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A NOTE ON ČECH AND KAN EXTENSIONS OF HOMOTOPY FUNCTORS

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In [1] we claimed the following result:

Theorem ([1], 2.8). *If $(\mathcal{T}, \mathcal{P})$ is a Čech extension pair of subcategories of* TOP *and $F : \mathcal{P} \to \mathcal{A}$ is a homotopy invariant functor into a complete category $\mathcal{A}$, then the Čech extension of F to $\mathcal{T}$ agrees with its Kan extension and the Kan extension taken in the homotopy category composed with the quotient functor.*

A key step in the proof was:

Theorem ([1], 2.4). *If $(\mathcal{T}, \mathcal{P})$ is a Čech extension pair then the codeterminate extension of homotopy from $\mathcal{P}$ to $\mathcal{T}$ is homotopy over $\mathcal{P}$.*

Frei points out in [3] that our proof of this second theorem is defective when $\mathcal{P}$ contains polyhedra which are not locally finite. This defect is of a technical rather than a conceptual nature. In fact, Theorem (1.7) of [1] implies that if the result is true, as indeed it is, there must be a proof using our calculus of codeterminate extensions.

Let $\hat{P} = \{(z_1, z_2) \in P \times P$: z_1 and z_2 be in a single simplex of $P\}$, where P is a polyhedron and $\hat{P}$ has the weak topology. The difficulty in the proof of 2.4 arises when P is not locally finite. In this situation product maps $(f, g) : X \to \hat{P}$ need not be continuous. This problem can be avoided by factoring through the strong topology.

We now give details. All numbered references and notation are from [1].

For a given polyhedron P, let $\kappa : P_s \to P$ be the map described in ([2], p. 354). κ is a homotopy inverse of $i : P \to P_s$, the identity (set) map.

Lemma. *Let $\bar{\kappa} = \kappa i$. Then $\psi = (\bar{\kappa} \times \bar{\kappa})(P \times P)_s \to P \times P$ is continuous and moreover $\psi(\hat{P}_s) \subseteq \hat{P}$.*

Proof. The continuity of ψ follows from the fact that it inherits from κ the property that every point of $(P\times P)_s$ has an open neighborhood whose image lies in a finite subcomplex of $P\times P$ where the two topologies agree.

That $\psi(\hat{P}_s)\subsetneq\hat{P}$ follows immediately from the various definitions of the spaces and maps involved.

Proof of 2.4. We need only show $h^k\supseteq h_{\mathscr{P}}$. Using the notation of 2.1 and 2.6, let $f,g:X\to Y$ be maps in $\mathscr{T}$ such that $fh_{\mathscr{P}}g$ and $\pi:Y\to P$ be a map in $\mathscr{T}$ such that $P\in\text{ob}\,\mathscr{P}$. Then $\pi f\sim\pi g$. Finally, let $Q\in\text{ob}\,\mathscr{P}$ be as in 2.5.

The data required by 1.7 is:

$$Q_1=P,\quad \phi_1=\pi f,\quad \pi_{1,0}=\text{id and } \pi_{1,1}=\bar{\kappa},$$

$$Q_2=\hat{P},\quad \phi_2(x)=\psi(\pi f(x),\theta(\pi'(x),0)),\quad \pi_{2,0}=p_1 \text{ and } \pi_{2,1}=p_2,$$

$$Q_3=Q,\quad \phi_3=\pi',\quad \pi_{3,0}=\bar{\kappa}\theta_0\pi' \text{ and } \pi_{3,1}=\bar{\kappa}\theta_1\pi',$$

$$Q_4=\hat{P},\quad \phi_4(x)=\psi(\theta(\pi'(x),1),\pi g(x)),\quad \pi_{4,0}=p_1 \text{ and } \pi_{4,1}=p_2,$$

$$Q_5=P,\quad \phi_5=\pi g,\quad \pi_{5,0}=\bar{\kappa},\quad \pi_{5,1}=\text{id}.$$

The result now follows from 1.7.

References

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