ELSEVIER Journal of Pure and Applied Algebra 119 (1997) 297–317

JOURNAL OF PURE AND APPLIED ALGEBRA

Actions on 2-complexes and the homotopical invariant Σ^2 of a group

Holger Meinert

Fachbereich Mathematik, Johann Wolfgang Goethe–Universität, Robert-Mayer-Str. 6–10, D-60054 Frankfurt a.M., Germany

Communicated by A. Dold; received 24 February 1995

Abstract

Given a group G acting on a 2-complex X we construct, depending on several choices, a 2-complex Y with free G-action together with a G-equivariant map $p : Y \twoheadrightarrow X$. From this we deduce two criteria for the computation of the homotopical invariant $\Sigma^2(G)$ introduced by B. Renz (Geometric invariants and HNN-extensions, in: K.N. Cheng and Y.K. Leong, Eds., Group Theory (de Gruyter, Berlin, 1989) 465–484). These results are used to complete the proof of the Σ^m-conjecture for metabelian groups of finite Prüfer rank. © 1997 Elsevier Science B.V.

1991 Math. Subj. Class.: 20F32, 57M60

1. Introduction

The observations made in [26] are the main motivation for the present paper. There we proved that the higher homological invariants $\Sigma^m(G;\mathbb{Z})$, defined by Bieri and Renz [7] for all $m \in \mathbb{N}$ (see Section 6), are completely determined by $\Sigma^1(G;\mathbb{Z})$ if G is a metabelian group of finite Prüfer rank (i.e. has a uniform bound on the number of generators needed to generate any finitely generated subgroup). We also gave a formula for the higher invariants in terms of $\Sigma^1(G;\mathbb{Z})$, and thus established the homological part of the so-called Σ^m-conjecture for metabelian groups G of finite Prüfer rank. However, the question whether $\Sigma^2(G;\mathbb{Z})$ coincides with the homotopical geometric invariant $\Sigma^2(G)$ defined by Renz (see below and [29, 30]) remained open. The aim of this article is to answer this question in the affirmative. We will achieve our objective by proving counterparts for the results obtained in [26] for the homological invariant $\Sigma^2(G;\mathbb{Z})$ – and then the proof in [26] goes through without changes.

The core of this article is a result on group actions on 2-dimensional CW-complexes ("2-complexes") which seems to be of independent interest. Given a cell-permuting

PII S0022-4049(96)00030-8

action of a group G on a 2-complex X we construct 2-complexes Y where G acts freely together with G-equivariant cellular maps $p: Y \twoheadrightarrow X$ mapping open cells onto open cells. Moreover, if X is finite mod G, if all stabilizers of vertices of X are finitely presented and all edge stabilizers are finitely generated, then Y can be chosen to be G-finite. Another important feature of p is that it induces isomorphisms $\pi_i(Z) \rightarrowtail\!\!\!\!\twoheadrightarrow \pi_i(p(Z))$ in homotopy for $i = 0, 1$ and any subcomplex $Z \subseteq Y$ provided that the fibres in Z of vertices of $p(Z)$ are 1-connected and that the fibres of "barycentres" of edges of $p(Z)$ are connected. In particular, this holds if Z is the full pre-image of a subcomplex of X because full pre-images of vertices of X will be homeomorphic to Cayley complexes of the vertex stabilizers and full pre-images of "barycentres" of edges will be homeomorphic to Cayley graphs of the edge stabilizers. All this is done in Section 3. For our application we have to be rather explicit. Similar, though less explicit, constructions have been given by Abels–Holz [1], Corson [13, 15], Geoghegan [19] and Haefliger [21].

We will apply this construction in Section 4 to deduce two criteria for the computation of the homotopical invariant $\Sigma^2(G)$ of a group G. This invariant is a conical subset of the real vector space $V(G) = \mathrm{Hom}(G; \mathbb{R}_{\mathrm{add}})$. A homomorphism $\chi : G \to \mathbb{R}$ belongs to the invariant $\Sigma^1(G)$ if there exists a finite generating system $\mathscr{X}$ of G so that the full subgraph of the Cayley graph $\Gamma(G; \mathscr{X})$, generated by all vertices in $G_\chi = \{g \in G \mid \chi(g) \geq 0\}$, is connected. The invariant $\Sigma^2(G)$ is defined to be the set of all $\chi \in V(G)$ for which there exists a finite presentation $\langle \mathscr{X} \mid \mathscr{R} \rangle$ of the group G so that the full subcomplex of the Cayley complex $C(\mathscr{X}; \mathscr{R})$, spanned by G_χ, is 1-connected. Then $\Sigma^2(G) \subseteq \Sigma^1(G) \subseteq V(G)$. These invariants have the remarkable property that they characterize all finitely presented, respectively all finitely generated, normal subgroups of G with abelian quotient (see [5, 7, 11, 30]).

In a sense, $\Sigma^2(G)$ has been defined using a free action of G on a 1-connected G-finite 2-complex, the Cayley complex. Now it turns out that it is possible to use non-free actions to "compute" $\Sigma^2(G)$.

Theorem A. *Suppose a group G acts on a 1-connected G-finite 2-complex X such that all vertex stabilizers are finitely presented and all edge stabilizers are finitely generated. Let $\chi : G \to \mathbb{R}$ be a homomorphism and assume that $h : X \to \mathbb{R}$ is a continuous map satisfying $h(gx) = \chi(g) + h(x)$ for all $g \in G$ and $x \in X$. Then $\chi \in \Sigma^2(G)$ if and only if there is a constant $d \geq 0$ such that the maps $\pi_i(h^{-1}([0, \infty))) \to \pi_i(h^{-1}([-d, \infty)))$ in homotopy, induced by inclusion, are trivial for $i = 0, 1$.*

To prove this theorem we use our construction of Section 3 and the result of Renz [30] that Theorem A is true if the action is free. It is not difficult to see that "χ-equivariant height functions" h as above always exist if all cell stabilizers are contained in the kernel of χ and fix cells pointwise, in particular, if G acts freely.

Making use of the flexibility of our construction in Section 3 we then prove a second criterion which is, in some sense, complementary to that given in Theorem A. Here we put $\Sigma^0(G) = V(G)$.

Theorem B. *Let a group G act on a 1-connected G-finite 2-complex X. If $\chi : G \to \mathbb{R}$ is a homomorphism such that $0 \neq \chi|_{G_\sigma} \in \Sigma^{2-\dim\sigma}(G_\sigma)$ for all cells σ of X, then $\chi \in \Sigma^2(G)$.*

There is an intimate connection between the geometric invariants and finiteness properties of metabelian groups, the starting point of the Σ-theory being the characterization of the finitely presented metabelian groups by Bieri and Strebel [8] in terms of Σ^1. For these groups it was known that $\operatorname{conv}_{\leq 2} \Sigma^1(G)^c \subseteq \Sigma^2(G)^c$ [17, 18]. Here the superscript "c" denotes the complement in $V(G)$, and $\operatorname{conv}_{\leq m} \Sigma^1(G)^c$ is the union of the convex hulls of all subsets of $\Sigma^1(G)^c$ with at most m elements. All the results above are needed to prove

Theorem C. *If G is a finitely presented metabelian group of finite Prüfer rank, then*

$$\operatorname*{conv}_{\leq 2} \Sigma^1(G)^c = \Sigma^2(G)^c.$$

As a consequence we obtain a positive answer for the question at the beginning: the homotopical invariant $\Sigma^2(G)$ coincides with the homological invariant $\Sigma^2(G;\mathbb{Z})$ whenever G is a finitely presented metabelian group of finite Prüfer rank. And combined with [26], Theorem C confirms the Σ^m-conjecture for all metabelian groups of finite Prüfer rank, one of the main results of the author's thesis [24]. In Section 6 we will give a definition of the homological invariants and a short summary of what has been proved.

1.1. Notation

By a *graph* (*2-complex*) we shall mean a 1-dimensional (2-dimensional) CW-complex. An *oriented CW-complex* is a CW-complex together with choices of characteristic maps for all cells. By a *cell* we always mean a closed cell, and we speak of *vertices* and *edges* instead of 0- and 1-cells. A subcomplex Y of a CW-complex X is *full* if Y is the largest subcomplex of X with vertex set Y^0.

A map between CW-complexes will be called *regular* if it is continuous, cellular and maps open cells onto open cells. By an *action of a group G on a CW-complex X* (X will be called a *G-complex*), we shall mean an action by regular homeomorphisms. We say that X is *G-finite*, or *finite* mod G, if there is only a finite number of orbits of cells of X.

2. The homotopical geometric invariants $\Sigma^m(G)$

In this section we introduce the homotopical geometric invariants $\Sigma^m(G)$ of a group G for all $m \in \mathbb{N}_0$. They form a descending chain of conical subsets of the real vector space $V(G) = \operatorname{Hom}(G;\mathbb{R})$. Although we will only use the invariants $\Sigma^1(G)$ and $\Sigma^2(G)$

later on, the general case is included here without extra effort. We will not give all details but refer to [11, 24, 26, 29, 30] for more information.

Throughout this section G will always denote a group and $\chi : G \to \mathbb{R}$ a homomorphism. For any real number λ we put $G_\chi^{[\lambda,\infty)} = \{g \in G \mid \chi(g) \geq \lambda\}$.

2.1. First definition of $\Sigma^1(G)$ and $\Sigma^2(G)$. Let $\mathscr{X}$ be a generating system of G. Recall that the *Cayley graph* $\Gamma(G;\mathscr{X})$ is the oriented graph with vertex set G and edge set $G \times \mathscr{X}$. The initial and terminal point of an edge (g,x) are given by g and gx, respectively. The full subgraph of $\Gamma(G;\mathscr{X})$ generated by all vertices in $G_\chi^{[\lambda,\infty)}$ will be denoted by $\Gamma(G;\mathscr{X})_\chi^{[\lambda,\infty)}$. Following Renz [29, 30] we define:

$$\chi \in \Sigma^1(G) \iff \textit{There is a finite generating system } \mathscr{X} \textit{ of } G \textit{ such that } \Gamma(G;\mathscr{X})_\chi^{[0,\infty)} \textit{ is connected.}$$

This does not depend upon the choice of a finite generating system $\mathscr{X}$ of G (see [29]), and $\Gamma(G;\mathscr{X})_\chi^{[\lambda,\infty)}$ is connected for all $\lambda \in \mathbb{R}$ if $\chi \neq 0$ and if $\Gamma(G;\mathscr{X})_\chi^{[0,\infty)}$ is connected. We remark that $\Sigma^1(G)$ coincides with the Bieri–Neumann–Strebel invariant $\Sigma_{G'}(G)$ introduced in [6] up to sign (see [7, Section 5]).

Let us define $\Sigma^2(G) \subseteq \Sigma^1(G)$ now. First, recall the definition of the *Cayley complex* $C(\mathscr{X};\mathscr{R})$ with respect to a presentation $\langle \mathscr{X} \mid \mathscr{R} \rangle$ of G. This is the 2-complex whose 1-skeleton is just the Cayley graph $\Gamma(G;\mathscr{X})$ and whose set of 2-cells is given by $G \times \mathscr{R}$. The 2-cell corresponding to $(g,r) \in G \times \mathscr{R}$ is attached along the closed edge path in $\Gamma(G;\mathscr{X})$ starting at the vertex g and spelling out the word $r \in \mathscr{R}$ (see [23, III.4]). By $C(\mathscr{X};\mathscr{R})_\chi^{[\lambda,\infty)}$ we denote the full subcomplex of $C(\mathscr{X};\mathscr{R})$ with vertex set $G_\chi^{[\lambda,\infty)}$. Then we put:

$$\chi \in \Sigma^2(G) \iff \textit{There is a finite presentation } \langle \mathscr{X} \mid \mathscr{R} \rangle \textit{ of } G \textit{ such that } C(\mathscr{X};\mathscr{R})_\chi^{[0,\infty)} \textit{ is 1-connected.}$$

The property of $C(\mathscr{X};\mathscr{R})_\chi^{[0,\infty)}$ of being 1-connected will, in general, depend on the chosen presentation. Below we will see a weaker property that characterizes $\Sigma^2(G)$ and is independent of the choice of a finite presentation.

The definition of the higher homotopical invariants $\Sigma^m(G)$ requires the concept of a "height function" which will be explained now. It plays a central role in the following.

2.2. Height functions. Let X be a G-complex. A *χ-equivariant height function on X* is a continuous map $h = h_\chi : X \to \mathbb{R}$ with the property that $h(gx) = \chi(g) + h(x)$ for all $g \in G$, $x \in X$. It is *regular* if $h(X^0) \subseteq \chi(G)$ and if h attains its minimum and maximum on each cell on the boundary of this cell. In other words, this is a valuation in the sense of [29, 30]. It is not difficult to see that regular height functions always exist if all cell stabilizers are contained in the kernel of χ and fix cells pointwise (see [26]), in particular, if G acts freely.

Given a height function $h = h_\chi : X \to \mathbb{R}$ and a real number λ, the maximal subcomplex of X contained in $h^{-1}([\lambda,\infty))$ will be called a *valuation subcomplex* and denoted by $X_h^{[\lambda,\infty)}$. If h is regular then $X_h^{[\lambda,\infty)}$ is just the full subcomplex of X generated by all vertices with h-value greater or equal to λ. In particular, if h is a regular height function on the Cayley complex $X = C(\mathscr{X};\mathscr{R})$ with respect to some presentation of G then $X_h^{[\lambda+\mu,\infty)} = C(\mathscr{X};\mathscr{R})_\chi^{[\lambda,\infty)}$ for all $\lambda \in \mathbb{R}$, where μ is the h-value of the vertex $1 \in G$.

Let $h = h_\chi : X \to \mathbb{R}$ be a height function. We say that $X_h^{[\lambda,\infty)}$ is *essentially k-connected in X*, for some $k \geq -1$, if there is a real number $d \geq 0$ so that the map

$$\pi_i(X_h^{[\lambda,\infty)}, *) \to \pi_i(X_h^{[\lambda-d,\infty)}, *)$$

in homotopy, induced by inclusion, is trivial for $i \leq k$ (the condition is empty for $k = -1$) and any choice of a basepoint $* \in X_h^{[\lambda,\infty)}$.

In all applications X will have finite m-skeleton mod G. So let us assume this. Choose a finite subcomplex $F \subseteq X^m$ such that $GF = X^m$ and put $c = \max h|_F - \min h|_F$. Then we find $(X^m)_h^{[\lambda,\infty)} \subseteq h^{-1}([\lambda,\infty)) \cap X^m \subseteq (X^m)_h^{[\lambda-c,\infty)}$ for all $\lambda \in \mathbb{R}$. Now the situation where $\chi = 0$ is easy to understand: X^m is contained in $X_h^{[\lambda,\infty)}$ for all sufficiently small $\lambda \in \mathbb{R}$. So let $\chi \neq 0$. Using the G-action it is not difficult to see that the following assertions are equivalent (cf. [26]):

(i) $X_h^{[\lambda,\infty)}$ is essentially $(m-1)$-connected in X for some λ.

(ii) $X_h^{[\lambda,\infty)}$ is essentially $(m-1)$-connected in X for all $\lambda \in \mathbb{R}$, with a uniform choice of d.

(iii) There is a $d \geq 0$ such that for some (or, equivalently, for all) $\lambda \in \mathbb{R}$ the maps $\pi_i(h^{-1}([\lambda,\infty)) \cap X^m) \to \pi_i(h^{-1}([\lambda-d,\infty)) \cap X^m)$ in homotopy, induced by inclusion, are trivial for all $i < m$.

Moreover, as height functions are completely determined on X^m by their values on the compact subcomplex F it follows that the conditions (i)–(iii) do not depend on the choice of the particular height function h. More details can be found in [4] and [26].

The classical approach to group cohomology uses Eilenberg–MacLane spaces to define new algebraic objects. The definition of the homotopical invariants is modelled on this approach. The next step is to study height functions on universal covers of $K(G,1)$-complexes with finite m-skeleton. It turns out that essential connectivity properties of the corresponding valuation subcomplexes are, in fact, group properties. This is the content of the following slightly more general result.

2.3. Theorem. *Let $\pi : G \twoheadrightarrow Q$ be a split epimorphism of groups and $\chi = \psi \circ \pi$ for some homomorphism $\psi : Q \to \mathbb{R}$. Let K be a $K(G,1)$-complex and L a $K(Q,1)$-complex, both with finite m-skeleton. Suppose $h = h_\chi : \tilde{K} \to \mathbb{R}$ and $H = H_\psi : \tilde{L} \to \mathbb{R}$ are χ- and ψ-equivariant height functions on the universal covering complexes, respectively. If*

$\tilde{K}_h^{[0,\infty)}$ is essentially $(m-1)$-connected in $\tilde{K}$ then $\tilde{L}_H^{[0,\infty)}$ is essentially $(m-1)$-connected in $\tilde{L}$.

Proof. If $\psi = 0$ or, equivalently, $\chi = 0$ then $\tilde{K}_h^{[\lambda,\infty)}$ and $\tilde{L}_H^{[\lambda,\infty)}$ contain the whole m-skeleta $\tilde{K}^m$ and $\tilde{L}^m$, respectively, for some $\lambda \le 0$. Thus both, $\tilde{K}_h^{[0,\infty)}$ and $\tilde{L}_H^{[0,\infty)}$, are essentially $(m-1)$-connected, and we will now assume that $\psi \neq 0$.

Let $\iota : Q \rightarrowtail G$ be a homomorphism such that $\pi \circ \iota = \mathrm{Id}_Q$. Then π and ι induce π- and ι-equivariant cellular maps $\tilde{\pi} : \tilde{K} \to \tilde{L}$ and $\tilde{\iota} : \tilde{L} \to \tilde{K}$, respectively, such that $\tilde{\pi} \circ \tilde{\iota}$ and $\mathrm{Id}_{\tilde{L}}$ are homotopic via a Q-equivariant cellular homotopy $\sigma : \tilde{L} \times I \to \tilde{L}$. Then $H \circ \tilde{\pi} : \tilde{K} \to \mathbb{R}$ is also a χ-equivariant height function on $\tilde{K}$. By the remarks above, $\tilde{K}_h^{[0,\infty)}$ is essentially $(m-1)$-connected in $\tilde{K}$ if and only if $\tilde{K}_{H\tilde{\pi}}^{[0,\infty)}$ has this property. Thus we may assume that $h = H \circ \tilde{\pi}$.

Since $\tilde{L}^m$ is Q-finite and σ is Q-equivariant in the first argument, there is a constant $a_1 \ge 0$ such that $|H(\sigma(y,t)) - H(y)| \le a_1$ for all $y \in \tilde{L}^m$ and all $t \in I$. Moreover, the image $\sigma(\tilde{L}^m, I)$ is contained in a Q-finite subcomplex M of $\tilde{L}^{m+1}$. It follows that $H^{-1}([\lambda,\infty)) \cap M \subseteq (\tilde{L}^{m+1})_H^{[\lambda-a_2,\infty)}$ for all $\lambda \in \mathbb{R}$ and some real number $a_2 \ge 0$. Putting $a = a_1 + a_2$ we see that $(\tilde{L}^m)_H^{[0,\infty)} \times I$ is mapped by σ into $(\tilde{L}^{m+1})_H^{[-a,\infty)}$.

Now, $h \circ \tilde{\iota} : \tilde{L} \to \mathbb{R}$ is also a ψ-equivariant height function. Therefore the difference $|h\tilde{\iota}(\cdot) - H(\cdot)|$ is bounded on $\tilde{L}^m$ and $\tilde{\iota}((\tilde{L}^m)_H^{[0,\infty)}) \subseteq (\tilde{K}^m)_h^{[-b,\infty)}$ for some $b \ge 0$. Now our assumptions imply that there is a real number $d \ge 0$ such that the inclusion $j : (\tilde{K}^m)_h^{[-b,\infty)} \rightarrowtail (\tilde{K}^m)_h^{[-b-d,\infty)}$ induces the trivial maps $\pi_i(j)$ in homotopy for all $i < m$. Finally, notice that $\tilde{\pi}((\tilde{K}^m)_h^{[-b-d,\infty)}) \subseteq (\tilde{L}^m)_H^{[-c-b-d,\infty)}$, where $c \ge 0$.

Let δ be the maximum of a and $b+c+d$. Then $\tilde{\pi} \circ j \circ \tilde{\iota}$ and the inclusion are homotopic via σ when considered as maps $(\tilde{L}^m)_H^{[0,\infty)} \to (\tilde{L}^{m+1})_H^{[-\delta,\infty)}$. But by construction $\tilde{\pi} \circ j \circ \tilde{\iota}$ induces the trivial maps $\pi_i((\tilde{L}^m)_H^{[0,\infty)}) \to \pi_i((\tilde{L}^m)_H^{[-\delta,\infty)}) = \pi_i((\tilde{L}^{m+1})_H^{[-\delta,\infty)})$ in homotopy for $i < m$. Thus $\tilde{L}_H^{[0,\infty)}$ is essentially $(m-1)$-connected in $\tilde{L}$. □

2.4. Theorem. (Renz [30]) *Let K be a $K(G,1)$-complex with finite m-skeleton, $\chi \neq 0$ and $h : \tilde{K} \to \mathbb{R}$ a χ-equivariant height function. If $\tilde{K}_h^{[0,\infty)}$ is essentially $(m-1)$-connected in $\tilde{K}$ then a $K(G,1)$-complex M with finite m-skeleton and a χ-equivariant height function $H = H_\chi : \tilde{M} \to \mathbb{R}$ exist so that $\tilde{M}_H^{[\lambda,\infty)}$ is $(m-1)$-connected for all $\lambda \in \mathbb{R}$.*

Sketch of proof for $m \le 2$. If $m = 1$ it follows from Theorem 2.3 that $\Gamma(G;\mathscr{X})_\chi^{[0,\infty)}$ is essentially 0-connected in the Cayley graph $\Gamma(G;\mathscr{X})$ for every choice of a finite generating system $\mathscr{X}$ of G. Then it is not difficult to see that all $\Gamma(G;\mathscr{X})_\chi^{[\lambda,\infty)}$ are connected in this situation. And if $m = 2$ we invoke Theorem 2.3 once again. It yields that $C(\mathscr{X};\mathscr{R})_\chi^{[0,\infty)}$ is essentially 1-connected in the Cayley complex $C(\mathscr{X};\mathscr{R})$ with respect to any finite presentation $\langle \mathscr{X} \mid \mathscr{R} \rangle$ of G. Then Renz ([29, Lemma 3]) has shown how to add a finite set $\mathscr{R}'$ of relators such that $C(\mathscr{X};\mathscr{R} \cup \mathscr{R}')_\chi^{[0,\infty)}$ is 1-connected. In fact, his proof shows that $C(\mathscr{X};\mathscr{R} \cup \mathscr{R}')_\chi^{[\lambda,\infty)}$ is 1-connected for all $\lambda \in \mathbb{R}$. □

We now come to the definition of the homotopical invariants. The definitions of $\Sigma^1(G)$ and $\Sigma^2(G)$ suggest the following generalization.

2.5. Definition of $\Sigma^m(G)$. *The homomorphism $\chi : G \to \mathbb{R}$ belongs to $\Sigma^m(G)$ if and only if there is a $K(G,1)$-complex K with finite m-skeleton and a χ-equivariant height function $h = h_\chi : \tilde{K} \to \mathbb{R}$ on the universal covering complex such that $\tilde{K}_h^{[0,\infty)}$ is $(m-1)$-connected.*

Note that, by the discussion above, the two definitions of $\Sigma^1(G)$ and $\Sigma^2(G)$ are equivalent and that $\Sigma^m(G) \neq \emptyset \Leftrightarrow 0 \in \Sigma^m(G) \Leftrightarrow G$ has an Eilenberg–MacLane complex with finite m-skeleton. Moreover, we always have $\Sigma^0(G) = V(G)$.

One of the main features of these invariants is that they characterize precisely the normal subgroups $G' \trianglelefteq N \trianglelefteq G$ admitting a $K(N,1)$-complex with finite m-skeleton. For details see [7], [30], and for an application [25].

We close this section with some immediate corollaries.

2.6. Corollary. *Let K be a $K(G,1)$-complex with finite m-skeleton. If $h : \tilde{K} \to \mathbb{R}$ is a χ-equivariant height function, then $\chi \in \Sigma^m(G)$ if and only if $\tilde{K}_h^{[0,\infty)}$ is essentially $(m-1)$-connected in $\tilde{K}$.*

2.7. Corollary. *Let H be a subgroup in G of finite index. Then $\chi \in \Sigma^m(G)$ if and only if $\chi|_H \in \Sigma^m(H)$.*

Proof. Choose a $K(G,1)$-complex K with finite m-skeleton and a χ-equivariant height function $h = h_\chi : \tilde{K} \to \mathbb{R}$. Since H has finite index in G, $\tilde{K}$ is also the universal covering complex of a $K(H,1)$-complex with finite m-skeleton, and h is a $\chi|_H$-equivariant height function. Now the result follows from the preceding corollary. □

2.8. Corollary. *Let $\pi : G \twoheadrightarrow Q$ be a split epimorphism and assume that $\chi = \psi \circ \pi$ for some homomorphism $\psi : Q \to \mathbb{R}$. If $\chi \in \Sigma^m(G)$ then $\psi \in \Sigma^m(Q)$.*

Proof. Since $\chi \in \Sigma^m(G)$ there is a $K(G,1)$-complex with finite m-skeleton. As is well known, this implies that Q also admits a $K(Q,1)$-complex L with finite m-skeleton (see [28], Theorem 6). Choose a χ-equivariant height function $h : \tilde{K} \to \mathbb{R}$ and a ψ-equivariant height function $H : \tilde{L} \to \mathbb{R}$ and apply Theorem 2.3 and Corollary 2.6. □

Remark. In [24] we gave another, more general proof of the last result. First of all, we proved the analogue for the homological invariants $\Sigma^m(G;\mathbb{Z})$ by different methods (see [26]). Then we showed that if $N \rightarrowtail G \overset{\pi}{\twoheadrightarrow} Q$ is any exact sequence of groups with $\psi \circ \pi = \chi \in \Sigma^2(G) \cap \Sigma'_N(G)$ then $\psi \in \Sigma^2(Q)$. Here $\Sigma'_N(G)$ is the (weak) Bieri–Neumann–Strebel invariant (see [6, 11]). Finally, we considered *split extensions*. It turned out that $\chi \in \Sigma^1(G)$ if and only if $\psi \in \Sigma^1(Q) \cap \Sigma'_N(Q)$. But $\psi \in \Sigma'_N(Q)$ implies $\chi \in \Sigma'_N(G)$ in this situation. This proves Corollary 2.8 for $m \leq 2$.

3. The construction of free actions on 2-complexes

In the previous section we defined a new object, $\Sigma^2(G)$, by using, in a sense, a free action of a group on a 2-complex. But often one has nice actions with non-trivial cell stabilizers. Therefore we work out a method for constructing free actions from non-free ones.

Let G be a group acting on a 2-complex X. Depending on several choices, we define a 2-complex Y and a cellular map $p : Y \twoheadrightarrow X$. There is some flexibility in this construction, but the main features can be summarized as follows.

3.1. Theorem. *The group G acts freely on Y, and $p : Y \twoheadrightarrow X$ is a G-equivariant regular surjection. In particular, images and pre-images of subcomplexes are again subcomplexes.*

If X is finite mod G, *if stabilizers of vertices of X are finitely presented and if edge stabilizers are finitely generated, then Y can be chosen to be G-finite.*

Let Z be a subcomplex of Y, $W = p(Z)$ and $p| : Z \twoheadrightarrow W$ the restriction of p. If the pre-image $(p|)^{-1}(\widehat{v})$ of each vertex $\widehat{v}$ of W is 1-connected and if the fibre $(p|)^{-1}(b_{\widehat{e}})$ over the "barycentre" $b_{\widehat{e}}$ of each edge $\widehat{e}$ of W is connected, then p induces isomorphisms $(p|)_* : \pi_i(Z) \rightarrowtail\!\!\!\!\twoheadrightarrow \pi_i(W)$ in homotopy for $i = 0, 1$. In particular, this holds if Z is the full pre-image of some subcomplex of X.

Remarks. (i) Intuitively speaking, Y is constructed by "blowing up" each vertex of X to a Cayley complex and attaching, instead of a single edge of X, a direct product of the Cayley graph of the edge stabilizer with a 1-cell. Finally, for every 2-cell of X and every element in its stabilizer, we glue in a new 2-cell lying over the original one. Then the cell stabilizers can act freely on the subcomplexes lying over the corresponding cells of X. However, this intuition works only if no edge of X is inverted under the G-action.

(ii) If the 2-cells of X are attached to the 1-skeleton along non-trivial closed edge paths and if cell stabilizers fix cells pointwise, then we could use the theory of complexes of groups, the higher dimensional generalization of the Bass–Serre theory developed independently by Corson [13–15] and Haefliger [20, 21], to construct Y. The action of G on X determines a 2-complex of groups, the 2-dimensional generalization of a graph of groups, over the quotient $G\backslash X$. Equipped with this data one can then form a complex of spaces $K \twoheadrightarrow G\backslash X$ by gluing together correctly direct products of $K(G_\sigma, 1)$-complexes of cell stabilizers with balls $\mathbb{B}^{\dim \sigma}$. Finally, one can take Y to be the 2-skeleton of the universal covering complex $\tilde{K}$. The details can be found in [13] (or [24]). Similar constructions can be found in [1] and in the forthcoming book of Geoghegan [19].

(iii) It is possible to use our construction to write down a presentation of G. As this has already been done in [12] and requires some additional work, we will not go into details.

3.2. If $\sigma = \sigma^n$ is an n-cell of X we put $\partial\sigma = \sigma \cap X^{n-1}$. Usually vertices (edges and 2-cells) of X are written as v, $\widehat{v}$ (e, $\widehat{e}$ and c, $\widehat{c}$, respectively).

We orient X by choosing, for every cell $\sigma = \sigma^n$ of X, a characteristic map $\Phi_\sigma : (\mathbb{B}^n, \mathbb{S}^{n-1}) \to (\sigma, \partial\sigma)$. The attaching map of σ will be denoted by $\varphi_\sigma : \mathbb{S}^{n-1} \to \partial\sigma \subseteq X^{n-1}$. It will be convenient to put $\imath(e) = \varphi_e(-1)$ and $\tau(e) = \varphi_e(1)$ for any edge e of X.

Recall that G acts on X by regular homeomorphisms. Let e be an edge of X and let $g \in G$. We say that the action of g on e is *orientation-preserving* if the two relative homeomorphisms Φ_{ge} and $g\,\Phi_e : (\mathbb{B}^1, \mathbb{S}^0) \to (ge, \partial ge)$ are homotopic as maps of pairs. Otherwise we say that the action of g on e is *orientation-reversing*. Note that in this case the maps Φ_{ge} and $g(\Phi_e \circ \rho)$ (resp. $\Phi_{ge} \circ \rho$ and $g\,\Phi_e$) are homotopic, where $\rho : \mathbb{B}^1 \rightarrowtail\!\!\!\!\rightarrow \mathbb{B}^1$ is inversion $b \mapsto -b$. Finally, we say that e is *inverted under the G-action* if there is an element in the stabilizer G_e with orientation-reversing action on e.

Now, we begin with the construction of our new oriented 2-complex Y, of the free G-action and of the map $p : Y \twoheadrightarrow X$. The characteristic and attaching maps of cells of Y will be denoted by Ψ_* and ψ_*, respectively.

3.3. First choices. For $n = 0, 1, 2$ we choose a G-transversal $\mathscr{T}_n$ for the set of n-cells of X such that $\imath(e) \in \mathscr{T}_0$ for all edges $e \in \mathscr{T}_1$. By $\mathscr{T}_1^-$ we denote the set of edges of $\mathscr{T}_1$ which are inverted under the G-action. Put $\mathscr{T}_1^+ = \mathscr{T}_1 \setminus \mathscr{T}_1^-$.

Now, let e be an edge in $\mathscr{T}_1$. Then we denote by $\widetilde{\tau}(e)$ the unique vertex in $\mathscr{T}_0$ which is equivalent to $\tau(e)$ mod G and choose an element $t_e \in G$ so that $\tau(e) = t_e \cdot \widetilde{\tau}(e)$. If $e \in \mathscr{T}_1^-$ take the element t_e in $G_e \setminus H_e$, where H_e is the subgroup of index 2 consisting of all elements of the stabilizer G_e with orientation-preserving action on e.

Finally, choose a presentation $\langle \mathscr{X}_v \mid \mathscr{R}_v \rangle$ of the stabilizer G_v for every vertex $v \in \mathscr{T}_0$ and choose a generating system $\mathscr{X}_e$ of G_e for any edge $e \in \mathscr{T}_1$. If $e \in \mathscr{T}_1^-$ we require $\mathscr{X}_e$ to consist of t_e together with a generating system $\mathscr{Y}_e$ of H_e.

3.4. The 1-skeleton Y^1. The vertices and edges of Y^1 are given by the sets

$$VY = G \times \mathscr{T}_0 \quad \text{and} \quad EY = (G \times \mathscr{T}_1) \coprod \{ (g, x_v, v) \mid g \in G,\ v \in \mathscr{T}_0,\ x_v \in \mathscr{X}_v \},$$

respectively. The edges come in two sorts: the ones in $G \times \mathscr{T}_1$, "lying over the edges of X", and the other ones, "making up Cayley graphs over the vertices of X". The attaching map $\psi_{(g,e)} : \mathbb{S}^0 \to Y^0$ of the edge $(g, e) \in G \times \mathscr{T}_1$ is defined by $-1 \mapsto (g, \imath(e))$ and $1 \mapsto (gt_e, \widetilde{\tau}(e))$. And if $v \in \mathscr{T}_0$ and $x_v \in \mathscr{X}_v$ we put $\psi_{(g,x_v,v)} : \mathbb{S}^0 \to Y^0$ to be $-1 \mapsto (g, v)$ and $1 \mapsto (gx_v, v)$. Then the 1-skeleton Y^1 is obtained in the usual way by gluing a 1-ball $\mathbb{B}^1$ to $Y^0 = G \times \mathscr{T}_0$ for every element in EY via the attaching map $\psi_* : \mathbb{S}^0 \to Y^0$. The characteristic maps Ψ_* of the edges are the canonical maps $\mathbb{B}^1 \to Y^1$.

The action of the group G is given, via the characteristic maps, by the obvious left action on the sets VY and EY of vertices and edges of Y^1. This makes Y^1 into a free

G-graph. Notice that a G-transversal of Y^1 is determined by all vertices of the form $(1,v)$, all edges $(1,e)$ and all edges $(1,x_v,v)$.

Finally, we want to define a continuous surjection $p_1 : Y^1 \twoheadrightarrow X^1$. On the set $Y^0 = G \times \mathscr{T}_0$ of vertices it is given by $(g,v) \mapsto gv$, and it maps each edge of the form (g,x_v,v) onto the vertex gv of X. Using the chosen characteristic maps, we map the edge $(1,e)$ of Y^1 onto the edge e of X. By the G-action we extend this to all edges of the form (g,e). Clearly, this gives a G-equivariant regular map $p_1 : Y^1 \twoheadrightarrow X^1$.

3.5. The fibres of p_1. We now examine the full pre-images of vertices and edges of X^1. Let $\widehat{v}$ be a vertex of X, $v \in \mathscr{T}_0$ and $\widehat{g} \in G$ so that $\widehat{v} = \widehat{g}v$. Then the pre-image of $\widehat{v}$ under p_1 is given by the set $\{(\widehat{g}g_v, v), (\widehat{g}g_v, x_v, v) \mid g_v \in G_v,\ x_v \in \mathscr{X}_v\}$ of vertices and edges of Y^1. Notice that this pre-image is just the $\widehat{g}$-translate of the copy $\Gamma(G_v; \mathscr{X}_v) \times \{v\} \subseteq Y^1$ of the Cayley graph of G_v with respect to $\mathscr{X}_v$ and is independent of the choice of $\widehat{g}$. In particular, $p_1^{-1}(\widehat{v})$ is connected.

Next, let $\widehat{e} = \widehat{g}e$ be an edge of X with $e \in \mathscr{T}_1$ and $\widehat{g} \in G$. Then the pre-image of $\widehat{e}$ is the union of the (possibly identical) subgraphs $p_1^{-1}(\iota(e))$ and $p_1^{-1}(\tau(e))$ together with the set $\{(\widehat{g}g_e, e) \mid g_e \in G_e\}$ of edges of Y^1. Since one end point of the edge $(\widehat{g}g_e, e)$ lies in $p_1^{-1}(\iota(e))$ and the other in $p_1^{-1}(\tau(e))$ we see that each edge lying over $\widehat{e}$ connects $p_1^{-1}(\iota(e))$ and $p_1^{-1}(\tau(e))$.

We can summarize the 1-dimensional case as follows.

3.6. Proposition. *The group G acts freely on the graph Y^1, and $p_1 : Y^1 \twoheadrightarrow X^1$ is a G-equivariant regular surjection.*

If X^1 is finite $\bmod G$ *and if all vertex stabilizers are finitely generated, then Y^1 can be chosen to be finite.*

Let $p_1| : Z \twoheadrightarrow p_1(Z) = W$ be the restriction of p_1 to a subgraph Z of Y. If the pre-image under $p_1|$ of each vertex of W is connected, then p_1 induces a bijection $(p_1|)_* : \pi_0(Z) \rightarrowtail\!\!\!\!\twoheadrightarrow \pi_0(W)$. This holds, in particular, if Z is the full pre-image of a subgraph of X^1.

3.7. The 2-complex Y. The set of 2-cells of Y is, by definition,

$$CY = (G \times \mathscr{T}_2) \coprod \{ (g,x_e,e) \mid g \in G,\ e \in \mathscr{T}_1,\ x_e \in \mathscr{X}_e \} \\ \coprod \{ (g,r_v,v) \mid g \in G,\ v \in \mathscr{T}_0,\ r_v \in \mathscr{R}_v \}.$$

The 2-cells come in three sorts: the first ones in $G \times \mathscr{T}_2$, "covering the 2-cells of X", the second ones of the form (g,x_e,e), "making up Cayley graphs over each point of an open 1-cell of X", and the third ones of the form (g,r_v,v), "the 2-cells of Cayley complexes over the vertices of X".

In 3.8–3.10 we will define the attaching maps for the 2-cells of Y. Since the G-action on Y^1 is already defined, we do this in such a way that the attaching map $\psi_{(g,c)} : \mathbb{S}^1 \to Y^1$ of the 2-cell $(g,c) \in G \times \mathscr{T}_2$ is just the g-translate of the attaching map $\psi_{(1,c)}$, that is, $\psi_{(g,c)}(s) = g\,\psi_{(1,c)}(s)$ for all $s \in \mathbb{S}^1$. Similarly, we proceed in the other cases. In other words, we only define the attaching maps for 2-cells of the form $(1,c)$, $(1,x_e,e)$ and $(1,r_v,v)$ and then refer to G-shifts. As usual, Y is the 2-complex obtained by gluing a 2-ball $\mathbb{B}^2$ to the 1-skeleton via the attaching map $\psi_*:\mathbb{S}^1 \to Y^1$ for every element in CY. The characteristic maps Ψ_* are the canonical maps $\mathbb{B}^2 \to Y$.

The G-action on Y is given by the action on the 1-skeleton and by the obvious left G-action on the set CY of 2-cells of Y (using the characteristic maps). Then G acts freely on Y, and a G-transversal for the set of 2-cells of Y is given by all 2-cells of the form $(1,c)$, $(1,x_e,e)$, $(1,r_v,v)$.

Simultaneously to the definition of the attaching maps we want to define the G-equivariant regular surjection $p:Y \twoheadrightarrow X$ extending $p_1:Y^1 \twoheadrightarrow X^1$. Again, we only define p on the G-transversal of 2-cells determined above and then refer to G-shifts. This will be done so that each 2-cell of the form $(1,c)$ is mapped regularly onto the 2-cell c of X, so that $(1,x_e,e)$ is mapped regularly onto the edge e of X and so that the image of $(1,r_v,v)$ is the vertex v. Then it will be clear that p has the desired properties.

3.8. The 2-cells of the form $(1,r_v,v)$**.** Let $v\in\mathscr{T}_0$, $r_v\in\mathscr{R}_v$, and recall that $\langle \mathscr{X}_v \mid \mathscr{R}_v \rangle$ was chosen to be a presentation of the stabilizer G_v of the vertex v. Moreover, we have seen in paragraph 3.5 that the pre-image of v under p_1 is just the copy $\Gamma(G_v;\mathscr{X}_v)\times\{v\}\subseteq Y^1$ of the Cayley graph with respect to the chosen generating system. Then the attaching map $\psi_{(1,r_v,v)}:\mathbb{S}^1 \to \Gamma(G_v;\mathscr{X}_v)\times\{v\}\subseteq Y^1$ is determined by the closed edge path starting at the vertex $(1,v)$ and spelling out the word $r_v\in\mathscr{R}_v$.

Note that the boundary path of the 2-cell $(1,r_v,v)$ of Y is mapped by p_1 onto the vertex $v\in X$. Now, we require p to map the whole 2-cell $(1,r_v,v)$ onto v.

3.9. The 2-cells of the form $(1,x_e,e)$**.** Here we consider two cases, either $e\in\mathscr{T}_1^+$ or $e\in\mathscr{T}_1^-$, and we start with the first one.

So let $x_e\in\mathscr{X}_e$ be an element of the chosen generating system of the stabilizer G_e. Since e is not inverted under the G-action, we find $x_e\in G_{\iota(e)}\cap G_{\tau(e)}$ and $t_e^{-1}x_et_e\in G_{\tilde\tau(e)}$. As the pre-images of $\iota(e)$ and $\tilde\tau(e)$ under p_1 are the copies $\Gamma(G_{\iota(e)};\mathscr{X}_{\iota(e)})\times\{\iota(e)\}$ and $\Gamma(G_{\tilde\tau(e)};\mathscr{X}_{\tilde\tau(e)})\times\{\tilde\tau(e)\}$ of the indicated Cayley graphs in Y^1, there are edge paths $\omega:I\to Y^1$ and $\tilde\omega:I\to Y^1$ in the pre-images of $\iota(e)$ and $\tilde\tau(e)$ leading from the vertices $(1,\iota(e))$ and $(1,\tilde\tau(e))$ to the vertices $(x_e,\iota(e))$ and $(t_e^{-1}x_et_e,\tilde\tau(e))$, respectively.

We now identify the 1-sphere $\mathbb{S}^1$ with the boundary $\partial(\mathbb{B}^1\times I)$ of the direct product $\mathbb{B}^1\times I$. Then the attaching map $\psi_{(1,x_e,e)}:\partial(\mathbb{B}^1\times I)\to Y^1$ is given by the characteristic map $\Psi_{(1,e)}$ on $\mathbb{B}^1\times\{0\}$, by the t_e-translate of the edge path $\tilde\omega$ on $\{1\}\times I$, by the characteristic map $\Psi_{(x_e,e)}$ on $\mathbb{B}^1\times\{1\}$, and by the edge path ω on $\{-1\}\times I$. By construction of ω and $\tilde\omega$ this is, in fact, a closed edge path in Y^1 (see Fig. 1).

Let us turn to the definition of p now. Consider the composite $p_1\circ\psi_{(1,x_e,e)}:\partial(\mathbb{B}^1\times I)\to X^1$. The images of $\{-1\}\times I$ and $\{1\}\times I$ are easily seen to be $\iota(e)$ and $\tau(e)$, and the images of $\mathbb{B}^1\times\{0\}$ and $\mathbb{B}^1\times\{1\}$ are given by Φ_e and $x_e\Phi_e:\mathbb{B}^1\to e\subseteq X^1$, respectively. Since e is not inverted under the G-action and since x_e stabilizes e, there is

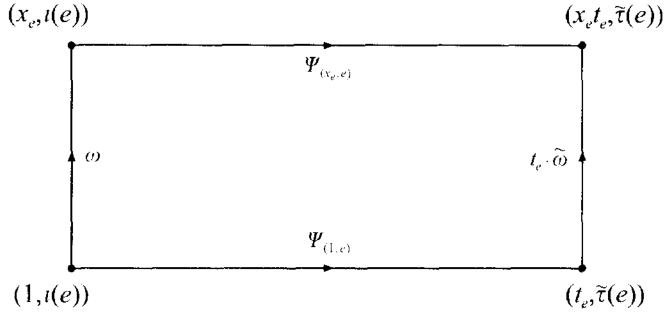

Fig. 1. The attaching map $\psi_{(1,x_e,e)} : \mathbb{S}^1 \to Y^1$ for $e \in \mathscr{T}_1^+$ and $x_e \in \mathscr{X}_e$.

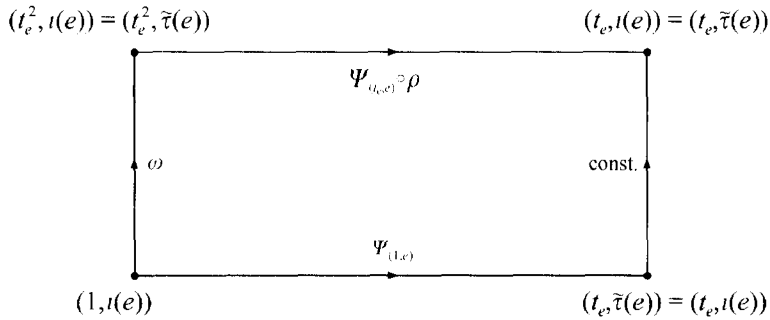

Fig. 2. The attaching map $\psi_{(1,t_e,e)} : \mathbb{S}^1 \to Y^1$ for $e \in \mathscr{T}_1^-$.

a homotopy $H : (\mathbb{B}^1 \times I, \mathbb{S}^0 \times I) \to (e, \partial e)$ such that $H(\cdot, 0) = \Phi_e(\cdot)$, $H(\cdot, 1) = x_e\, \Phi_e(\cdot)$, $H(-1, I) = \iota(e)$ and $H(1, I) = \tau(e)$. Moreover, H can be chosen so that the pre-image in $\mathbb{B}^1 \times I$ of each point in the open cell $e - \partial e$ is homeomorphic to a closed interval between the bottom and the top of $\mathbb{B}^1 \times I$ meeting each slice $\mathbb{B}^1 \times \{t\}$ exactly once. Now, the homotopy H induces the desired map p on the 2-cell $(1, x_e, e)$ which coincides with p_1 on the boundary.

Now, let $e \in \mathscr{T}_1^-$. If $x_e \in \mathscr{X}_e = \mathscr{Y}_e \cup \{t_e\}$ is in $\mathscr{Y}_e$, we can proceed as before since the action of any element in $\mathscr{Y}_e$ on the edge e is orientation-preserving. Hence it only remains to deal with $x_e = t_e$.

Notice first, that $\iota(e) = \tilde{\tau}(e) \in \mathscr{T}_0$ and that $t_e^2 \in H_e \leq G_{\iota(e)} \cap G_{\tilde{\tau}(e)}$. Then there exists an edge path $\omega : I \to \Gamma(G_{\iota(e)}; \mathscr{X}_{\iota(e)}) \times \{\iota(e)\} \subseteq Y^1$ with initial point $(1, \iota(e))$ and terminal point $(t_e^2, \iota(e))$ in the pre-image of $\iota(e)$ under p_1. Again, we identify the ball-sphere pair $(\mathbb{B}^2, \mathbb{S}^1)$ with the pair $(\mathbb{B}^1 \times I, \partial(\mathbb{B}^1 \times I))$. Then the attaching map $\psi_{(1,t_e,e)} : \partial(\mathbb{B}^1 \times I) \to Y^1$ is given by the characteristic map $\Psi_{(1,e)}$ on $\mathbb{B}^1 \times \{0\}$, by the constant map $\{1\} \times I \to (t_e, \iota(e))$, by the inversion $\rho : \mathbb{B}^1 \to \mathbb{B}^1$, $b \mapsto -b$, followed by the characteristic map $\Psi_{(t_e,e)}$ on $\mathbb{B}^1 \times \{1\}$, and by the edge path ω on $\{-1\} \times I$. Again, this gives a well-defined closed edge path (see Fig. 2).

Finally, we have to give the definition of p on the 2-cell $(1, t_e, e)$. As before we consider $p_1 \circ \psi_{(1,t_e,e)} : \partial(\mathbb{B}^1 \times I) \to X^1$. Then the sides $\{-1\} \times I$ and $\{1\} \times I$ are mapped onto the vertices $\iota(e)$ and $\tau(e)$, respectively. The images of the bottom and top are given by $\Phi_e : \mathbb{B}^1 \to e$ and $t_e(\Phi_e \circ \rho) : \mathbb{B}^1 \to e$, respectively. As t_e reverts the orientation of e, we see that Φ_e and $t_e(\Phi_e \circ \rho)$ determine the same orientation of e. In other words, there is a homotopy $H : (\mathbb{B}^1 \times I, \mathbb{S}^0 \times I) \to (e, \partial e)$ such that $H(\cdot, 0) = \Phi_e(\cdot)$, $H(\cdot, 1) = t_e \Phi_e \rho(\cdot)$, $H(-1, I) = \iota(e)$ and $H(1, I) = \tau(e)$. Again, we can do this in such a way that $H(\cdot, t) : (\mathbb{B}^1 - \mathbb{S}^0) \times \{t\} \to e$ is a homeomorphism onto the open cell $e - \partial e$ for all $t \in I$. This homotopy gives our map p on the 2-cell $(1, t_e, e)$.

3.10. The 2-cells of the form $(1, c)$. Let c be a 2-cell in $\mathscr{T}_2$ and consider its attaching map $\varphi_c : \mathbb{S}^1 \to \partial c \subseteq X^1$. Suppose first that $\varphi_c^{-1}(X^0) = \emptyset$. Then there is an edge $\widehat{e} \subseteq X^1$ such that $\varphi_c(\mathbb{S}^1) \subseteq \widehat{e} - \partial \widehat{e}$. Since the pre-image under p_1 of the open 1-cell $\widehat{e} - \partial \widehat{e}$ is a disjoint union of open 1-cells of Y^1 which are mapped homeomorphically onto $\widehat{e} - \partial \widehat{e}$, we can find a lift $\psi_{(1,c)} : \mathbb{S}^1 \to Y^1$ of φ_c. This lift is used to glue the 2-cell $(1, c)$ to the 1-skeleton Y^1. Now, we can define $p : (1, c) \to c$ so that it coincides on the boundary with p_1 and maps the open 2-cell $(1, c) - \partial(1, c)$ homeomorphically onto $c - \partial c$.

Next, assume that $\varphi_c^{-1}(X^0) \neq \emptyset$. Then each path component of $\mathbb{S}^1 - \varphi_c^{-1}(X^0)$ is an open interval mapped by φ_c into some open 1-cell of X. There are at most countably many such open intervals; this follows for example from the fact that $\mathbb{S}^1 \subseteq \mathbb{R}^2$ satisfies the second axiom of countability (see, for example, [16, p. 180]). Label these intervals I_n, $n \in N \subseteq \mathbb{N}$. As above, it is easy to find a lift $\psi_{(1,c)}^n : \overline{I_n} \to Y^1$ of φ_c on the closed interval $\overline{I_n}$ such that the image of I_n is contained in an open 1-cell of Y. Now, let J_n be the closed interval which follows I_n in the cyclic order determined by some orientation of the circle and is mapped onto a vertex of X. If J_n consists of a single point then we "stretch" it to a closed interval J_n' of length 2^{-n}; otherwise we put $J_n' = J_n$. Since there are only countably many intervals, we then find a homeomorphic copy S^1 of the circle $\mathbb{S}^1$ and an identification $q_{(1,c)} : S^1 \twoheadrightarrow \mathbb{S}^1$ which is "identity" on the intervals I_n and "collapse" $J_n' \twoheadrightarrow J_n$.

As the J_n are mapped by φ_c onto vertices of X and as pre-images under p_1 of vertices are connected subgraphs of Y^1, there is a map $\psi_{(1,c)} : S^1 \to Y^1$ which coincides with the $\psi_{(1,c)}^n$ on the closed intervals $\overline{I_n}$ and satisfies $p_1 \circ \psi_{(1,c)} = \varphi_c \circ q_{(1,c)} : S^1 \to \partial c$. We use this map $\psi_{(1,c)}$ to attach the 2-cell $(1, c)$ to the 1-skeleton.

Finally, $q_{(1,c)}$ also induces an identification $Q_{(1,c)} : B^2 \twoheadrightarrow \mathbb{B}^2$, where B^2 is a homeomorphic copy of the 2-ball $\mathbb{B}^2$, such that the interior of B^2 is mapped homeomorphically onto the interior of $\mathbb{B}^2$. Then the composite $\Phi_c \circ Q_{(1,c)} : B^2 \to c$ induces a map p on the 2-cell $(1, c)$ with the desired properties.

This completes the construction of Y and of the map $p : Y \twoheadrightarrow X$. In the following proposition we summarize some properties of this construction.

3.11. Proposition. *The group G acts freely on Y, and $p : Y \twoheadrightarrow X$ is a G-equivariant regular surjection.*

If X is finite mod G, *if vertex stabilizers are finitely presented and if edge stabilizers are finitely generated, then Y can be chosen to be G-finite.*

3.12. The fibres of p. For later use let us quickly determine the full pre-images of vertices, edges and open 2-cells of X under p.

So let $\widehat{v} = \widehat{g}v$ be a vertex of X with $\widehat{g} \in G$ and $v \in \mathscr{T}_0$. Then $p^{-1}(\widehat{v})$ is the $\widehat{g}$-translate of the copy $C(\mathscr{X}_v; \mathscr{R}_v) \times \{v\} \subseteq Y$ of the Cayley complex of the stabilizer G_v in Y.

And the full pre-image of an edge $\widehat{e} = \widehat{g}e$ of X, where $e \in \mathscr{T}_1$, consists of the subcomplex $p^{-1}(\iota(\widehat{e})) \cup p^{-1}(\tau(\widehat{e})) \subseteq Y$ together with the set $\{(\widehat{g}g_e, e) \mid g_e \in G_e\}$ of edges of Y and the set $\{(\widehat{g}g_e, x_e, e) \mid g_e \in G_e,\ x_e \in \mathscr{X}_e\}$ of 2-cells of Y lying over $\widehat{e}$.

Now, the boundary path of the 2-cell $(\widehat{g}g_e, x_e, e)$ is given by two paths in $p^{-1}(\iota(\widehat{e})) \cup p^{-1}(\tau(\widehat{e}))$ together with the two edges $(\widehat{g}g_e, e)$ and $(\widehat{g}g_e x_e, e)$. Then our construction of the map p allows one to identify the full pre-image $p^{-1}(b_{\widehat{e}})$ of the "barycentre" $b_{\widehat{e}} \in \widehat{e} - \partial\widehat{e}$ with the Cayley graph $\Gamma(G_e; \mathscr{X}_e)$ (see Figs. 1 and 2). Furthermore, if $p| : Z \twoheadrightarrow p(Z)$ is the restriction of p to a subcomplex $Z \subseteq Y$ we can identify $(p|)^{-1}(b_{\widehat{e}})$ with a subgraph $Z_{\widehat{e}}$ of $\Gamma(G_e; \mathscr{X}_e)$. More explicitly, a vertex $g_e \in G_e$ (resp. an edge $(g_e, x) \in G_e \times \mathscr{X}_e$) of the Cayley graph $\Gamma(G_e; \mathscr{X}_e)$ lies in $Z_{\widehat{e}}$ if and only if the edge $(\widehat{g}g_e, e)$ (resp. the 2-cell $(\widehat{g}g_e, x_e, e)$) of Y lying over $\widehat{e}$ belongs to the subcomplex Z.

Finally, we exhibit the pre-image of an open 2-cell $\widehat{c} - \partial\widehat{c}$ of X. Let $\widehat{c} = \widehat{g}c$ with $c \in \mathscr{T}_2$. Then this pre-image is made up of all open 2-cells of Y determined by the set $\{(\widehat{g}g_c, c) \mid g_c \in G_c\}$ of (closed) 2-cells.

We close this section with the proof of the following result:

3.13. Proposition. *Let Z be a subcomplex of Y, $W = p(Z)$ and $p| : Z \twoheadrightarrow W$ the restriction of p. If the pre-image $(p|)^{-1}(\widehat{v})$ of each vertex $\widehat{v}$ of W is 1-connected and the pre-image $(p|)^{-1}(b_{\widehat{e}})$ of the "barycentre" $b_{\widehat{e}}$ of each edge $\widehat{e}$ of W is connected, then p induces an isomorphism $(p|)_* : \pi_1(Z, z) \rightarrowtail\!\!\!\twoheadrightarrow \pi_1(W, p(z))$ for any choice of a basepoint $z \in Z$. This holds, in particular, if Z is the full pre-image of some subcomplex in X.*

Proof. One can use the arguments of the proof of Proposition 3.1 in [13]. For the benefit of the reader we indicate a second, more direct proof.

Assume first that $Z \subseteq p^{-1}(X^1)$, and hence $W \subseteq X^1$ is a graph. We define $j : W \to Z$ by first choosing for each vertex $\widehat{v}$ (resp. edge $\widehat{e}$) of W a vertex $\widehat{V} \in (p|)^{-1}(\widehat{v})$ (resp. an edge $\widehat{E} \subseteq (p|)^{-1}(\widehat{e})$). Then divide each edge $\widehat{e}$ of W into three parts: j maps the middle part to $\widehat{E}$ and the end parts to edge paths in the connected complexes $p^{-1}(\iota(\widehat{e}))$ and $p^{-1}(\tau(\widehat{e}))$ joining the corresponding end points of $\widehat{E}$ with the chosen lifts of $\iota(\widehat{e})$ and $\tau(\widehat{e})$. Then $p \circ j$ is homotopic to the identity.

Let α be a closed edge path in Z. Suppose that α travels along an edge of Z lying over some edge $\widehat{e}$ of W. As $(p|)^{-1}(b_{\widehat{e}})$ or, equivalently, the subgraph $Z_{\widehat{e}}$ of 3.12 are connected, one can use the 2-cells in $(p|)^{-1}(\widehat{e})$ lying over $\widehat{e}$ to homotop this part of α to an edge path of the form $\alpha_1 \widehat{E}^{\pm 1} \alpha_2$, where α_1 and α_2 are contained in the pre-images of the end points of $\widehat{e}$. Thus α is homotopic to a closed edge path $\widetilde{\alpha}$ in Z, made up of subpaths either lying in the pre-image of a vertex of W or travelling along a chosen lift $\widehat{E}$ of an edge $\widehat{e} \subseteq W$. But now the pre-images of vertices of W are 1-connected.

Hence we can homotop $\widetilde{\alpha}$ to a closed edge path in $j(W)$. From these arguments one can infer that $jp(\alpha)$ is freely homotopic to α. It follows that $(p|)_* : \pi_1(Z) \to \pi_1(W)$ is an isomorphism if $Z \subseteq p^{-1}(X^1)$.

Next, we deal with the general case. We know already that p induces an isomorphism $(p|)_* : \pi_1(Z \cap p^{-1}(X^1)) \rightarrowtail\!\!\!\!\twoheadrightarrow \pi_1(W \cap X^1)$. Then $\pi_1(Z)$ is the quotient of $\pi_1(Z \cap p^{-1}(X^1))$ by the normal subgroup generated by the homotopy classes of the attaching maps of all 2-cells of Z lying over some 2-cell of W. Let $\widehat{c}$ be a 2-cell of W and let $\widehat{C}$ be a 2-cell in $(p|)^{-1}(\widehat{c})$. Then the composite of the characteristic map $\Psi_{\widehat{C}}$ with p is a characteristic map for $\widehat{c}$. Thus the homotopy class $[\psi_{\widehat{C}}] \in \pi_1(Z \cap p^{-1}(X^1))$ is mapped by $(p|)_*$ into the normal subgroup of $\pi_1(W \cap X^1)$ generated by the homotopy class $[\varphi_{\widehat{c}}]$. From this we infer that $(p|)_*$ identifies a presentation of $\pi_1(Z)$ with the standard presentation of $\pi_1(W)$.

Finally, we know from the discussion in 3.12 that full pre-images of vertices and "barycentres" of edges of X are homeomorphic to Cayley complexes and Cayley graphs, respectively. This proves the final assertion of our proposition. □

4. Criteria for the homotopical geometric invariant $\Sigma^2(G)$

In this section we will apply the construction above in order to prove Theorem A and Theorem B of the introduction. We leave it to the reader to write down the corresponding results for the invariant Σ^1.

We begin with Theorem A. Our original proof in [24] was based on the theory of complexes of groups as indicated in the remark after Theorem 3.1 and required some additional assumptions. Also, some additional mild assumptions were necessary to prove the corresponding results for all homological invariants $\Sigma^m(G;\mathbb{Z})$ in [26].

4.1. Theorem. *Let G be a group acting on a 1-connected 2-complex X such that X is finite* mod G, *all vertex stabilizers are finitely presented and all edge stabilizers are finitely generated. Let $\chi : G \to \mathbb{R}$ be a homomorphism and assume that X admits a χ-equivariant height function $h = h_\chi : X \to \mathbb{R}$. Then $\chi \in \Sigma^2(G)$ if and only if $X_h^{[0,\infty)}$ is essentially 1-connected in X.*

Proof. Let Y be a G-free complex constructed from the action of G on X, as in the preceding section. Our assumptions imply that Y is 1-connected and can be chosen to be G-finite. Denote by $p : Y \twoheadrightarrow X$ the corresponding G-equivariant regular surjection.

We now define a χ-equivariant height function $H : Y \to \mathbb{R}$ by putting $H = h \circ p$. As p is regular, we find that $Y_H^{[\lambda,\infty)} = p^{-1}(X_h^{[\lambda,\infty)})$ for all $\lambda \in \mathbb{R}$. According to Theorem 3.1 the restriction of p induces isomorphisms $(p|)_* : \pi_i(Y_H^{[\lambda,\infty)}) \rightarrowtail\!\!\!\!\twoheadrightarrow \pi_i(X_h^{[\lambda,\infty)})$ in homotopy for $i = 0, 1$. Thus $X_h^{[0,\infty)}$ is essentially 1-connected in X if and only if $Y_H^{[\lambda,\infty)}$ is essentially 1-connected in Y. Since Y is the 2-skeleton of the universal cover of a $K(G,1)$-complex with finite 2-skeleton, the latter is equivalent to $\chi \in \Sigma^2(G)$ by Corollary 2.6. □

The homological version of the following consequence has been obtained in [25] with the aid of a certain spectral sequence.

4.2. Corollary. *Let $N \rightarrowtail G \overset{\pi}{\twoheadrightarrow} Q$ be a short exact sequence of finitely presented groups and let $\psi \in V(Q)$. Then $\psi \in \Sigma^2(Q)$ if and only if $\psi \circ \pi \in \Sigma^2(G)$.*

Proof. Consider the free action of Q on the Cayley complex $C(\mathscr{Y};\mathscr{S})$ with respect to some finite presentation $\langle \mathscr{Y} \mid \mathscr{S} \rangle$ of Q. Then there exists a ψ-equivariant height function $h: C(\mathscr{Y};\mathscr{S}) \to \mathbb{R}$. Now, G acts via π on $C(\mathscr{Y};\mathscr{S})$, and h is also a $(\psi \circ \pi)$-equivariant height function. Since N is finitely presented, we can apply Theorem 4.1 to obtain the desired result. □

We close this section with a new proof of Theorem B of the introduction. Its original proof in [24] was based on Brown's paper [12] and on Renz' combinatorial Σ^2-criterion [29]. Here we apply our results of Section 3 once again.

4.3. Theorem. *Suppose that a group G acts on a 1-connected, G-finite 2-complex X. If $0 \neq \chi|_{G_\sigma} \in \Sigma^{2-\dim\sigma}(G_\sigma)$ for all cells σ of X, then $\chi \in \Sigma^2(G)$.*

Proof. We set $\chi_\sigma = \chi|_{G_\sigma}$ for every cell σ of X. Again, let Y be a G-free complex constructed from the action on X according to Section 3. This complex depends on several choices, some of which will be specified now. We retain the notation and terminology of the preceding section.

If v is a vertex of X in the G-transversal $\mathscr{T}_0$ then we choose a finite presentation $\langle \mathscr{X}_v \mid \mathscr{R}_v \rangle$ of its stabilizer G_v such that the valuation subcomplexes $C(\mathscr{X}_v;\mathscr{R}_v)_{\chi_v}^{[\lambda,\infty)}$ are 1-connected for every $\lambda \in \mathbb{R}$. As $0 \neq \chi_v \in \Sigma^2(G_v)$, we know that such presentations do exist (see Theorem 2.4). And for all edges $e \in \mathscr{T}_1$ we choose a finite generating system $\mathscr{X}_e$ of the stabilizer G_e. Since $0 \neq \chi_e \in \Sigma^1(G_e)$, the valuation subgraphs $\Gamma(G_e;\mathscr{X}_e)_{\chi_e}^{[\lambda,\infty)}$ of the Cayley graph $\Gamma(G_e;\mathscr{X}_e)$ are connected for all $\lambda \in \mathbb{R}$. Moreover, we can always assume that $\chi(x) \geq 0$ for all $x \in \bigcup_{v\in\mathscr{T}_0} \mathscr{X}_v \cup \bigcup_{e\in\mathscr{T}_1} \mathscr{X}_e$. And we assume that $\chi(t_e) \geq 0$ if $e \in \mathscr{T}_1$; this can be achieved by right multiplication with elements of the stabilizer G_e (resp. H_e, if $e \in \mathscr{T}_1^-$) with sufficiently high χ-value.

Finally, in the definition of Y we had to choose, for all $e \in \mathscr{T}_1^+$ and all $x_e \in \mathscr{X}_e$, edge paths $\omega : I \to \Gamma(G_{\iota(e)};\mathscr{X}_{\iota(e)}) \times \{\iota(e)\} \subseteq Y^1$ and $\tilde{\omega} : I \to \Gamma(G_{\tilde{\tau}(e)};\mathscr{X}_{\tilde{\tau}(e)}) \times \{\tilde{\tau}(e)\}$ with initial points $(1,\iota(e))$ and $(1,\tilde{\tau}(e))$ and terminal points $(x_e,\iota(e))$ and $(t_e^{-1}x_et_e,\tilde{\tau}(e))$, respectively (see 3.9). We make our choices so that these edge paths are completely contained in $\Gamma(G_{\iota(e)};\mathscr{X}_{\iota(e)})_{\chi_{\iota(e)}}^{[0,\infty)} \times \{\iota(e)\}$ and $\Gamma(G_{\tilde{\tau}(e)};\mathscr{X}_{\tilde{\tau}(e)})_{\chi_{\tilde{\tau}(e)}}^{[0,\infty)} \times \{\tilde{\tau}(e)\}$, respectively. This is possible because $\chi(x_e) = \chi(t_e^{-1}x_et_e) \geq 0$. We proceed similarly in the cases where $e \in \mathscr{T}_1^-$. In other words, we assume that the minimum of the χ-values of the vertices on each chosen edge path is always attained on the initial vertex.

All other choices are made arbitrarily. Thus we obtain "the" G-free complex Y together with the G-equivariant regular map $p : Y \twoheadrightarrow X$. We define $h : Y^0 \to \mathbb{R}$ by putting $h(g,v) = \chi(g)$ for all $(g,v) \in G \times \mathscr{T}_0 = Y^0$. Then this map can be extended to a regular χ-equivariant height function $h : Y \to \mathbb{R}$ (see [26]). As h is regular, $Y_h^{[0,\infty)}$ is just the full subcomplex of Y whose vertices $(g,v) \in Y^0$ have the property $\chi(g) \geq 0$.

Let us determine the pre-images of vertices, edges and 2-cells of X under the restriction map $p| : Y_h^{[0,\infty)} \to X$. If $\widehat{v} = \widehat{g}v$ is a vertex of X with $\widehat{g} \in G$ and $v \in \mathscr{T}_0$ then $(p|)^{-1}(\widehat{v})$ is the $\widehat{g}$-translate of the copy $C(\mathscr{X}_v; \mathscr{R}_v)_{\chi_v}^{[-\chi(\widehat{g}),\infty)} \times \{v\}$ of the indicated Cayley complex in Y. Note that the assumption $\chi_v \neq 0$ guarantees that this pre-image is non-void. Furthermore, we see that $(p|)^{-1}(\widehat{v})$ is 1-connected.

Now, let $\widehat{e} = \widehat{g}e$ be an edge of X and $e \in \mathscr{T}_1$. Then the pre-image of $\widehat{e}$ under $p|$ consists of the pre-images of its end points together with the set $\{(\widehat{g}g_e, e) \mid g_e \in G_e,\ \chi(g_e) \geq -\chi(\widehat{g})\}$ of edges and the set $\{(\widehat{g}g_e, x_e, e) \mid g_e \in G_e,\ x_e \in \mathscr{X}_e,\ \chi(g_e) \geq -\chi(\widehat{g})\}$ of 2-cells of Y. This can be seen as follows. We have $h(\widehat{g}g_e, \iota(e)) \leq h(\widehat{g}g_e t_e, \tau(e))$. Thus the edge $(\widehat{g}g_e, e)$ belongs to $Y_h^{[0,\infty)}$ if and only if $\chi(\widehat{g}g_e) \geq 0$. And our choices above imply that the minimum of the h-values of all vertices on the boundary path of the 2-cell $(\widehat{g}g_e, x_e, e)$ is attained on the initial vertex $(\widehat{g}g_e, \iota(e))$ (this is the lower left vertex in Figs. 1 and 2). This proves our claim. Because $\chi_e \neq 0$ we see that the pre-image of each open edge is non-empty. Moreover, we infer that the fibre $(p|)^{-1}(b_{\widehat{e}})$ of the "barycentre" of $\widehat{e}$ is connected. To see this, note that this fibre is homeomorphic to the valuation subgraph $\Gamma(G_e; \mathscr{X}_e)_{\chi_e}^{[-\chi(\widehat{g}),\infty)}$.

Finally, recall that the 2-cells of Y in the pre-image of a 2-cell $\widehat{c} = \widehat{g}c$ of X, where $c \in \mathscr{T}_2$, are given by $\{(\widehat{g}g_c, c) \mid g_c \in G_c\}$. As there are elements $g_c \in G_c$ with arbitrarily high χ-value, we see that at least one of these 2-cells belongs to $Y_h^{[0,\infty)}$.

We can summarize this discussion by saying that $p| : Y_h^{[0,\infty)} \to X$ is surjective and that the pre-images of vertices (resp. "barycentres" of edges) are 1-connected (resp. connected). By Theorem 3.1 it follows that $Y_h^{[0,\infty)}$ is 1-connected. Consequently $\chi \in \Sigma^2(G)$ by Theorem 2.4 or Corollary 2.6. □

5. Metabelian groups of finite Prüfer rank

We want to sketch how Theorem C follows from the results previously obtained. All proofs are given in detail in [26] for the homological invariants.

The first step consists of an application of Theorem 4.1 to the standard tree of a descending HNN-extension (see [26, Proposition 4.2]).

5.1. Proposition. *Let* $G = \langle B, t \,|\, t^{-1}bt = \phi(b)\ (b \in B) \rangle$ *be a descending HNN-extension, where* $\phi : B \rightarrowtail B$ *is a monomorphism. If* $\chi : G \to \mathbb{R}$ *is a homomorphism such that* $\chi(B) = \{0\}$ *and* $\chi(t) \geq 0$, *and if* B *is finitely presented, then* $\chi \in \Sigma^2(G)$.

Next, one uses Theorem 4.3, Corollary 2.7 and Proposition 5.1 to generalize this result to iterated descending HNN-extensions ([26], Theorem 4.6). Using results of Bieri and Strebel [9, 10] one then concludes (see [26, Theorem 4.8]):

5.2. Theorem. *For every constructible nilpotent-by-abelian group G we have* $\Sigma^2(G)^c \subseteq \operatorname{conv} \Sigma^1(G)^c$.

Of course, $\operatorname{conv} \Sigma^1(G)^c$ is the convex hull of $\Sigma^1(G)^c$ in the finite dimensional vector space $V(G)$, and $\Sigma^m(G)^c$ is the complement of $\Sigma^m(G)$ in $V(G)$. Recall that a soluble-by-finite group G is *constructible* (*in the sense of* [3]) if it admits a finite chain $1 = H_0 \leq H_1 \leq \cdots \leq H_k = G$ of subgroups such that either H_i has finite index in H_{i+1} or H_{i+1} is an ascending HNN-extension of the form $H_{i+1} = \langle H_i, t \mid tH_it^{-1} = \vartheta(H_i) \rangle$ with a monomorphism $\vartheta : H_i \rightarrowtail H_i$. Constructible soluble-by-finite groups are finitely presented and of finite Prüfer rank [3], and so they are nilpotent-by-abelian-by-finite (see [31, Part (a) of the proof of Theorem 10.38]).

Then one can invoke the Bieri–Strebel result [8] (also see [6, 11]) that $0 \notin \operatorname{conv}_{\leq 2} \Sigma^1(G)^c$ holds for any finitely presented nilpotent-by-abelian group G and Renz' result [30] on finitely presented normal subgroups with abelian quotient in order to obtain (cf. [26, Theorem 5.1]):

5.3. Theorem. *The inclusion* $\operatorname{conv}_{\leq 2} \Sigma^1(G)^c \subseteq \Sigma^2(G)^c$ *holds for all constructible soluble-by-finite groups G.*

Here $\operatorname{conv}_{\leq m} \Sigma^1(G)^c$ is the union of the convex hulls of all subsets of $\Sigma^1(G)^c$ of at most m elements. Since $\Sigma^1(G)^c$ is invariant under multiplication by positive reals, $\operatorname{conv}_{\leq m} \Sigma^1(G)^c$ equals the m-fold sum $\Sigma^1(G)^c + \cdots + \Sigma^1(G)^c$ in $V(G)$.

Finally, we sketch the core of the proof of Theorem C of the introduction.

5.4. Theorem. *If G is a finitely presented metabelian group of finite Prüfer rank, then* $\operatorname{conv}_{\leq 2} \Sigma^1(G)^c = \Sigma^2(G)^c$.

Sketch of proof. Firstly, one reduces the problem to groups of the following type ([26, Subsection 6.2]): $G = M \rtimes Q$ is a semi-direct product of a free abelian group Q of finite rank and a Q-module M which is torsion-free as abelian group and has finite torsion-free rank. This follows from results of D.J.S. Robinson along with Corollaries 2.7 and 4.2.

Next, by the proof of Theorem 8 in [3] there is a constructible metabelian descending HNN-extension $G^* = \langle G, t \mid t^{-1}Gt \leq G \rangle$ with base group G. With the aid of Theorem 5.3, Proposition 5.1 and a theorem of Bieri–Strebel [10] one can conclude that $\operatorname{conv}_{\leq 2} \Sigma^1(G)^c \subseteq \Sigma^2(G)^c$ (see [26, Subsection 6.3]).

Following Åberg [2] (also see [32]) one introduces, then, a semi-direct product $\mathcal{G} = \mathcal{M} \rtimes Q$ such that M is a Q-submodule of $\mathcal{M}$ and such that the split epimorphism $\mathcal{G} \twoheadrightarrow Q$ factors through a split epimorphism $\varrho : \mathcal{G} \twoheadrightarrow G$. Moreover, $\mathcal{G}$ acts on the finite dimensional, 1-connected CW-complex Y constructed by Åberg such that Y is finite mod $\mathcal{G}$ and all cell stabilizers are polycyclic. For more details see [2, 32, 26].

Next, one considers homomorphisms $\chi : G \to \mathbb{R}$ that are not contained in the linear subspace of $V(G)$ spanned by $\Sigma^1(G)^c$. One defines $\chi^{\mathcal{G}} : \mathcal{G} \to \mathbb{R}$ by putting $\chi^{\mathcal{G}} = \chi \circ \varrho$.

Then one finds that the restrictions of $\chi^{\mathscr{G}}$ to all cell stabilizers do not vanish (see [26, Subsection 6.5]). Moreover, since cell stabilizers are polycyclic, one has $V(\mathscr{G}_\sigma) = \Sigma^2(\mathscr{G}_\sigma)$ for all cells σ of Y. An appeal to Theorem 4.3 yields $\chi^{\mathscr{G}} \in \Sigma^2(\mathscr{G})$. Hence $\chi \in \Sigma^2(G)$ by Corollary 2.8.

This leaves us with homomorphisms $\chi : G \to \mathbb{R}$ which are in span $\Sigma^1(G)^c$ but not in $\mathrm{conv}_{\leq 2}\, \Sigma^1(G)^c$. One defines $\chi^{\mathscr{G}}$ as before. In this situation one can show that there exists a $\chi^{\mathscr{G}}$-equivariant height function $h = h_{\chi^{\mathscr{G}}} : Y \to \mathbb{R}$ and that the valuation subspaces $h^{-1}([\lambda, \infty))$ are 1-connected for all $\lambda \in \mathbb{R}$ (cf. [26, Subsections 6.6 and 6.7]). Then one invokes Theorem 4.1 to see that $\chi^{\mathscr{G}} \in \Sigma^2(\mathscr{G})$. As above, $\chi \in \Sigma^2(G)$ follows. □

6. Summary

Here we summarize what has been proved in [26], in [27], and in the present paper about the higher geometric invariants of soluble-by-finite groups of finite Prüfer rank. For more details on the connection between finiteness properties of metabelian groups and the (higher) geometric invariants and on the FP_m- and Σ^m-conjecture the reader is referred to [26] or to the survey [27].

Recall first that a monoid M is said to be of type FP_m if the trivial $\mathbb{Z}M$-module $\mathbb{Z}$ admits a projective resolution with finitely generated modules in all dimensions $\leq m$. Given a group G, one defines $\Sigma^m(G;\mathbb{Z})$ to be the set of all homomorphisms $\chi : G \to \mathbb{R}$ such that the submonoid $G_\chi = \chi^{-1}([0,\infty))$ is of type FP_m. Then $\Sigma^0(G;\mathbb{Z}) = V(G)$, $\Sigma^1(G;\mathbb{Z}) = \Sigma^1(G)$ (see [7, Section 5], or [30]), and $\Sigma^m(G) = \Sigma^2(G) \cap \Sigma^m(G;\mathbb{Z})$ for all $m \geq 2$ ([30]; a sketch of proof can also be found in [5]).

Recall that metabelian groups of type FP_2 are finitely presented by the main result of [8]. Thus a metabelian group is of type FP_m if and only if it admits an Eilenberg–MacLane complex with finite m-skeleton. Then [26] and the present paper yield one of the main results of the author's thesis [24], the truth of a part of the Σ^m-conjecture:

6.1. Theorem. *Let G be a metabelian group of finite Prüfer rank. If G is of type FP_m, then the complements, $\Sigma^m(G;\mathbb{Z})^c$ and $\Sigma^m(G)^c$, of the homological and the homotopical geometric invariant coincide and are given by the formula*

$$\operatorname*{conv}_{\leq m} \Sigma^1(G)^c = \Sigma^m(G;\mathbb{Z})^c = \Sigma^m(G)^c.$$

Finally, let us state what we know about the higher geometric invariants of soluble-by-finite groups of finite Prüfer rank (see [27]). Recall that this class includes the class of all constructible soluble-by-finite groups, or, equivalently, the class of all soluble-by-finite groups of type FP_∞ as Kropholler has shown [22].

6.2. Theorem. *Let G be a soluble-by-finite group of finite Prüfer rank. If G is of type FP_m then*

$$\operatorname*{conv}_{\leq m} \Sigma^1(G)^c \subseteq \Sigma^m(G;\mathbb{Z})^c \subseteq \Sigma^m(G)^c.$$

If G is a constructible nilpotent-by-abelian group then $\bigcup_{m\geq 1}\Sigma^m(G)^c$ *is contained in a rationally defined open half space of* $V(G)$, *and*

$$\underset{\leq m}{\operatorname{conv}}\,\Sigma^1(G)^c \subseteq \Sigma^m(G;\mathbb{Z})^c \subseteq \Sigma^m(G)^c \subseteq \operatorname{conv}\Sigma^1(G)^c$$

holds for all $m\in\mathbb{N}$.

References

[1] H. Abels and S. Holz, Higher generation by subgroups, J. Algebra 160 (1993) 287–309.

[2] H. Åberg, Bieri–Strebel valuations (of finite rank), Proc. London Math. Soc. (3) 52 (1986) 269–304.

[3] G. Baumslag and R. Bieri, Constructable solvable groups, Math. Z. 151 (1976) 249–257.

[4] R. Bieri, The homotopical geometric invariant revisited, Preprint, Universität Frankfurt a.M., 1992.

[5] R. Bieri, The geometric invariants of a group – a survey with emphasis on the homotopical approach, in: G.A. Niblo and M.A. Roller, Eds., Geometric Group Theory, Vol. 1, London Math. Soc. Lecture Note Series, Vol. 181 (Cambridge University Press, Cambridge, 1993) 24–36.

[6] R. Bieri, W.D. Neumann and R. Strebel, A geometric invariant for discrete groups, Invent. Math. 90 (1987) 451–477.

[7] R. Bieri and B. Renz, Valuations on free resolutions and higher geometric invariants of groups, Comment. Math. Helv. 63 (1988) 464–497.

[8] R. Bieri and R. Strebel, Valuations and finitely presented metabelian groups, Proc. London Math. Soc. (3) 41 (1980) 439–464.

[9] R. Bieri and R. Strebel, A geometric invariant for modules over an abelian group, J. Reine Angew. Math. 322 (1981) 170–189.

[10] R. Bieri and R. Strebel, A geometric invariant for nilpotent-by-abelian-by-finite groups, J. Pure Appl. Algebra 25 (1982) 1–20.

[11] R. Bieri and R. Strebel, Geometric invariants for discrete groups, Preprint of a monograph (to appear, W. de Gruyter), Universität Frankfurt a.M., 1992.

[12] K.S. Brown, Presentations for groups acting on simply-connected complexes, J. Pure Appl. Algebra 32 (1984) 1–10.

[13] J.M. Corson, Complexes of groups, Proc. London Math. Soc. (3) 65 (1992) 199–224.

[14] J.M. Corson, Groups acting on complexes and complexes of groups, in: R. Charney, M. Davis and M. Shapiro, Eds., Geometric Group Theory, Ohio State University, Math. Research Institute Publ. 3 (de Gruyter, Berlin, 1995).

[15] J.M. Corson, Complexes of groups II and finiteness properties of groups, Preprint, University of Alabama, Tuscaloosa, 1992.

[16] J. Dugundji, Topology (Allyn and Bacon, Boston, 1966).

[17] R. Gehrke, Die höheren geometrischen Invarianten für Gruppen mit Kommutatorrelationen, Dissertation, Universität Frankfurt a.M., 1992.

[18] R. Gehrke, The higher geometric invariants for groups with sufficient commutativity, Preprint, Universität Frankfurt a.M., 1993.

[19] R. Geoghegan, Topological methods in group theory, Draft chapters 1–6 of a book, State University of New York, Binghamton, 1994.

[20] A. Haefliger, Complexes of groups and orbihedra, in: E. Gyhs, A. Haefliger and A. Verjovsky, Eds., Group Theory from a Geometrical Viewpoint (World Scientific, Singapore, 1991) 504–542.

[21] A. Haefliger, Extensions of complexes of groups, Ann. Inst. Fourier 42 (1992) 275–311.

[22] P.H. Kropholler, On groups of type $(FP)_\infty$, J. Pure Appl. Algebra 90 (1993) 55–67.

[23] R.C. Lyndon and P.E. Schupp, Combinatorial group theory, Ergebnisse der Mathematik und ihrer Grenzgebiete, Vol. 89 (Springer, Berlin, 1977).

[24] H. Meinert, Die höheren geometrischen Invarianten Σ^m von Gruppen via Operationen auf CW-Komplexen und der Beweis der Σ^m-Vermutung für metabelsche Gruppen endlichen Prüfer-Ranges, Dissertation, Universität Frankfurt a.M., 1993.

[25] H. Meinert, The higher geometric invariants of direct products of virtually free groups, Comment. Math. Helv. 69 (1994) 39–48.

[26] H. Meinert, The homological invariants for metabelian groups of finite Prüfer rank: a proof of the Σ^m-conjecture, Proc. London Math. Soc. (3) 72 (1996) 385–424.
[27] H. Meinert, On the geometric invariants of soluble groups of finite Prüfer rank, Preprint (to appear in Proceedings of the LMS Durham Symposium on Geometry and Cohomology in Group Theory, London Math. Soc. Lecture Note Series, Cambridge University Press), Universität Frankfurt a.M., 1995.
[28] J.G. Ratcliffe, Finiteness conditions for groups, J. Pure Appl. Algebra 27 (1983) 173–185.
[29] B. Renz, Geometric invariants and HNN-extensions, in: K.N. Cheng and Y.K. Leong, Eds., Group Theory (Proceedings of the Singapore Group Theory Conference, June 1987) (W. de Gruyter, Berlin, 1989) 465–484.
[30] B. Renz, Geometrische Invarianten und Endlichkeitseigenschaften von Gruppen, Dissertation, Universität Frankfurt a.M. (1988).
[31] D.J.S. Robinson, Finiteness Conditions for Soluble Groups, Part 2 (Springer, Berlin, 1972).
[32] H. Schmitt, Endlichkeitsbedingungen für metabelsche Gruppen von endlichem Rang, Diplomarbeit, Universität Frankfurt a.M., 1985.