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Actions of G_a on A^3 defined by homogeneous derivations

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Abstract

The first example of an algebraic action of G_a on affine 3-space having maximal rank 3 is produced. Its fixed points consist of a single line in A^3, and G_a is realized as an algebraic subgroup of $Aut_k(A^3)$ whose non-trivial elements are of degree 41. The corresponding derivation is homogeneous and irreducible of degree 4. Since triangulable actions are never of maximal rank, this action is non-triangulable. This action is embedded, for each $n \geq 3$, into a G_a-action on A^n, in such a way that the resulting action has rank n, thus showing that algebraic G_a-actions on A^n having maximal rank exist for each $n \geq 3$.

Also considered is the general case of a homogeneous locally nilpotent derivation on $k^{[3]}$. The main tool here is the exponent of a polynomial relative to the derivation. By describing such derivations of type $(2, d+1)$, where d is the degree of the derivation, it is shown that actions induced by homogeneous derivations of degree less than four have rank at most 2. The rank 3 example mentioned above appears as a special case of Theorem 4.2. © 1998 Elsevier Science B.V.

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1. Introduction

If k is a field of characteristic 0, then algebraic actions of $G_a = (k, +)$ on affine n-space A_k^n are equivalent to k-derivations on the polynomial ring $k[X_1, \ldots, X_n]$ which are locally nilpotent (cf. [12]). The *rank* of such an action is defined in [5], and the question is asked whether there exist actions of maximal rank n for $n \geq 3$. It is known that for $n = 2$, the rank is at most one [11], and for $n \geq 3$ all previously constructed examples were of rank less than n. It will be shown that, for all $n \geq 3$, actions of

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rank n in dimension n exist. Note well, however, that the actions constructed below have fixed points: the important question of whether all fixed-point free actions on A^3 are equivariantly trivial remains open, and is equivalent to deciding whether such an action can have rank 3. (This equivalence is implied by the results of [4].) For $n \geq 4$, fixed-point free algebraic G_a-actions on A^n which are not equivariantly trivial were first constructed by Winkelmann [13] and Smith (cf. [12]).

In Section 2 we construct an action of rank 3 in dimension 3, and demonstrate that it has the desired properties. Its fixed points consist of a single line in A^3, and G_a is realized as an algebraic subgroup of $Aut_k(A^3)$ whose non-trivial elements are of degree 41. The corresponding derivation is homogeneous and irreducible of degree 4. Since triangulable actions are never of maximal rank, it is immediate that this action is non-triangulable (cf. [1, 10]).

In Section 3, the action of Section 2 is embedded, for each $n \geq 3$, into an action in dimension n, in such a way that the resulting action has rank n. In the remainder of the paper, certain general results are given concerning homogeneous locally nilpotent derivations in dimension 3. These results are used to show that, for such derivations of degree less than 4, the rank is always at most 2.

The following notation and definitions will be used. Let R be any integral k-domain, and let D be a k-derivation of R. Denote the kernel of D by $Ker(D)$, and let

$$Nil(D) = \{ f \in R | D^s f = 0 \quad \text{for } s \gg 0 \}.$$

Both $Ker(D)$ and $Nil(D)$ are subalgebras of R [9]. We say D is *locally nilpotent* iff $Nil(D) = R$.

Let $k^{[n]}$ denote the polynomial ring in n variables over k. When $R = k^{[n]}$, the *rank* of D is the least integer $r \geq 0$ for which there exists a system of variables $(X_1, \dots, X_n)$ of $k^{[n]}$ satisfying $k[X_{r+1}, \dots, X_n] \subset Ker(D)$. When D is locally nilpotent, the corresponding algebraic action of G_a on A^n is given by $\exp(tD)$ ($t \in G_a$). Conversely, every such action corresponds to a locally nilpotent derivation of $k^{[n]}$ in an essentially unique way. Hence, the *rank* of an algebraic action of G_a on A^n is taken to be the rank of the corresponding derivation on $k^{[n]}$.

Given a system of variables on $k^{[n]}$, together with a corresponding system of weights on those variables, let V_i denote the k-vector space of weighted i-forms in $k^{[n]}$. We say D is *weighted-homogeneous* with respect to the given variables, of *weighted degree d*, if $D(V_i) \subset V_{i+d}$ for each $i \in \mathbb{Z}$. Weighted-homogeneous derivations have the property that, if $f \in Ker(D)$ decomposes as $f = \sum_{i \in \mathbb{Z}} f_i$ for $f_i \in V_i$, then $f_i \in Ker(D)$ for each i.

If each of the given variables has weight 1, we say simply that D is *homogeneous* with respect to the given variables. Since every variable of $k^{[n]}$ has a non-trivial linear part, we can observe the following simple, but crucial, fact: If D is a homogeneous k-derivation of $k^{[n]}$ (with respect to any system of variables), then $rank(D) < n$ iff there exists $L \in V_1$ for which $DL = 0$.

2. A rank three action in dimension three

Throughout this section, $B = k^{[3]} = k[X,Y,Z]$. Define a k-derivation $\Delta : B \to B$ as follows. For any $H \in B$,

$$\Delta(H) = \frac{\partial(F,G,H)}{\partial(X,Y,Z)},$$

where $F = (XZ + Y^2)$ and $G = (ZF^2 + 2X^2YF - X^5)$. Set $A = k[F,G]$, noting that $A \subset Ker(\Delta)$. Direct computation shows

1. $\Delta X = -2Fr \quad$ where $r = (X^3 - FY)$,
2. $\Delta Y = -6X^2r - G$,
3. $\Delta r = FG$.

Therefore, $r \in Nil(\Delta)$, and $A[r] \subset Nil(\Delta)$. Since $\Delta X \in A[r]$, we have $A[X,r] \subset Nil(\Delta)$. Since $\Delta Y \in A[X,r]$, we have $A[X,Y,r] \subset Nil(\Delta)$, and since $ZF^2 = (G - 2X^2YF + X^5) \in A[X,Y,r]$, we have $\Delta^s(ZF^2) = F^2 \cdot \Delta^s(Z) = 0$ for $s \gg 0$. Therefore, $Z \in Nil(\Delta)$, and $B = A[X,Y,Z,r] \subset Nil(\Delta)$. In other words, Δ is locally nilpotent on B.

Next, it must be demonstrated that the rank of Δ is three. By the observation at the end of Section 1, it suffices to show that, if $\Delta L = 0$ for $L \in V_1$, then $L = 0$. Suppose $L = \alpha X + \beta Y + \gamma Z$ for $\alpha, \beta, \gamma \in k$, and $\Delta L = 0$. Then $\alpha\Delta X + \beta\Delta Y + \gamma\Delta Z = 0$. Modulo (F), we obtain

$$\begin{aligned} \alpha(0) + \beta(-5X^5) + \gamma(10X^4Y) = 0 &\Longrightarrow \beta X - 2\gamma Y = 0 \\ &\Longrightarrow \beta = \gamma = 0 \\ &\Longrightarrow L = 0. \end{aligned}$$

Therefore Δ has rank 3.

The foregoing results may be stated in other terms. If $GA_3(k)$ denotes the group of k-automorphisms of A^3 (the *affine Cremona group*), and if $GA_2(k[X])$ is the subgroup which fixes the plane $X = 0$, then the one-parameter subgroup $\Gamma = \exp(t\Delta)$ of $GA_3(k)$ cannot be conjugated into $GA_2(k[X])$. However, if Ω denotes the *non-linear orthogonal subgroup* of $GA_3(k)$ (i.e., automorphisms fixing the non-degenerate quadratic form F), then $\Gamma \subset \Omega$. It is straightforward to show $\Delta^3X = \Delta^7Y = \Delta^{11}Z = 0$, while Δ^2X, Δ^6Y, and $\Delta^{10}Z$ are not zero. Thus, the degree of any automorphism in Γ (other than 1) equals the degree of $\Delta^{10}Z$, namely 41.

2.1. Kernel

We show that $A = Ker(\Delta)$ (from which it is also clear that the rank of Δ is 3). To prove this, we use the following.

Theorem 1 (Zurkowski [14, Theorem 5] and Daigle [2, Corollary 3.2]). *Let K be a field of characteristic 0. If D is a non-zero weighted homogeneous K-derivation of*

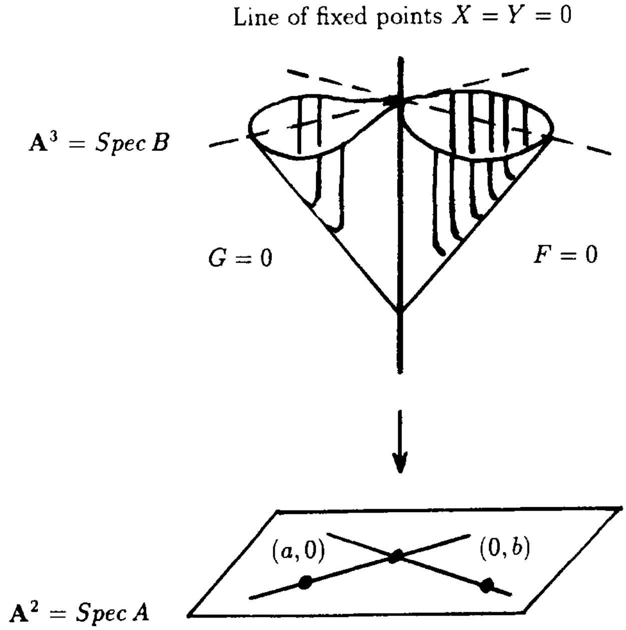

Fig. 1.

$K[X_1, X_2, X_3] = K^{[3]}$ which is locally nilpotent, and if the weights of X_1, X_2, and X_3 are positive, then there exist weighted homogeneous polynomials f and g such that $Ker(D) = K[f, g]$.

(This was first proved by Zurkowski in the case K is algebraically closed; Zurkowski does not assert the homogeneity of f and g in his statement of the theorem, but this is included in his proof.)

By this theorem, there exist homogeneous $f, g \in B$ such that $Ker(\Delta) = k[f, g]$. Suppose F is neither in (f) nor in (g). Then $F = ag^m + fh$ for some $a \in k^*$ and $h \in B$. By homogeneity, $m\,deg(g) = 2$. Since $rank(\Delta) = 3$, $deg(g) > 1$, and thus $deg(g) = 2$. But the same reasoning also shows $deg(f) = 2$, which is impossible, since G (having degree 5) could not be a polynomial in f and g. Therefore, F lies in either (f) or (g), and since F is irreducible, we may assume $F = f$.

We now have $G \in k[F, g]$ and $G \notin (F)$. Write $G = bg^n + FH$ for $b \in k^*$ and $H \in B$. Then $n\,deg(g) = 5$, and since $deg(g) > 1$, it follows that $deg(g) = 5$. Now, up to constant multiples, g is the only element of $k[F, g]$ of degree 5. Therefore, $k[F, G] = k[F, g] = Ker(\Delta)$, as claimed.

2.2. Orbits

The set of points fixed by the G_a-action on A^3 induced by Δ is precisely the set of points where ΔX, ΔY, and ΔZ vanish simultaneously, and it is easy to check that this set is the line defined by $X = Y = 0$. Every other orbit is a line, i.e., isomorphic to $\mathbf{G}_a \cong \mathbf{A}^1$. Let $\pi : \mathbf{A}^3 \to \mathbf{A}^2$ be the morphism induced by the inclusion $A \hookrightarrow B$. Then the fiber over the point $(a, b) \in \mathbf{A}^2$ is defined by the ideal $(f - a, g - b)$ in B, and each fiber is a union of orbits. The fiber over the origin $(0,0)$ is the line of fixed points mentioned above. If neither a nor b is 0, then the fiber over (a,b) is a single (coordinate) line in $\mathbf{A}^3$. The most interesting fibers lie over points $(0, b)$ and $(a, 0)$ for $a \neq 0$ and $b \neq 0$. Over $(0, b)$, $b \neq 0$, the fiber consists of five (coordinate) lines lying on the surface defined by F. And over $(a, 0)$, $a \neq 0$, the fiber consists of two (coordinate) lines lying on the surface defined by G. The situation is depicted in Fig. 1.

3. Extensions of Δ to higher dimensions

In the event $n = 3m$ ($m \geq 1$), Δ may be extended to the locally nilpotent derivation $\Delta \times \cdots \times \Delta$ on $k^{[n]} = k[X_1,\ldots,X_n] = k^{[3]} \times \cdots \times k^{[3]}$. The extended derivation is again homogeneous, and it is of rank n, since the images of $X_1,\ldots,X_n$ are linearly independent.

The following more general construction is due to Daigle. Given $n \geq 3$, identify $B = k[X, Y, Z]$ as a subring of $\tilde{B} = k^{[n]} = k[X_1,\ldots,X_n]$ via $X = X_1$, $Y = X_2$, and $Z = X_3$. Define $\tilde{\Delta}$ on $\tilde{B}$ by letting $\tilde{\Delta} = \Delta$ on B, and

$$\tilde{\Delta}(X_i) = X_{i-1}^5 \quad \text{for } i = 4,5,\ldots,n.$$

Then $\tilde{\Delta}$ is again locally nilpotent and homogeneous. To see that $\tilde{\Delta}$ has maximal rank, it suffices to check that $\tilde{\Delta}X_1,\ldots,\tilde{\Delta}X_n$ are linearly independent. Suppose

$$a_1\tilde{\Delta}(X_1) + \cdots + a_n\tilde{\Delta}(X_n) = 0 \quad (a_i \in k).$$

Modulo $(X_1,\ldots,X_{n-2})$, this equation becomes $a_n\tilde{\Delta}(X_n) = 0$, so $a_n = 0$. By induction (since $\tilde{\Delta}X_1$, $\tilde{\Delta}X_2$, $\tilde{\Delta}X_3$ are linearly independent), we obtain $a_1 = \cdots = a_n = 0$.

4. Homogeneous locally nilpotent derivations in dimension three

Let R be any integral k-domain, and let D be a k-derivation of R. D is *irreducible* if its image is contained in no proper principal ideal. An ideal $I \subset R$ is an *integral* ideal for D if $D(I) \subset I$ [8]. If D is locally nilpotent, then the following facts are well-known.

1. *Ker*(D) is factorially closed (other terminology: *inert*; *saturated*).
2. For any localization $S^{-1}R$ of R, $S^{-1}D$ is a k-derivation on $S^{-1}R$, and $S^{-1}D$ is locally nilpotent iff $S \subset (Ker(D) - 0)$.
3. For any ideal $I \subset R$, $D \bmod I$ is a well-defined (locally nilpotent) k-derivation on $R \bmod I$ iff I is an integral ideal of D.

The following result is also required.

Theorem 2 (Daigle [3, Corollary 2.5]). *Let K be any field of characteristic* 0, *and let D be a locally nilpotent K-derivation of $K^{[n]} = K[X_1,\dots,X_n]$ such that $Ker(D) = K[f_1,\dots,f_{n-1}] \cong K^{[n-1]}$. Define a K-derivation T on $K^{[n]}$ as follows: for each $h \in K^{[n]}$,*

$$T(h) = \frac{\partial(f_1,\dots,f_{n-1},h)}{\partial(X_1,\dots,X_n)}.$$

Then T is irreducible and locally nilpotent, and $D = fT$ for some $f \in Ker(D)$.

It should be noted that Miyanishi [7] has shown that when K is algebraically closed of characteristic 0, any non-zero locally nilpotent derivation on $K^{[3]}$ has kernel isomorphic to $K^{[2]}$. (Zurkowski's proof of Theorem 1 is independent of Miyanishi's result.) For the record, Miyanishi's result can be generalized as follows.

Theorem 3. *If K is any field of characteristic* 0, *and if D is any non-zero locally nilpotent K-derivation of $B = K^{[3]}$, then $Ker(D) \cong K^{[2]}$.*

Proof. (This proof, in essentially the same form, was given independently by Daigle.) Let L be an algebraically closed field containing K. Then D extends uniquely to a locally nilpotent L-derivation D_L on $B_L = (L \otimes_K B) \cong L^{[3]}$. By exactness, $Ker(D_L) = L \otimes_K Ker(D)$, and by Miyanishi's result, $Ker(D_L) \cong L^{[2]}$. Under these conditions, Kambayashi [6] has shown $Ker(D) \cong K^{[2]}$. □

4.1. Exponents

As above, let R be any integral k-domain. For any locally nilpotent derivation D on R, and for any $f \neq 0$ in R, define the *exponent* of f by

$$\nu_D(f) = max\{\, s \in \mathbb{Z} | D^s f \neq 0\} = deg_t(\exp tD(f)).$$

Also, define $\nu_D(0) = -\infty$. The following properties hold:

(P1) $\nu_D(fg) = \nu_D(f) + \nu_D(g)$,

(P2) $\nu_D(Df) = \nu_D(f) - 1$ if $Df \neq 0$,

(P3) $Df \in (f) \Leftrightarrow Df = 0 \Leftrightarrow \nu_D(f) \leq 0$,

(P4) $\nu_D(f+g) \leq max\{\nu_D(f), \nu_D(g)\}$,

(P5) $\nu_D(f+g) < max\{\nu_D(f), \nu_D(g)\} \Longrightarrow \nu_D(f) = \nu_D(g)$.

In addition, the following are required.

Lemma 1. *If D is any locally nilpotent derivation of $B = k^{[3]} = k[x,y,z]$ for which $I = (x,y)$ is an integral ideal of D, then Dz is constant modulo I.*

Proof. Let $\bar{D}$ be the induced k-derivation on $\bar{B} = B \bmod I \cong k[z]$. Then since $\bar{D}$ is locally nilpotent, $\bar{D} = \lambda \cdot (d/dz)$ for some $\lambda \in k$, and $\bar{D}z = \lambda$. □

Lemma 2. *If D is any irreducible locally nilpotent derivation of $B = k^{[3]} = k[x,y,z]$ having $Ker(D) = k[f,g]$, and if $g \in (m,f)$ for some linear form m in x,y,z, then $Dm \in (f)$.*

Proof. By a linear change of coordinates, we may assume $m = x$, and $g \in (x,f)$. Write $g = ax + bf$ for $a,b \in B$. By Theorem 2,

$$\begin{aligned} Dx &= f_y g_z - f_z g_y \\ &= f_y(xa_z + bf_z + fb_z) - f_z(xa_y + bf_y + fb_y) \\ &= x(f_y a_z - f_z a_y) + f(f_y b_z - f_z b_y). \end{aligned}$$

Thus, $Dx \in (x,f)$. Passing to the quotient modulo the integral ideal (f), we have $\bar{D}\bar{x} \in (\bar{x})$. Since $Ker(D)$ is factorially closed, f is irreducible, and $B \bmod (f)$ is a domain. By (P3), $\bar{D}\bar{x} = 0$, i.e., $Dx \in (f)$. □

Now suppose D is a non-zero homogeneous locally nilpotent derivation of $B = k^{[3]} = k[X,Y,Z]$. By Theorem 1, there exist irreducible homogeneous polynomials F and G such that

$$Ker(D) = k[F,G] \quad \text{and} \quad deg F \le deg G .$$

If $e_1 = deg(F)$ and $e_2 = deg(G)$, we will say D is of *type* (e_1,e_2).

Note that the type of D depends only on $Ker(D)$, and is uniquely determined. Suppose $k[F,G] = k[F',G']$, where F' and G' are again homogeneous. Since $k[F,G]$ is a $k^{[2]}$, $F' = aF + bG + H_1$ and $G' = cF + dG + H_2$, where $ad - bc \neq 0$, and $H_1,H_2 \in (F,G)^2$. From this, it follows easily that $\{deg F', deg G'\} = \{deg F, deg G\}$.

Moreover, Theorem 2 shows we may as well assume D is irreducible, given by

$$D = \frac{\partial(F, G, \cdot)}{\partial(X, Y, Z)}.$$

Under such assumption, observe that $e_1 \le e_2$ and $e_1 + e_2 = d + 3$, where d is the degree of D.

Proposition 1. *If p is a prime integer dividing both e_1 and e_2, then for every homogeneous polynomial $H \neq 0$,*

$$3v_D(H) \equiv deg H \pmod{p}.$$

Proof. Since $e_1 + e_2 = d + 3$, $d \equiv -3 \pmod{p}$. Given H, if $s = v_D(H)$, then

$$D^s H \in Ker(D) = k[F,G] \Longrightarrow p \text{ divides } deg D^s H.$$

But by homogeneity,

$$deg D^s H = deg H + sd \Longrightarrow 0 \equiv deg H - 3s \pmod{p}. \qquad □$$

In the case $p = 3$ above, the proposition implies $3\nu_D(L) \equiv 1 \pmod 3$ for every non-zero $L \in V_1$, which is absurd. We thus obtain the following.

Corollary 1. *For any pair of positive integers $e_1 \le e_2$ such that $e_1 \equiv e_2 \equiv 0$ modulo 3, the (e_1, e_2) case cannot occur.*

Note next that, for any $n \in \mathbb{Z}$, $W_n^i = \{ f \in V_i \mid \nu_D(f) \le n \}$ is a vector subspace of V_i, and that $W_m^i \subseteq W_n^i$ when $m \le n$. Since $dim\, V_1 = 3$, there exist non-negative integers u, v, w such that

$$u \le v \le w \quad \text{and} \quad \{ \nu_D(L) \mid L \in (V_1 - 0) \} = \{u, v, w\}.$$

Observe that, by (P1) and (P4), $\nu_D(\cdot)$ is bounded on V_i by iw. Note further that $rank(D) < 3$ iff $u = 0$. There exists a basis $\{X', Y', Z'\}$ of V_1 for which

$$\nu_D(X') = u, \qquad \nu_D(Y') = v, \qquad \nu_D(Z') = w.$$

So we may as well assume $u = \nu_D(X)$, $v = \nu_D(Y)$, and $w = \nu_D(Z)$.

Proposition 2. *If $d = -1$ or $d = 0$, then $u = 0$.*

Proof. If $d = 0$, D: $V_1 \to V_1$ is a nilpotent linear map, and there exists $L \in (V_1 - 0)$ with $DL = 0$. If $d = -1$, then $DX, DY \in k$, and $D(X \cdot DY - Y \cdot DX) = 0$. □

Proposition 3. *If k is algebraically closed and $u \neq 0$, then either $u = v = w$ or $u < v < w$.*

Proof. Since $u \neq 0$, we may assume $d \geq 1$. Consider the case u, v, w are not all equal, so that $u < w$. Let $\{L, M\}$ be any basis of $V_1 \cap k[Y, Z]$. If $DX = aM^t$ for some $a \in k[X, L]$ $(t \geq 0)$, then some non-zero element of V_1 divides DX. But if $DX \in (L')$ for $L' \in (V_1 - 0)$, then by (P1) and (P2),

$$u - 1 = \nu_D(DX) \geq \nu_D(L') \geq u,$$

a contradiction. Thus, if we write

$$DX = \sum_{s=0}^{d+1} a_s M^{d+1-s} \quad (a_s \in V_s \cap k[X, L]),$$

then $a_s \neq 0$ for at least two distinct values of s. By (P5), we see that there exists a distinct pair s_1, s_2 such that $\nu_D(a_{s_1} M^{d+1-s_1}) = \nu_D(a_{s_2} M^{d+1-s_2})$.

If $\nu_D(L) = u$, then for every non-zero a_{s_i}, $\nu_D(a_{s_i}) = s_i u$, since a_{s_i} factors as a product of linears in X and L. Thus,

$$s_1 u + (d + 1 - s_1)\nu_D(M) = s_2 u + (d + 1 - s_2)\nu_D(M) \Longrightarrow u = \nu_D(M).$$

But then W_u^1 contains a basis of V_1, i.e., $W_u^1 = V_1$, contradicting the assumption that $u < w$. Therefore, $u < \nu_D(L)$ for *every* non-zero $L \in V_1 \cap k[Y, Z]$, i.e., $\nu_D(L) \geq v > u$.

Next, write $DX = XQ + T$ for $Q \in V_d$ and $T \in V_{d+1} \cap k[Y, Z]$. If $T = 0$, then $DX \in (X)$, which implies $u = 0$, a contradiction. Thus, $T \neq 0$, and T factors as a product of linears in Y and Z. It follows from (P1) and the preceding paragraph that $\nu_D(T) \geq (d+1)v$.

Since $Q \in V_d$, $\nu_D(XQ) = u + \nu_D(Q) \leq u + dw < (d+1)w$. If $v = w$, then $\nu_D(T) \geq (d+1)v = (d+1)w > \nu_D(XQ)$, which by (P5) implies $\nu_D(DX) = \nu_D(T)$. But this is absurd, since then $u - 1 \geq (d+1)v = (d+1)w > (d+1)u \geq 2u$. Therefore, $v < w$. □

4.2. Fixed points

Since $e_1 = \deg F \geq 1$ and $e_2 = \deg G \geq 1$, and since F and G are homogeneous, F and G vanish along some line ℓ in A^3 containing the origin. (If $\pi: \operatorname{Spec} B \to \operatorname{Spec} A$ is the morphism induced by the inclusion $A \hookrightarrow B$, then ℓ is contained in the fiber of π lying over the origin of $\operatorname{Spec} A$.) Suppose ℓ is defined by the ideal $I = (x, y)$ for $x, y \in V_1$, so that $F, G \in I$. Direct computation then shows $Dx, Dy \in I$, i.e., I is an integral ideal for D. By Lemma 1, if $\{x, y, z\}$ is a basis for V_1, then $Dz = a + f$ for $a \in k$ and $f \in I$; and by homogeneity, $a = 0$. Thus, $Dz \in I$. It follows that $DB \subset I$, and ℓ is a line of fixed points for the action on A^3 induced by D.

Lemma 3. *In the notation of the preceding paragraph, x and y may be chosen so that $F, G \in (x, y^2)$, and (x, y^2) is an integral ideal for D.*

Proof. Write

$$\begin{pmatrix} F \\ G \end{pmatrix} = N \begin{pmatrix} x \\ y \end{pmatrix}, \qquad N \in \mathcal{M}_2(B).$$

Since $Dz \in I$, $\det N \in I$, so that if

$$N = \begin{pmatrix} f_1 + az^m & f_2 + bz^m \\ g_1 + cz^n & g_2 + dz^n \end{pmatrix} \quad \text{for } f_1, f_2, g_1, g_2 \in I; \quad a, b, c, d \in k,$$

then

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = 0.$$

Multiplying G by an appropriate constant, we may assume $a = c$ and $b = d$:

$$\begin{pmatrix} F \\ G \end{pmatrix} = M \begin{pmatrix} x \\ y \end{pmatrix} + (ax + by) \begin{pmatrix} z^m \\ z^n \end{pmatrix},$$

where $M \in \mathcal{M}_2(I)$ and $a, b \in k$.

If $(ax + by) \neq 0$, let $x' = (ax + by)$ and $y' = (a'x + b'y)$, where $ab' - ba' \neq 0$. Replacing x, y by x', y' preserves I, so we may assume $b = 0$ in this case.

In the case $(ax + by) = 0$, b must again equal 0. So in either case, we may assume

$$\begin{pmatrix} F \\ G \end{pmatrix} = M \begin{pmatrix} x \\ y \end{pmatrix} + ax \begin{pmatrix} z^m \\ z^n \end{pmatrix} \quad (M \in \mathcal{M}_2(I), a \in k). \tag{$*$}$$

This shows $F, G \in (x, y^2)$. Since $(\partial/\partial z)$ preserves this ideal, $F_z, G_z \in (x, y^2)$ as well. Therefore, $Dx = F_yG_z - F_zG_y$ and $Dy = F_zG_x - F_xG_z$ lie in (x, y^2), and (x, y^2) is an integral ideal for D. □

5. The (2, d + 1) case

The main purpose of this section is to prove the following.

Theorem 4. *Let k be an algebraically closed field of characteristic 0, and let $B = k^{[3]}$.*

1. *There exist no homogeneous locally nilpotent derivations of B of type $(2, d+1)$ if $d = 1, 2$, or 3.*
2. *Let D be a homogeneous locally nilpotent derivation on $B = k^{[3]}$ of type $(2, d+1)$, with $d \geq 4$. Then there exists a basis X, Y, Z of V_1, together with polynomials $F \in V_2$ and $G \in V_{d+1}$, such that $Ker(D) = k[F, G]$, where for some $a_1 \in V_{d-3}$, a_2, $a_3 \in V_{d-4}$, and $a_4 \in k$,*

$$F = XZ + Y^2 \quad \textit{and} \quad G = a_1F^2 + a_2X^3F + a_3X^2YF + a_4X^{d+1}.$$

Moreover, $v_D(X) < v_D(Y) < v_D(Z)$.

Proof. Suppose D is a homogeneous locally nilpotent derivation of B of type $(2, d+1)$ for $d \geq 1$. Note that $Ker(D) = k[F, G]$ clearly contains no linear form, so $u > 0$ in this case. By the proof of Lemma 3, we may assume F and G have the form $(*)$ above. In particular, $F \in (x, y^2)$ for some $x, y \in V_1$, and we can write $F = xL + ty^2$ for some $t \in k^*$ and some $L \in V_1$ having $L_z \neq 0$. Replacing L by z, and y by an appropriate multiple of y, we may assume $F = xz + y^2$. (Such replacement does not affect the form $(*)$ above, and it preserves the ideal (x, y^2).) Since $G \in (x, y^2) = (x, F)$, we conclude, by Lemma 2, that $Dx \in (F)$.

If $S = k[x,z] \oplus k[x,z] \cdot y$, then $B = (F) \oplus S$. Hence, there exist $P \in B$ and $Q \in S$ for which $G = PF + Q$. Direct computation shows

$$0 \equiv Dx \equiv 2yQ_z - xQ_y \pmod{F}.$$

Since $deg_y(2yQ_z - xQ_y) \leq 2$, we conclude $(2yQ_z - xQ_y) = Fh$ for some $h \in k[x,z]$.

If $Q = p + qy$ for $p, q \in k[x,z]$, then

$$(2yQ_z - xQ_y) = (2q_z)y^2 + (2p_z)y - (xq) = (2q_z)F + (2p_z)y - (2xzq_z + xq).$$

Therefore, $p_z = 2zq_z + q = 0$. By homogeneity, this forces $q = 0$ (easy exercise using Euler's lemma), and $p = mx^{d+1}$ for some $m \in k$. Since G is irreducible, $m \neq 0$, so rescaling G if necessary, we obtain $G = PF + x^{d+1}$. It follows that $D = F \cdot D_1 + \lambda \cdot D_2$, where $\lambda = (d+1)x^d$, and D_1 and D_2 are the k-derivations of B defined by

$$D_1 = \frac{\partial(F, P, \cdot)}{\partial(x, y, z)}, \qquad D_2 = \frac{\partial(F, x, \cdot)}{\partial(x, y, z)}.$$

If $d = 1$, then $P \in k$ and $Ker(D) = k[F, PF + x^2] = k[F, x^2]$, which is impossible, since this ring is not factorially closed. Thus, $d \geq 2$.

Since $P \in V_{d-1}$, $\nu_D(P) \leq (d-1)w$, and since $\nu_D(G) = 0$, (P5) implies $\nu_D(PF) = \nu_D(x^{d+1})$. Therefore,

$$(d-1)w \geq \nu_D(P) = \nu_D(PF) = \nu_D(x^{d+1}) = (d+1)\nu_D(x) \Longrightarrow \nu_D(x) < w.$$

So the "$u = v = w$" case cannot occur when $e_1 = 2$, and by Propostion 3, $u < v < w$ in this case. Moreover, since $\nu_D(xz + y^2) = 0$, $\nu_D(x) + \nu_D(z) = 2\nu_D(y)$. From this, it follows readily that $u = \nu_D(x)$, $v = \nu_D(y)$, and $w = \nu_D(z)$.

Consider the case $d = 2$: since $Dx \in (F)$, $Dx = LF$ for some non-zero $L \in V_1$. But then $u - 1 = \nu_D(L) \geq u$, which is absurd. Thus, $d \geq 3$.

Consider the case $d = 3$: since $Dx \in (F)$, $Dx = QF$ for some non-zero $Q \in V_2$, and therefore $\nu_D(Q) = u - 1$. Consider the following subspaces of V_2:

$$0 \subset W_0^2 \subset W_{2u}^2 \subset W_{u+v}^2 \subset W_{2v}^2 \subset W_{v+w}^2 \subset W_{2w}^2 = V_2.$$

Since every inclusion is proper, and since $dim\, V_2 = 6$, there are only six exponents possible for elements of $(V_2 - 0)$. Since $u - 1 < 2u$, we conclude that $Q \in W_0 = k \cdot F$. But then $Dx = F \cdot D_1 x \in (F^2)$, which implies $F \in \mathrm{im}\,(D_2)$ (since $D_1 x = -D_2 P$), and it is easy to show that F is not in the image of D_2. Therefore $d \geq 4$.

Finally, if we write $P = HF + (\alpha + \beta y)$ for $\alpha, \beta \in k[x, z]$, then x^3 divides α, and x^2 divides β. To see this, consider $D_2 P$, which has exponent $u - 1$ since $DP = \lambda D_2 P$. Direct computation yields $D_2 P = f + g - h$, where $f = F(xH_y - 2yH_z - 2\beta_z)$, $g = x(\beta + 2z\beta_z)$, and $h = (2\alpha_z)y$.

If $\alpha \notin (x)$, then $h \notin (x)$ as well, and $\nu_D(h) = v + (d-2)w$. Since $\nu_D(f) \leq (d-3)w$ and $\nu_D(g) \leq u + (d-2)w$, this would imply $\nu_D(D_2 P) = \nu_D(h)$, a contradiction. Thus, $\alpha \in (x)$, and $\nu_D(h) \leq u + v + (d-3)w$.

If $\beta \notin (x)$, then $g \notin (x)$, and $\nu_D(g) = u + (d-2)w$. But this would exceed the exponent of both f and h, implying $\nu_D(D_2 P) = \nu_D(g)$, a contradiction. Thus, $\beta \in (x)$, and $\nu_D(g) = ju + (d-1-j)w$ for some $j \geq 2$.

If $\alpha \notin (x^3)$, then $h \notin (x^3)$ as well, and $\nu_D(h) = iu + v + (d-2-i)w$ for $i = 1$ or 2. This dominates $\nu_D(f)$, and we conclude that $\nu_D(g) = \nu_D(h)$: otherwise $\nu_D(D_2 P)$ equals either $\nu_D(g)$ or $\nu_D(h)$, which is impossible. We therefore have

$$iu + v + (d-2-i)w = ju + (d-1-j)w.$$

Since $u + w = 2v$, this implies $(2i - 2j + 1)(u - w) = 0$. But this is impossible, since $u < w$, and the equation $2i - 2j + 1 = 0$ has no integral solutions. Therefore $\alpha \in (x^3)$.

If $\beta \notin (x^2)$, then $\nu_D(g) = 2u + (d-3)w$, which would dominate $\nu_D(f)$ and $\nu_D(h)$. But then $\nu_D(D_2 P) = \nu_D(g)$, a contradiction. Therefore, $\beta \in (x^2)$, completing the proof. □

Corollary 2. *If k is an algebraically closed field of characteristic 0, and if D is a homogeneous locally nilpotent k-derivation of $B = k^{[3]}$ of degree at most 3, then $rank(D) \leq 2$.*

Proof. Let d be the degree of D. It suffices to assume D is irreducible and non-zero. If $d = -1$ or $d = 0$, then Proposition 2 implies the desired result, so assume $1 \leq d \leq 3$. Since $d = e_1 + e_2 - 3$, D must be of one of the following types:

$$(1, 3),\ (1, 4),\ (1, 5),\ (2, 2),\ (2, 3),\ (2, 4),\ (3, 3).$$

The first three have a linear form in the kernel, and hence are of rank at most 2. The next three are shown by Theorem 4 to be impossible, and the last type is shown by Corollary 1 to be impossible. □

A final related question is the following.

Question. For $F = XZ + Y^2$, can we classify all locally nilpotent derivations of the domain $B \bmod (F)$? Equivalently, what are the algebraic $\boldsymbol{G}_a$-actions on the affine surface defined by F?

For certain other polynomials, this question is answered by a recent theorem of Miyanishi.

Theorem 5 (Miyanishi [8, Theorem 2.6]). *Let K be an algebraically closed field of characteristic 0, and suppose R is a factorial K-domain of dimension 2, finitely generated over K, with $R^* = K^*$. The following are equivalent.*

1. *There exists a non-zero locally nilpotent K-derivation on R.*
2. $R \cong K^{[2]}$.

For example, if $f = (X^2 + Y^3 + Z^5)$, the ring $\bar{B} = B \bmod (f)$ admits no non-zero k-derivation which is locally nilpotent, since $\bar{B}$ is factorial, and the surface defined by f is clearly not an $\boldsymbol{A}^2$. In particular, f lies in the kernel of no non-zero locally nilpotent derivation of B. (It should be noted that there exist non-algebraically closed fields for which this theorem fails.)

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