} \right\}. \tag{4.11}$$

Take the case $p=1$ for convenience (and without loss of generality, since the ensuing analysis may be carried out within each set of aggregation). Then

[cf. (4.6) and (4.8)]

$$\|(I-\Pi)\epsilon\|^2 = \begin{cases} \sum_i \left(\epsilon_i - \dfrac{\sum_j \epsilon_j}{\sum_j \bar{x}_j} \bar{x}_i \right)^2, & \text{example (a)}, \\ \sum_i \left[\epsilon_i - \dfrac{1}{n} \left(\sum_j \dfrac{\epsilon_j}{\bar{x}_j} \right) \bar{x}_i \right]^2, & \text{example (b)}, \\ \sum_i \left(\epsilon_i - \dfrac{\epsilon_{i*}}{\bar{x}_{i*}} \bar{x}_i \right)^2, & \text{example (c)}, \\ \sum_i \left(\epsilon_i - \dfrac{1}{n} \sum_j \epsilon_j \right)^2, & \text{example (e)}. \end{cases} \tag{4.12}$$

Here, since $p = 1$, all sums are taken from 1 to n.

Although in each of these four examples the distribution of z_a, z_b, z_c, and z_e (we use the subscripts a, b, c, and e to distinguish among the four examples being discussed) is in principle determinable from the measure imposed on the class of problems (as a subset of $\mathbb{R}^{n(n+1)}$), we will for convenience simply assume that the ϵ_i and the $\bar{x}_i$, $i = 1, \ldots, n$, which appear in (4.12) are each randomly chosen samples of independent random variables u_ϵ and u_x, respectively. We shall show that there are distributions of u_ϵ and u_x for which (4.11) is less than unity. This suggests that the typical a/d step is better on the average than a simple relaxation step (1.2) for a matrix A whose spectral radius $\rho(A)$ is near unity.

If u is a random variable with mean $E(u)$ and variance $\sigma^2(u)$, let

$$cv(u) = \frac{\sigma(u)}{E(u)} \quad \text{and} \quad s(u) = E(u)E(u^{-1}).$$

Using (4.11) and (4.12) and assuming independence of the random variables u_ϵ and u_x, straightforward but lengthy computations show that

asymptotically in the limit for n large

$$E(z^2)=\begin{cases} \dfrac{cv^2(u_\epsilon)+cv^2(u_x)}{1+cv^2(u_\epsilon)}, \\ \dfrac{cv^2(u_\epsilon)+s^2(u_x)cv^2(u_x)+[1-s(u_x)]^2}{1+cv^2(u_\epsilon)}, \\ \dfrac{cv^2(u_\epsilon)+s^2(u_x)\{[1+cv^2(u_x)][1+cv^2(u_x^{-1})] \times[1+cv^2(u_\epsilon)]-1\}+[1-s(u_x)]^2}{1+cv^2(u_\epsilon)} \\ \dfrac{cv^2(u_\epsilon)}{1+cv^2(u_\epsilon)}, \end{cases} \tag{4.13}$$

for the examples (a), (b), (c), and (e), respectively. We see that

$$E(z_e^2)\leqslant E(z_a^2),$$

$$E(z_e^2)\leqslant E(z_b^2)\leqslant E(z_c^2).$$

These inequalities show that among our examples aggregation onto a typical direction, viz e, may very well give the best result, on the average. This is the case in the absence of additional information, so that all possible problems are equally likely. When information of the type described following (4.10) is available, more special aggregation techniques will give better results, as the deterministic estimates of Section 4.2.3 show.

In order to perceive the possible values of $E(z^2)$ [for comparison with a value of $\rho(A)$ near unity], let us make a choice for the distributions of the random variables u_x and u_ϵ. In the case that the components of A and b are nonnegative, (1.2) shows that for sufficiently large iteration index k the components of ϵ and the components of x are of one sign. (Indeed, this is the case for any $k>0$ if $x^0=0$.) Thus suppose that u_ϵ is uniformly distributed on the interval $[0,r]$ $(r>0)$, and that u_x is uniformly distributed on the interval $[s,t]$ $(t>s>0)$. Homogeneity properties of the quantities $cv(u)$ and $s(u)$ show that without loss of generality the intervals $[0,r]$ and $[s,t]$ may be replaced by $[0,1]$ and $[1,\gamma]$, respectively, where $\gamma=t/s$. Then a direct

computation shows that $cv^2(u_\epsilon)=\frac{1}{3}$ and $s(u_x)=[(\gamma+1)\log\gamma]/[2(\gamma-1)]$. Thus we obtain

$$E(z^2)=\begin{cases}\dfrac{\gamma}{2(\gamma+1)},\\[2ex] 1-\dfrac{(\gamma+1)\log\gamma}{\gamma-1}\left[\dfrac{3}{4}-\dfrac{2\gamma+1}{8(\gamma-1)}\log\gamma\right],\\[2ex] 1-\dfrac{3}{4}(\gamma+1)\left[\dfrac{\log\gamma}{\gamma-1}-\dfrac{2}{9}\dfrac{2\gamma+1}{\gamma}\right],\\[2ex] \dfrac{1}{4},\end{cases} \tag{4.14}$$

for the four examples, respectively. Notice that in the present context (all components positive) the direction e is indeed typical. The corresponding values of $E(z^2)$ reflect this.

A more explicit perception of the values of $E(z^2)$ is obtained in turn by considering the limiting case $\gamma=1+\delta$ with $\delta>0$ and small. We obtain

$$E(z^2)=\frac{1}{4}\begin{cases}1+\dfrac{\delta}{2}-\dfrac{\delta^2}{4}+O(\delta^3),\\[2ex] 1+\dfrac{\delta}{2}-\dfrac{2}{9}\delta^2+O(\delta^3),\\[2ex] 2-3\delta+O(\delta^2),\\[2ex] 1,\end{cases}$$

for the four examples, respectively. In the asymptotic context introduced here, $\bar{x}$ is quasiproportional to the vector e. Thus all the components of $\bar{x}$ play a similar role in the aggregation process. Indeed the expressions for examples (a) and (b) in (4.12) approach the expression for the last example. This is confirmed by the values of $E(z^2)$ just derived.

4.2.4 Iterates Following the a/d Step. Our computational experience has shown that in many cases not only does the a/d process produce an improvement for $\epsilon(\tilde{x})$ [beyond the gain of $\rho(A)$ provided by a simple relaxation step], but an increased rate of improvement persists for several iterates following the a/d step. Denote by $\hat{x}$ the first such iterate, i.e.,

$$\hat{x}=A\tilde{x}+b. \tag{4.15}$$

(Cf. (2.7) and Sloan [12].)

Using the identity

$$(I-A\Pi)^{-1}-(I-A)^{-1}=(I-A\Pi)^{-1}(A\Pi-A)(I-A)^{-1}$$

and the relation $(I-\Pi)^2=I-\Pi$, we obtain

$$\begin{aligned} x^*-\hat{x}&=(I-A)^{-1}b-(I-A\Pi)^{-1}b \\ &=(I-A\Pi)^{-1}A(I-\Pi)x^* \\ &=(I-A\Pi)^{-1}A(I-\Pi)(I-\Pi)x^*. \end{aligned} \tag{4.16}$$

[Compare (4.1).] Then in the case of multiplicative aggregation, in which the relation $\Pi\bar{x}=\bar{x}$ also holds, we get

$$x^*-\hat{x}=(I-A\Pi)^{-1}A(I-\Pi)(I-\Pi)(x^*-\bar{x}). \tag{4.17}$$

[Compare (4.3).]

In the additive case, we correspondingly define

$$\hat{e}=A\bar{e}+r.$$

Then $\bar{x}+\hat{\epsilon}=A\bar{x}+b=\hat{x}$. Thus by applying (4.16) to the equation $(I-A)\epsilon=r$, we obtain

$$\begin{aligned} \epsilon-\hat{\epsilon}&=(x^*-\bar{x})-(\hat{x}-\bar{x})=x^*-\hat{x} \\ &=(I-A\Pi)^{-1}A(I-\Pi)(I-\Pi)\epsilon, \end{aligned}$$

and in turn, the identity corresponding to (4.17) follows in the additive case.

Let $\|(I-A\Pi)^{-1}\|\leqslant\hat{K}$. [Compare (4.4).] Let

$$\hat{\alpha}:=\frac{\|x^*-\hat{x}\|}{\|x^*-\bar{x}\|}. \tag{4.18}$$

[Compare (4.10).] Then from (4.17), (4.18), and (4.10), we have

$$\hat{\alpha}=\frac{\|x^*-\hat{x}\|}{\|x^*-\bar{x}\|}=\frac{\|x^*-\hat{x}\|}{\|(\Pi-I)\epsilon\|}\,\frac{\|(\Pi-I)\epsilon\|}{\|x^*-\bar{x}\|}\leqslant\hat{K}\|A(I-\Pi)\|\,\|I-\Pi A\|,$$

since

$$\|(\Pi-I)\epsilon\|\leqslant\|I-\Pi A\|\,\|x^*-\bar{x}\|.$$

(Cf. Section 4.2.1.) Thus, the first iteration after the a/d step may also provide a gain over a straightforward step of successive approximation if

$$\rho_1=\rho(A(I-\Pi)) \tag{4.19}$$

is significantly smaller than $\rho(A)$. This is illustrated in cases (i) and (ii) of the numerical experiments which follow.

5. NUMERICAL EXPERIMENTS

In this section, we display the results of numerical experiments employing the a/d strategies (a) and (b) of Section 2.5.2 and (e) of Section 2.5.1. Strategies (a) and (b) are used multiplicatively, while strategy (e) is used additively. Each strategy is applied in three cases, corresponding to different matrix-vector pairs A and b.

Case (i): A is an input-output matrix of order $n = 9$ and spectral radius $\rho(A) = 0.7$. The number of aggregates is $p = 3$. b is the associated demand vector. (This matrix-vector pair is more explicitly identified in [9].)

Case (ii): A is a randomly chosen positive matrix: $n = 15$, $\rho(A) = 0.897$, and $p = 3$. b is chosen randomly also.

Case (iii): A is a matrix arising from a finite difference approximation to a boundary value problem: $n = 20$, $\rho(A) = 0.982$, and $p = 5$. (A is identically zero except in the two principal off-diagonals, where all the entries equal $\frac{1}{2}$.) b is identically zero except for its first entry, which is unity.

The aggregates are chosen lexicographically in all cases. For example, in the case $n=9$, $p=3$, we set $\mathcal{K}_1=\{1,2,3\}$, $\mathcal{K}_2=\{4,5,6\}$, and $\mathcal{K}_3=\{7,8,9\}$ (cf. Section 2.1).

We display graphs and tables illustrating the computations. An explanation of the information displayed in these graphs and tables now follows.

We perform m simple relaxation steps followed by an a/d step and repeat. This policy is applied in all three cases for both $m=5$ and $m=10$. ν indexes the simple relaxation steps, which are N in total number. The sequence m repeated N/m times is called the (a/d) policy. r indexes the a/d steps. In every circumstance, the initial vector $x^0=(1,2,\ldots,n)^t$ is used.

In summary, we deal with three a/d strategies, three matrix-vector cases, and two (a/d) policies, i.e., 18 different situations.

We define

$$\alpha_\nu := \frac{\|x^* - x^\nu\|}{\|x^* - x^{\nu-1}\|}, \qquad \nu = 1,2,\ldots,$$

and corresponding to the rth a/d step, $r = 1,2,\ldots$, we define

$$\tilde{\alpha}^r := \frac{\|x^* - \tilde{x}^r\|}{\|x^* - \bar{x}^r\|}$$

[compare (4.10)],

$$\hat{\alpha}^r := \frac{\|x^* - \hat{x}^r\|}{\|x^* - \tilde{x}^r\|}$$

[compare (4.18)],

$$\tilde{K}^r := \frac{\|(I - \Pi^r)x^*\|}{\|x^* - \tilde{x}^r\|}$$

[compare (4.4)],

$$\hat{K}^r := \frac{\|A(I - \Pi^r)x^*\|}{\|x^* - \hat{x}^r\|}$$

[compare (4.18)], and

$$\rho_1^r := \rho(A(I - \Pi^r))$$

[compare (4.19)].

In each case and policy we plot superimposed graphs of the points (ν, α_ν), $\nu = 1,2,\ldots$, and $(r, \tilde{\alpha}_r)$, $r = 1,2,\ldots$ for the three examples. [Of course the r scale is either 5 or 10 times the ν scale depending on the policy (value of $m = 5$ or 10) of a/d steps.] The solid line plot corresponds to a/d (a), the short dashed line plot corresponds to a/d (b), and the long dashed line plot corresponds to a/d (e). All the plots are discontinuous at the abcissae $r = 1,2,\ldots$, and to facilitate comprehension, we place dotted vertical portions at these abcissae.

We also tabulate $(r, \tilde{\alpha}^r, \hat{\alpha}^r, \rho_1^r, \tilde{K}^r, \hat{K}^r)$. The tables and graphs for the various alphas are useful as measures of convergence rate per step in the relaxation and a/d steps as the case may be. They show the degree to which acceleration is achieved. The remaining tabulated quantities are useful as follows:

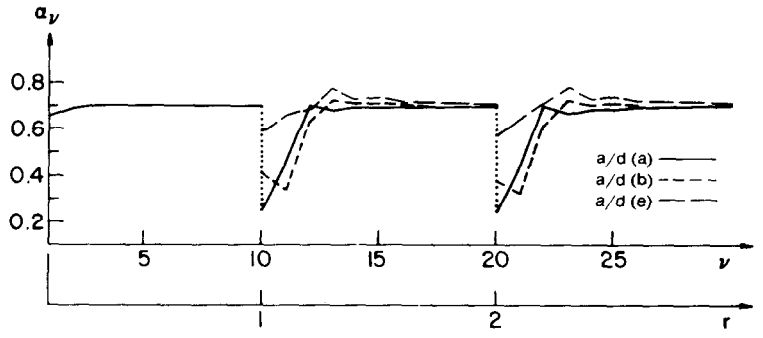
ρ_1^r small compared to ρ shows the favorable tendency of α_ν to return more slowly to its limit value ρ.

$\tilde{K}^r$ [$\hat{K}^r$] shows the usefulness of characterizing $\|x^* - \tilde{x}\|$ [$\|x^* - \hat{x}\|$] in terms of $\|(I - \Pi)(x^* - \tilde{x})\|$ [$\|A(I - \Pi)(x^* - \tilde{x})\|$]. Compare (4.4) [(4.17)–(4.18)].

We also tabulate the Euclidean norm of the following error vectors: initial (the initial error norm $\|x^{(0)} - x^*\|$); simple(N) (the final error norm after N simple relaxation steps $\|x^{(N)} - x^*\|$); and a/d (a), a/d (b), and a/d (e) (the final error norms after N simple relaxation steps during which the a/d process is applied after each m such steps, i.e., according to the (a/d) policy (a), (b), and (e), respectively). According to our conventions the last three quantities correspond to the a/d examples (a), (b), and (e), respectively [cf. Sections 2.5.2(a) and 2.5.2(b), and 2.5.1, respectively]. These tables indicate the global difference between simple relaxation and the various accelerated versions.

TABLE 1
CASE (i) $N = 30$; POLICY = 10 10 10

	r	1	2
(a)	$\tilde{\alpha}_r$	0.25	0.24
	$\hat{\alpha}_r$	0.44	0.43
	ρ_1^r	0.48	0.48
	$\tilde{K}_r$	1.00	1.00
	$\hat{K}_r$	0.42	0.43
(b)	$\tilde{\alpha}_r$	0.40	0.38
	$\hat{\alpha}_r$	0.34	0.33
	ρ_1^r	0.43	0.43
	$\tilde{K}_r$	0.76	0.77
	$\hat{K}_r$	0.43	0.45
(e)	$\tilde{\alpha}_r$	0.59	0.59
	$\hat{\alpha}_r$	0.66	0.66
	ρ_1^r	0.48	0.48
	$\tilde{K}_r$	0.42	0.41
	$\hat{K}_r$	0.12	0.11

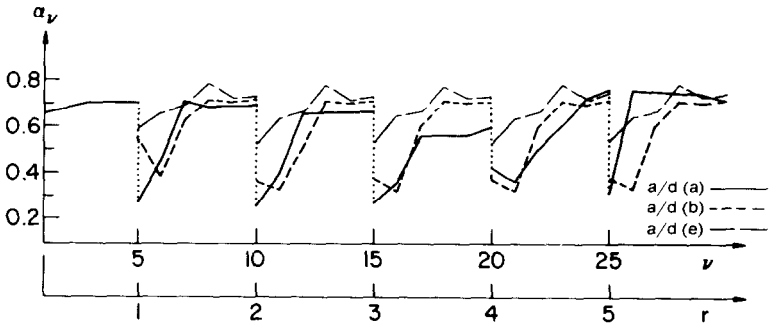


	initial	simple(30)	a/d (a)	a/d (b)	a/d (e)
Error norm	5×10^4	1.13	0.021	0.036	0.057

TABLE 2

CASE (i): $N=30$; POLICY=5 5 5 5 5

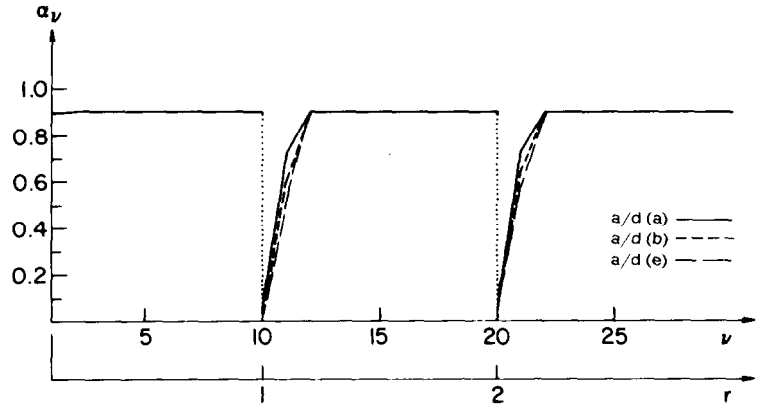
	r	1	2	3	4	5
(a)	$\tilde{\alpha}_r$	0.28	0.25	0.28	0.42	0.32
	$\hat{\alpha}_r$	0.45	0.40	0.35	0.37	0.75
	ρ_1^r	0.48	0.48	0.48	0.48	0.48
	$\tilde{K}_r$	1.01	1.00	1.00	1.00	0.97
	$\hat{K}_r$	0.42	0.50	0.71	1.01	0.61
(b)	$\tilde{\alpha}_r$	0.55	0.37	0.38	0.38	0.38
	$\hat{\alpha}_r$	0.39	0.33	0.33	0.33	0.33
	ρ_1^r	0.43	0.43	0.43	0.43	0.43
	$\tilde{K}_r$	0.68	0.76	0.76	0.76	0.76
	$\hat{K}_r$	0.55	0.47	0.38	0.38	0.38
(e)	$\tilde{\alpha}_r$	0.59	0.53	0.54	0.54	0.54
	$\hat{\alpha}_r$	0.66	0.64	0.65	0.64	0.64
	ρ_1^r	0.48	0.48	0.48	0.48	0.48
	$\tilde{K}_r$	0.48	0.42	0.42	0.42	0.42
	$\hat{K}_r$	0.13	0.16	0.16	0.16	0.16



	initial	simple(30)	a/d (a)	a/d (b)	a/d (e)
Error norm	5×10^4	1.13	0.9×10^{-4}	2×10^{-4}	0.084

TABLE 3
CASE (ii): $N=30$; POLICY $=10\ 10\ 10$

	r	1	2
(a)	$\tilde{\alpha}_r$	.068	.049
	$\hat{\alpha}_r$	.73	.73
	ρ_1^r	.14	.14
	$\tilde{K}_r$	.72	.71
	$\hat{K}_r$	.19	.19
(b)	$\tilde{\alpha}_r$	.058	.044
	$\hat{\alpha}_r$	.60	.65
	ρ_1^r	.15	.15
	$\tilde{K}_r$	.86	.80
	$\hat{K}_r$	.25	.22
(e)	$\tilde{\alpha}_r$	.093	.093
	$\hat{\alpha}_r$	.59	.59
	ρ_1^r	.14	.14
	$\tilde{K}_r$	.53	.37
	$\hat{K}_r$	.18	.13

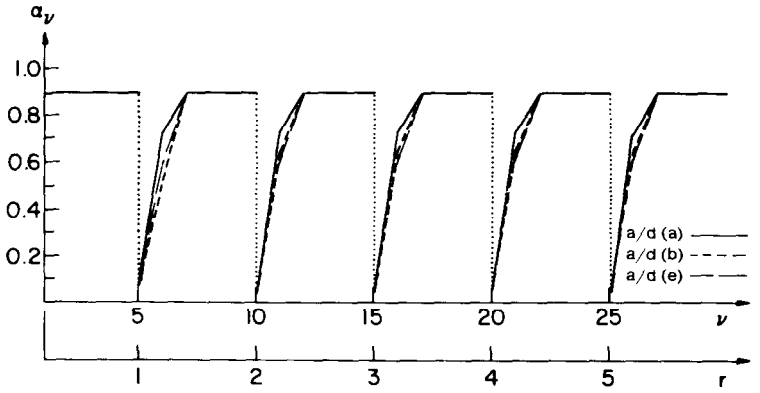


	initial	simple(30)	a/d (a)	a/d (b)	a/d (e)
Error norm	215	8.07	0.018	0.017	0.029

TABLE 4

CASE (ii): $N=30$; POLICY $=5\ 5\ 5\ 5\ 5$

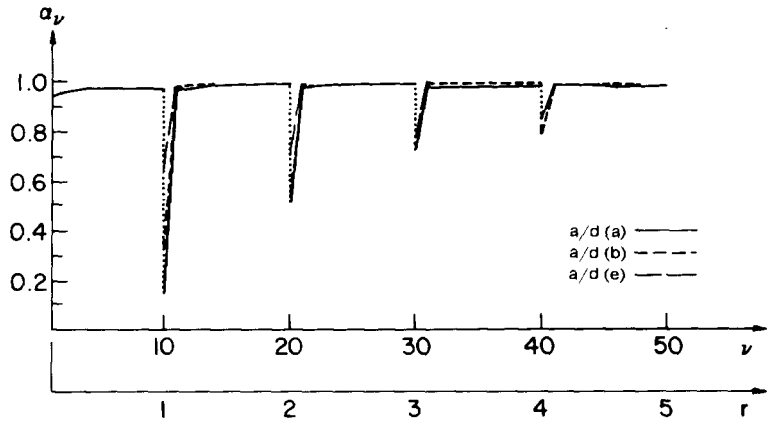
	r	1	2	3	4	5
(a)	$\tilde{\alpha}_r$	.095	.050	.049	.049	.049
	$\hat{\alpha}_r$	.73	.73	.73	.73	.73
	ρ_1^r	.14	.14	.14	.48	.14
	$\tilde{K}_r$	.74	.72	.71	.71	.71
	$\hat{K}_r$	2.0	.19	.19	.19	.19
(b)	$\tilde{\alpha}_r$	.075	.044	.044	.044	.044
	$\hat{\alpha}_r$	.50	.65	.65	.65	.65
	ρ_1^r	.15	.15	.13	.15	.15
	$\tilde{K}_r$	.94	.80	.80	.80	.80
	$\hat{K}_r$	.32	.23	.22	.22	.22
(e)	$\tilde{\alpha}_r$	.093	.093	.093	.093	.093
	$\hat{\alpha}_r$	.56	.56	.56	.56	.56
	ρ_1^r	.14	.14	.14	.14	.14
	$\tilde{K}_r$	.75	.37	.38	.38	.38
	$\hat{K}_r$	.25	.12	.13	.13	.13



	initial	simple(30)	a/d (a)	a/d (b)	a/d (e)
Error norm	215	8.07	2×10^{-6}	3×10^{-7}	6×10^{-6}

TABLE 5
CASE (iii): $N = 50$; POLICY = 10 10 10 10 10

	r	1	2	3	4
(a)	$\tilde{\alpha}_r$	.15	.51	.73	.84
	$\hat{\alpha}_r$	.97	.98	.98	.98
	ρ_1^r	.98	.98	.98	.98
	$\tilde{K}_r$	.22	.24	.27	.29
	$\hat{K}_r$	.10	.17	.22	.26
(b)	$\tilde{\alpha}_r$	.33	.55	.73	.79
	$\hat{\alpha}_r$	.98	.99	.99	.99
	ρ_1^r	.99	.98	.98	.98
	$\tilde{K}_r$	.10	.15	.16	.18
	$\hat{K}_r$	.052	.083	.11	.14
(e)	$\tilde{\alpha}_r$	.63	.70	.75	.78
	$\hat{\alpha}_r$	.98	.98	.98	.99
	ρ_1^r	.97	.97	.97	.97
	$\tilde{K}_r$	.052	.075	.10	.18
	$\hat{K}_r$	.024	.042	.063	.088

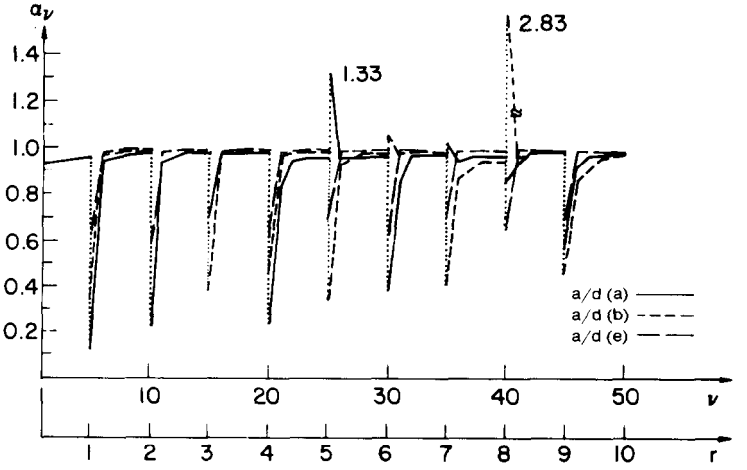


	initial	simple(50)	a/d (a)	a/d (b)	a/d (e)
Error norm	51	22.4	0.96	2.5	5.46

TABLE 6

CASE (iii): $N = 50$; POLICY = 5 5 5 5 5 5 5 5 5 5

	r	1	2	3	4	5	6	7	8	9
(a)	$\tilde{\alpha}_r$	.13	.23	.68	.24	1.33	.39	1.02	.86	.58
	$\hat{\alpha}_r$	.96	.94	.98	.84	.98	.86	.95	.94	.92
	ρ_1^r	.98	.96	.98	.98	.98	.98	.98	.98	.98
	$\tilde{K}_r$	.24	.39	.29	.70	.39	.71	.55	.53	.69
	$\hat{K}_r$	.11	.085	.22	.36	.36	.60	.55	.50	.70
(b)	$\tilde{\alpha}_r$	.33	.34	.39	.46	.034	1.07	.41	2.83	.46
	$\hat{\alpha}_r$	.98	.98	.98	.98	.94	.98	.88	.97	.88
	ρ_1^r	.95	.99	.99	.99	.99	.99	.99	.99	.99
	$\tilde{K}_r$	.096	.18	.28	.27	.52	.23	.76	.26	.81
	$\hat{K}_r$	.046	.063	.12	.064	.35	.18	.76	.26	.84
(e)	$\tilde{\alpha}_r$	.64	.60	.69	.62	.69	.63	.69	.64	.69
	$\hat{\alpha}_r$	.97	.98	.98	.98	.98	.98	.99	.98	.99
	ρ_1^r	.97	.97	.97	.97	.97	.97	.97	.97	.97
	$\tilde{K}_r$	.047	.068	.088	.12	.14	.17	.19	.21	.23
	$\hat{K}_r$	.021	.025	.041	.044	.067	.065	.095	.075	.12



	initial	simple(50)	a/d (a)	a/d (b)	a/d (e)
Error norm	51	22.4	0.0125	0.0517	0.462

6. DISCUSSION

These computational experiments confirm the error estimates developed in Section 4. They also show that a/d processes can be extremely effective in accelerating the convergence of simple relaxation.

For the examples treated, the multiplicative a/d processes (a) and (b) appear superior to the additive a/d process (e). The stochastic estimate of Section 4.2.3 supplies an explanation for this: the vectors on which the aggregation steps are based are not randomly chosen, but are in fact the approximations $\bar{x}$ themselves, and the $\bar{x}$, of course, are converging to x^*.

By examining the graphs in the examples, it can be seen that the a/d steps have been employed only after the iteration process settles down to its asymptotic rate of convergence (as represented by the horizontal portions of the graphs). This suggests that the a/d step should be used as soon as this happens. The second example of case (i) is a case in point. Indeed, compare the error norms of the two examples of case (i). Timing of the a/d step is a delicate matter, as the following observation suggests.

In case (iii) we see that it is possible for a particular a/d step to be counterproductive (see the graph of the last example). However, the global performance is unimpaired by these events. A guess at the reason for such a counterproductive step is the premature application of the a/d process at that point. Thus in actual practice such a counterproductive step should be rejected in favor of one or more ordinary relaxation steps. Thereupon the a/d step should be retried. Strategies for policy choices remain an open question.

In cases (i) and (ii), ρ_1 is significantly less than unity, so that $\hat{\alpha}_r$, the measure of improvement of the iteration step following the a/d step, is correspondingly small. In case (iii), both ρ_1 and $\hat{\alpha}_r$ are near unity.

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