Journal of Pure and Applied Algebra 28 (1983) 265-277
North-Holland

DUALITY FOR QUOTIENT MODULES AND A CHARACTERIZATION OF REFLEXIVE MODULES

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Communicated by H. Bass
Received 8 November 1982
Revised 30 November 1982

Introduction

Let R be a left Artinian ring, U a left R-module and $T=\mathrm{End}({}_RU)$. From a well known theorem of Morita duality the following conditions are equivalent (see [1], [11], [18]):

(i) There exists a duality between the category $\mathcal{F}_1$ of all finitely generated left R-modules and the category $\mathcal{F}_2$ of all finitely generated right T-modules via functors $\mathrm{Hom}(-, {}_RU): \mathcal{F}_1 \to \mathcal{F}_2$ and $\mathrm{Hom}(-, U_T): \mathcal{F}_2 \to \mathcal{F}_1$.

(ii) ${}_RU$ is a finitely generated injective cogenerator (in the category of all left R-modules).

When one of the above equivalent conditions is satisfied, T is right Artinian and the right T-module U_T is a finitely generated injective cogenerator.

In this paper we shall extend this concept of Morita duality (of Artinian rings) to a situation of certain hereditary torsion theories. Let R and T be rings and U an (R,T)-bimodule. Let us denote $I({}_RU)$ (resp. $I(U_T)$) the injective hull of ${}_RU$ (resp. U_T). Throughout this paper τ_1 (resp. τ_2) denotes a hereditary torsion theory (see [4], [15]) with respect to the ring R (resp. T) cogenerated by $I({}_RU)$ (resp. $I(U_T)$). The τ_1-*torsion free class* means a class of all left R-modules which are embedded in direct products of copies of $I({}_RU)$. A submodule M of a left R-module N is τ_1-*closed* if N/M is τ_1-torsion free. On the other hand, M is called τ_1-*dense* in N, if $\mathrm{Hom}({}_RN/M, {}_RI({}_RU))=0$. If N has a finitely generated τ_1-dense submodule, N is said to be τ_1-*finitely generated.* Let $Q_{\tau_1}(\ )$ (resp. $Q_{\tau_2}(\ )$) be the localization functor with respect to τ_1 (resp. τ_2). We shall say that a left R-module M is a τ_1-*quotient module*, if $Q_{\tau_1}(M)=M$. Let L be a homomorphic image of a finite direct sum of copies of a left R-module K. Then, L is called *finitely generated by K.*

Now, assume R satisfies the descending chain condition on τ_1-closed left ideals. When ${}_RU$ is faithful and $T=\mathrm{End}({}_RU)$, in Section 2 it will be proved that the following conditions are equivalent.

(i) (a) $Q_{\tau_1}(U)=Q_{\tau_2}(U)(=\bar{U}$, say).

(b) There exists a duality between the category $\mathscr{C}_1$ of all τ_1-finitely generated τ_1-quotient left R-modules and the category $\mathscr{C}_2$ of all τ_2-finitely generated τ_2-quotient right T-modules via functors

$$\mathrm{Hom}(-, {}_R\bar{U}):\mathscr{C}_1\to\mathscr{C}_2 \quad \text{and} \quad \mathrm{Hom}(-, \bar{U}_T):\mathscr{C}_2\to\mathscr{C}_1.$$

(ii) ${}_RU$ is τ_1-finitely generated and every τ_1-torsion free left R-module which is finitely generated by $R\oplus U$ is embedded in a direct product of copies of ${}_RU$.

Furthermore, any of these two equivalent statements implies that T satisfies the descending chain condition on τ_2-closed right ideals, U_T is τ_2-finitely generated and every τ_2-torsion free right T-module which is finitely generated by $T\oplus U$ is embedded in a direct product of copies of U_T.

It is to be noted that this result is closely connected with a condition for left QF-3 rings, i.e., rings with minimal faithful left modules (cf. [17]), to be right QF-3 ([3], [12]).

A typical example of ${}_RU$ in this result is a ring R such that every finitely generated submodule of $I({}_RR)$ is torsionless and R satisfies descending chain condition on annihilator left ideals, i.e., a ring R with a semi-primary (left and right) QF-3 maximal two-sided quotient ring [9]. When R has this condition, in Section 3 (as an application of the considerations in Section 2) we shall give a necessary and sufficient condition for a finitely generated left R-module to be reflexive. Especially, if R is a QF-3 ring with the ascending chain condition on annihilator left (and hence right) ideals, it will be proved that every reflexive module is contained in a finitely generated projective module. Consequently, every reflexive left module over a left Artinian QF-3 ring is finitely generated.

Throughout this paper every ring has an identity, every homomorphism between modules will be written on the opposite side of scalars and DCC (ACC) means the descending (ascending) chain condition.

1. Preliminaries

In this section R, T are rings and U is a left R-module. A left R-module M is *U-torsionless*, if M is embedded in a direct product of copies of ${}_RU$.

Consider the following conditions.

(i) Every finitely generated submodule of $I({}_RU)$ is U-torsionless.

(ii) Every finitely generated τ_1-torsion free module is U-torsionless.

(iii) Every τ_1-torsion free module which is finitely generated by $R\oplus U$ is U-torsionless.

Then, (i)$\Leftrightarrow$(ii)$\Leftarrow$(iii). If ${}_RU$ is finitely generated, these three conditions are equivalent.

Proposition 1.1. *Assume that every τ_1-torsion free left R-module which is finitely generated by U is U-torsionless and $T = \mathrm{End}({}_RU)$. Then, T is embedded in $S = \mathrm{End}({}_RQ_{\tau_1}(U))$ as a τ_2-dense essential right T-submodule.*

Proof. Since ${}_RU$ is τ_1-torsion free, $Q_{\tau_1}(U) \supseteq U$. Assume $S \ni f \neq 0$ and $U \ni x \neq 0$. Since $U + Uf$ is a homomorphic image of $U \oplus U$, it is U-torsionless. Then, there exists $g_1, g_2 \in \mathrm{Hom}({}_RU + Uf, {}_RU)$ such that $(x)g_1 \neq 0$ and $(Uf)g_2 \neq 0$. Since $Q_{\tau_1}(U + Uf) = Q_{\tau_1}(U)$, g_1 is extended to an element $\bar{g}_1 \in S$ uniquely. The fact that $U\bar{g}_1 \subset U$ and $Uf\bar{g}_1 \subset U$ implies $\bar{g}_1$ and $f\bar{g}_1$ are elements of T. As $(x)\bar{g}_1 \neq 0$, we have $\mathrm{Hom}(S/T_T, I(U_T)_T) = 0$ by [7, p. 3, Proposition 0.3]. From a same argument g_2 is extended to an element $\bar{g}_2$ of S such that $\bar{g}_2 \in T$ and $0 \neq f\bar{g}_2 \in T$. It follows S_T is an essential extension of T_T and this completes the proof.

Let M be a τ_1-torsion free left R-module. By [5] M has DCC and ACC on τ_1-closed submodules, if and only if there exists a maximal chain

$$M = M_0 \supsetneqq M_1 \supsetneqq \cdots \supsetneqq M_k = 0 \tag{$*$}$$

of τ_1-closed submodules. In this case M is said to have *τ_1-finite length* and we denote *τ_1-length* ${}_RM = k$. The chain $(*)$ is called *τ_1-composition series* of M. When R has DCC on τ_1-closed left ideals, by a result of Miller and Teply [10] every left R-module with DCC on τ_1-closed submodules has ACC on τ_1-closed submodules.

Assume ${}_RM$ has DCC on τ_1-closed submodules and there exists an R-monomorphism $\alpha : M \to \prod_{i \in \Lambda} K^{(i)}$, where $K^{(i)}$ is a copy of a τ_1-torsion free left R-module K. Let α_i be the canonical mapping $M \to K^{(i)}$, $i \in \Lambda$. Since $\ker \alpha_i$ is τ_1-closed in M, there exists a finite subset F of Λ such that $\bigcap_{i \in F} \ker \alpha_i = 0$, i.e., M is embedded in a finite direct sum of copies of K.

Lemma 1.2. *Assume R has DCC on τ_1-closed left ideals. If ${}_RU$ is τ_1-finitely generated and every finitely generated submodule of $I({}_RU)$ is U-torsionless, $Q_{\tau_1}(U)$ contains an injective R-submodule E such that $I({}_RU)$ is E-torsionless. Consequently, every τ_1-torsion free module is $Q_{\tau_1}(U)$-torsionless.*

Proof. It is easily checked that ${}_RU$ has τ_1-finite length, since ${}_RU$ is τ_1-finitely generated. Then, $I({}_RU)$ is finite Goldie dimensional. Put $I({}_RU) = M_1 \oplus \cdots \oplus M_n$, where M_i is injective uniform. Suppose there is a M_i which is not τ_1-finitely generated. Then, we have an infinite chain $0 = L_0 \subsetneqq L_1 \subsetneqq \cdots \subsetneqq L_j \subsetneqq \cdots$ of finitely generated submodule of M_i such that L_j is not τ_1-dense in L_{j+1}, $j = 0, 1, \ldots$. Assume that τ_1-length ${}_RU = k$. As L_{k+1} is a finitely generated submodule of $I({}_RU)$, it is $Q_{\tau_1}(U)$-torsionless. This implies L_{k+1} is embedded in a finite direct sum of copies of $Q_{\tau_1}(U)$, since L_{k+1} has DCC on τ_1-closed submodules. Clearly L_{k+1} is a uniform left R-module. Hence there exists an R-monomorphism $f : L_{k+1} \to Q_{\tau_1}(U)$. We have a chain $0 = L_0f \subsetneqq L_1f \subsetneqq \cdots \subsetneqq L_{k+1}f$ of submodules of $Q_{\tau_1}(U)$ such that L_jf is not τ_1-dense in $L_{j+1}f$, $j = 0, 1, \ldots, k$. However, this is a contradiction, since

τ_1-length $Q_{\tau_1}(U)=k$. It follows that each M_i has a finitely generated τ_1-dense submodule V_i, say. As is shown above there exists an R-monomorphism from V_i to $Q_{\tau_1}(U)$ and hence from M_i to $Q_{\tau_1}(U)$. Then, we can see that there exists a subclass $\{M_{p_1}, M_{p_2}, \dots, M_{p_t}\}$ of $\{M_1, M_2, \dots, M_n\}$ such $E=\bigoplus_{j=1}^{t} M_{p_j}$ is embedded in $Q_{\tau_1}(U)$ and $I({}_RU)$ is E-torsionless.

Now, in the following a τ_1-torsion free left R-module M is called *τ_1-cocritical*, if $M\neq 0$ and every non-zero submodule is τ_1-dense.

Lemma 1.3. *Assume every τ_1-torsion free left R-module which is finitely generated by $R\oplus U$ is U-torsionless and $T=\operatorname{End}({}_RU)$. Then,*

(i) *If M is a cyclic τ_2-cocritical right T-module,* $\operatorname{Hom}(M_T, U_T)$ *is a τ_1-cocritical left R-module.*

(ii) *If R has DCC on τ_1-closed left ideals and N is a τ_1-cocritical left R-module,* $\operatorname{Hom}({}_RN, {}_RU)$ *is a τ_2-cocritical right T-module, when* $\operatorname{Hom}({}_RN, {}_RU)\neq 0$.

Proof. (i) There exists a maximal τ_2-closed right ideal J of T such that $M\cong T/J$. Put $l_U(J)=\{x\in U; xJ=0\}$. Then, ${}_R\operatorname{Hom}(M_T, U_T)\cong {}_Rl_U(J)$. It is easy to see that $U/l_U(J)$ is ${}_RU$-torsionless and then $l_U(J)$ is a τ_1-closed submodule of ${}_RU$. Suppose there exists a proper τ_1-closed submodule V $(\neq 0)$ of $l_U(J)$. As V is τ_1-closed in U, U/V is U-torsionless. Hence there exists $t\in T$ such that $l_U(J)t\neq 0$ and $Vt=0$. Clearly $t\notin J$ and $J+tT$ is a τ_2-dense right ideal of T. However, this is a contradiction, since $V(J+tT)=0$. It follows that $l_U(J)$ is a τ_1-cocritical left R-module.

(ii) Assume $\operatorname{Hom}({}_RN, {}_RU)\ni f_1$ $(\neq 0)$, f_2 and $U\ni u\neq 0$. By Lemma 1.2 there exists an injective left R-submodule E of $Q_1(U)$ such that $I({}_RU)$ is E-torsionless. There exists an R-homomorphism $\theta: Q_{\tau_1}(U)\to E$ such that $(u)\theta\neq 0$. As ${}_RN$ is τ_1-cocritical and $\operatorname{Im} f_1$ is a non-zero τ_1-torsion free module, f_1 is a monomorphism. Hence there exists an R-homomorphism $\phi: U\to E$ such that $f_2\theta=f_1\phi$, since ${}_RE$ is injective. As θ and ϕ can be regarded to be elements of $\operatorname{End}({}_RQ_{\tau_1}(U))$, by Proposition 1.1 there exists a τ_2-dense right ideal D of T such that $\theta D\subseteq T$ and $\phi D\subseteq T$. We can choose $d\in D$ such that $(u)\theta d\neq 0$. Since $f_2\cdot\theta d=f_1\cdot\phi d$, we have $f_2\cdot\theta d\in f_1T$. This implies $\operatorname{Hom}([\operatorname{Hom}({}_RN, {}_RU)/f_1T]_T, I(U_T)_T)=0$. Thus $\operatorname{Hom}({}_RN, {}_RU)_T$, which is U_T-torsionless, is τ_2-cocritical unless it is 0.

Assume U is an (R, T)-bimodule. Let X be a left R-module and Y a right T-module. We shall say that X and Y form an *orthogonal pair* with respect to ${}_RU_T$, provided there exists a bilinear mapping $({}_RX, Y_T)\to {}_RU_T$ denoted by (,) such that $(x, Y)=0$ implies $x=0$ and $(X, y)=0$ implies $y=0$. If W is a subset of X (resp. Y), we denote by $r(W)$ (resp. $l(W)$) the annihilator $\{y\in Y;\ (W, y)\}=0$ (resp. $\{x\in X; (x, W)\}=0$). It is easily checked that $Y/r(W)$ is U-torsionless. Furthermore, it is evident that ${}_RX$ (resp. Y_T) is τ_1-torsion free (resp. τ_2-torsion free).

Proposition 1.4. *Let* $T=\operatorname{End}({}_R(U)$. *Assume that R satisfies DCC on τ_1-closed left* *ideals, $_RU$ is τ_1-finitely generated and every τ_1-torsion free left R-module which is finitely generated by $R\oplus U$ is U-torsionless. If a left R-module X and a right T-module Y form an orthogonal pair with respect to $_RU_T$, then τ_1-length $_RX=$ τ_2-length Y_T in the case where τ_1-length $_RX<\infty$ or τ_2-length $Y_T<\infty$.*

Proof. Assume $_RX$ has τ_1-finite length. Let $X=X_0\supsetneqq X_1\supsetneqq\cdots\supsetneqq X_n=0$ be a τ_1-composition series. Then, we have a chain $0=r(X_0)\subseteq r(X_1)\subseteq\cdots\subseteq r(X_n)=Y$ of τ_2-closed submodules of Y_T. One can check that $r(X_{i+1})/r(X_i)$ is embedded in the right T-module $\mathrm{Hom}(_RX_i/X_{i+1}, {}_RU)$ canonically, $i=0,\dots,n-1$. Since X_i/X_{i+1} is τ_1-cocritical, by lemma 1.3 $r(X_{i+1})/r(X_i)$ is τ_2-cocritical unless it is 0. This implies τ_1-length $_RX\geq\tau_2$-length Y_T. Conversely, assume there exists a τ_2-composition series $Y=Y_0\supsetneqq Y_1\supsetneqq\cdots\supsetneqq Y_m=0$. Let $y\in Y_i\setminus Y_{i+1}$. Suppose $l(Y_i)\subsetneqq l(yT+Y_{i+1})$. Then, there exists $x\in X$ such that $(x,Y_i)\neq 0$ and $(x,yT+Y_{i+1})=0$. Hence $\mathrm{Hom}([Y_i/yT+Y_{i+1}]_T, U_T)\neq 0$. This is a contradiction, since $yT+Y_{i+1}$ is a τ_2-dense submodule of Y_i. It follows that

$$l(Y_i)=l(yT+Y_{i+1})\quad\text{and}\quad l(Y_{i+1})/l(Y_i)=l(Y_{i+1})/l(yT+Y_{i+1}).$$

So $l(Y_{i+1})/l(Y_i)$ is embedded in $\mathrm{Hom}([yT+Y_{i+1}/Y_{i+1}]_T, U_T)$. As $yT+Y_{i+1}/Y_{i+1}$ is a cyclic τ_2-cocritical right T-module, by Lemma 1.3 $\mathrm{Hom}([yT+Y_{i+1}/Y_{i+1}]_T, U_T)$ is a τ_1-cocritical left R-module. Thus we have τ_2-length $Y_T\geq\tau_1$-length $_RX$ and this completes the proof.

Lemma 1.5. *Assume U is an (R, T)-bimodule and M is a left R-module. Then, the right T-module* $\mathrm{Hom}(_RM, {}_RU)$ *can be embedded in a direct product of copies of U_T as a τ_2-closed submodule.*

Proof. There exists a T-monomorphism $\theta:\mathrm{Hom}(_RM, {}_RU)\to\prod_{x\in M}U^{(x)}$ *defined by* $p_m\theta(f)=(m)f$, $f\in\mathrm{Hom}(_RM, {}_RU)$ and $m\in M$, where $U^{(x)}$ is a copy of U_T and p_m is the projection $\prod_{x\in M}U^{(x)}\to U^{(m)}$. Suppose $y\in\prod_{x\in M}U^{(x)}$ and there exists a τ_2-dense right ideal D of T such that $yD\subseteq\mathrm{Im}\,\theta$. Define a mapping $\phi:M\to U$ by $(x)\phi=(y)p_x$, $x\in M$. If we show that ϕ is an R-homomorphism, then $y=\theta(\phi)\in\mathrm{Im}\,\theta$ and the proof will be completed. Let $d\in D$ be an arbitrary element. We can select $f\in\mathrm{Hom}(_RM, {}_RU)$ such that $(y)p_xd=(yd)p_x=(x)f$, $x\in M$. Therefore, for every $r\in R$ and $x\in M$ we have

$$\{(rx)\phi-r\cdot(x)\phi\}d=\{(y)p_{rx}-r\cdot(y)p_x\}d=(rx)f-r\cdot(x)f=0.$$

Since D is a τ_2-dense right ideal of T, $(rx)\phi=r\cdot(x)\phi$. Moreover, for every $x_1,x_2\in M$

$$\begin{aligned}[(x_1+x_2)\phi-\{(x_1)\phi+(x_2)\phi\}]d&=[(y)p_{x_1+x_2}-\{(y)p_{x_1}+(y)p_{x_2}\}]d\\&=(x_1+x_2)f-\{(x_1)f+(x_2)f\}=0.\end{aligned}$$

It follows that ϕ is an R-homomorphism.

2. Duality for quotient modules

A left R-module M has ACC on τ_1-closed submodules, if and only if every submodule of M is τ_1-finitely generated (see [15, p. 263]). Now, we are able to prove the folowing:

Theorem 2.1. *Let R be a ring, U a faithful left R-module and $T=\mathrm{End}(_RU)$. Assume R has DCC on τ_1-closed left ideals, $_RU$ is τ_1-finitely generated and every τ_1-torsion free left R-module which is finitely generated by $R\oplus U$ is U-torsionless. Then:*

(i) T has DCC on τ_2-closed right ideals, U_T is τ_2-finitely generated and every τ_2-torsion free right T-module which is finitely generated by $T\oplus U$ is U-torsionless.

(ii) $Q_{\tau_1}(U)=Q_{\tau_2}(U)$ $(=\bar{U}$, say), $\mathrm{End}(_R\bar{U})=Q_{\tau_2}(T)$ and $\mathrm{End}(\bar{U}_T)=Q_{\tau_1}(R)$.

Proof. (i) Since $_RU$ and T_T (resp. $_RR$ and U_T) form an orthogonal pair with respect to $_RU_T$, by Proposition 1.4 T_T (resp. U_T) has DCC on τ_2-closed submodules. U_T also has ACC on τ_2-closed submodules and hence is τ_2-finitely generated. Let P be a finite direct sum of copies of the right T-module $T\oplus U$. Assume N is a τ_2-closed submodule of P. It is sufficient to show that P/N is U-torsionless. Set $P^*=\mathrm{Hom}(P_T,U_T)$. Then, $_RP^*$ and P_T form an orthogonal pair with respect to $_RU_T$. Clearly P_T has τ_2-finite length and we can check that there exist a τ_2-composition series $P=P_0\supsetneqq P_1\supsetneqq\cdots\supsetneqq P_k=N\supsetneqq\cdots\supsetneqq P_n=0$. By the same argument as in the proof of Proposition 1.4

$$0=l(P_0)\subsetneqq l(P_1)\subsetneqq\cdots\subsetneqq l(P_k)=l(N)\subsetneqq\cdots\subsetneqq l(P_n)=P^*$$

is a τ_1-composition series and we can deduce $r(l(P_i))=P_i$, $i=0,\dots,n$. Since $r(l(N))=N$, there exists a T-monomorphism $\phi:P/N\to\prod_{f\in l(N)}U^{(f)}$ defined by $p_g\phi(x+N)=g(x)$, $x\in P$ and $g\in l(N)$, where p_g is the projection $\prod_{f\in l(N)}U^{(f)}\to U^{(g)}$.

(ii) Assume $Q_{\tau_1}(U)\ni x\neq 0$ and $U\ni y\neq 0$. As $Rx+U$ is a homomorphic image of the left R-module $R\oplus U$ and hence U-torsionless, there exists $g_1,g_2\in\mathrm{Hom}(_RRx+U,{}_RU)$ such that $(y)g_1\neq 0$, $(x)g_2\neq 0$. By a same argument as in the proof of Proposition 1.1 g_1,g_2 are extended to $\bar{g}_1,\bar{g}_2\in\mathrm{End}(_RQ_{\tau_1}(U))$, which are contained in T. Since $(x)\bar{g}_1\subseteq U$ and $(y)\bar{g}_1\neq 0$, we have $\mathrm{Hom}(Q_{\tau_1}(U)/U_T, I(U_T)_T)=0$. Furthermore, $Q_{\tau_1}(U)_T$ is an essential extension of U_T, since $0\neq(x)\bar{g}_2\in U$. This implies $Q_{\tau_1}(U)\subseteq Q_{\tau_2}(U)$. Next, put $R'=\mathrm{End}(U_T)$. Let τ_1' be the hereditary torsion theory with respect to the ring R' cogenerated by $I(_{R'}U)$. Since $_RR'$ and U_T form an orthogonal pair with respect to $_RU_T$, by Proposition 1.4 and its left right symmetry, τ_1-length $_RR'=\tau_2$-length $U_T=\tau_1$-length $_RR$. Hence $_RR$ is a τ_1-dense submodule of $_RR'$. Then, we can check that $Q_{\tau_1}(U)=Q_{\tau_1'}(U)$, where $Q_{\tau_1'}(\ )$ is the localization functor with respect to τ_1'. Now, as is shown above we can deduce that $Q_{\tau_2}(U)\subseteq Q_{\tau_1'}(U)$. Thus $Q_{\tau_1}(U)=Q_{\tau_2}(U)$ $(=\bar{U}$, say). Put $S=\mathrm{End}(_R\bar{U})$ and $Q=\mathrm{End}(\bar{U}_T)$. By Lemma 1.5 S_T is embedded in a direct product of copies of the

τ_2-quotient module $\bar{U}_T$ as a τ_2-closed submodule. Hence S_T is a τ_2-quotient module and it follows from Proposition 1.1 that $S=Q_{\tau_2}(T)$. Moreover, R' is a τ_1'-dense essential left R-submodule of Q by Proposition 1.1. Hence R is a τ_1-dense essential left R-submodule of Q, too, and $Q=Q_{\tau_1}(R)$.

Remark 1. If in Theorem 2.1 R has a minimal τ_1-dense left ideal D, then $D\bar{U}\subseteq U$ and hence $I(U_T)$ is U-torsionless, since by left right symmetry of Lemma 1.2 $I(U_T)$ is $\bar{U}$-torsionless.

Recently it is proved in [9] that a ring R has a semi-primary QF-3 maximal two-sided quotient ring, if and only if R has DCC on annihilator left ideals and every finitely generated submodule of $I({}_RR)$ is torsionless (where QF-3 means left and right QF-3). Assume R satisfies this condition and U is a finitely generated torsionless faithful left R-module. Then, annihilator left ideals coincide with τ_1-closed left ideals. Furthermore, every finitely generated submodule of $I({}_RU)$ is torsionless and hence U-torsionless. Let $T=\mathrm{End}({}_RU)$. By Theorem 2.1 T has DCC on τ_2-closed right ideals and every finitely generated submodule of $I(T_T)$ is U-torsionless. If the trace ideal of ${}_RU$ has no nonzero right annihilator in R, we can deduce that the fiathful right T-module U is torsionlesss. This implies T has DCC on annihilator right ideals and every finitely generated submodule of $I(T_T)$ is torsionless. Thus we have the following result.

Corollary 2.2. *Let U be a finitely generated torsionless faithful left module over a ring R and the trace ideal of U has no non-zero right annihilator in R. If R has a semi-primary* QF-3 *maximal two-sided quotient ring, then so does* $\mathrm{End}({}_RU)$.

Now, using Theorem 2.1 we can prove:

Theorem 2.3. *Let U be a faithful left R-module over a ring R and $T=\mathrm{End}({}_RU)$. If R satisfies DCC on τ_1-closed left ideals, then the following conditions are equivalent.*

(i) (a) $Q_{\tau_1}(U)=Q_{\tau_2}(U)$ $(=\bar{U}$, *say).*

(b) *There exists a duality between the category $\mathscr{C}_1$ of all τ_1-finitely generated τ_1-quotient left R-modules and the category $\mathscr{C}_2$ of all τ_2-finitely generated τ_2-quotient right T-modules via functors*

$$\mathrm{Hom}(-,{}_R\bar{U}):\mathscr{C}_1\to\mathscr{C}_2 \quad \textit{and} \quad \mathrm{Hom}(-,\bar{U}_T):\mathscr{C}_2\to\mathscr{C}_1.$$

(ii) ${}_RU$ *is τ_1-finitely generated and every τ_1-torsion free left R-module which is finitely generated by $R\oplus U$ is U-torsionless.*

Proof. (i)$\Rightarrow$(ii) Put $F=\mathrm{Hom}(-,{}_R\bar{U})$ and $G=\mathrm{Hom}(-,\bar{U}_T)$, the canonical contravariant functors between the category of all left R-modules and the category of all right T-modules. It is evident that ${}_R\bar{U}\cong{}_RG(Q_{\tau_2}(T))\in\mathscr{C}_1$. Hence ${}_R\bar{U}$ is

τ_1-finitely generated. Since R has DCC on τ_1-closed left ideals, ${}_R\bar{U}$ has DCC on τ_1-closed submodules and then ${}_RU$ is τ_1-finitely generated. Put $S=\mathrm{End}({}_R\bar{U})$. Since $Q_{\tau_2}(T)\cong FG(Q_{\tau_2}(T))\cong F(\bar{U})=S$, S is the ring of quotient of T with respect to τ_2. Let L be a τ_1-torsion free left R-module such that there exists an R-epimorphism

$$f:\bigoplus_{i=1}^{n}(R^{(i)}\oplus U^{(i)})\to L,$$

where $R^{(i)}$ and $U^{(i)}$ are copies of ${}_RR$ and ${}_RU$ respectively. Assume $L\ni a\neq 0$. Since ${}_RL$ has DCC on τ_1-closed submodules and hence τ_1-finitely generated, $Q_{\tau_1}(L)\in\mathscr{C}_1$. It follows that $Q_{\tau_1}(L)\cong GF(Q_{\tau_1}(L))$ and hence we can see that L is ${}_R\bar{U}$-torsionless. Therefore, there exists an R-homomorphism $h:L\to\bar{U}$ such that $(a)h\neq 0$. Let ϕ_i (resp. θ_i) be the canonical mapping $R^{(i)}\to\bar{U}$ (resp. $U^{(i)}\to\bar{U}$) induced from

$$f\cdot h:\bigoplus_{i=1}^{n}(R^{(i)}\oplus U^{(i)})\to\bar{U},$$

$i=1,\dots,n$. Write $u_i=(1_i)\phi_i$, where 1_i is the identity of $R^{(i)}$. On other hand, θ_i is extended to an element $\bar{\theta}_i$ of S. We can select a τ_2-dense right ideal D of T such that $u_iD\subseteq U$ and $\bar{\theta}_iD\subseteq T$, $i=1,\dots,n$. Now, there exists $d\in D$ such that $(a)h\cdot d\neq 0$. We can easily check that $h\cdot d\in\mathrm{Hom}({}_RL,{}_RU)$ and hence L is ${}_RU$-torsionless.

(ii)$\Rightarrow$(i) By Theorem 2.1 $Q_{\tau_1}(U)=Q_{\tau_2}(U)$ $(=\bar{U})$ and $S=\mathrm{End}({}_R\bar{U})$ is a ring of quotient of T with respect to τ_2. Assume $M\in\mathscr{C}_1$. By Lemma 1.2 the τ_1-torsion free module M is ${}_R\bar{U}$-torsionless. So ${}_RM$ and $F(M)_S$ form an orthogonal pair with respect to ${}_R\bar{U}_S$. It is evident that τ_1 is the hereditary torsion theory cogenerated by $I({}_R\bar{U})$. Let $\bar{\tau}_2$ be the hereditary torsion theory with respect to the ring S cogenerated by $I(\bar{U}_S)$. Since ${}_R\bar{U}$ is τ_1-finitely generated, by Proposition 1.4 τ_1-length ${}_RM=\bar{\tau}_2$-length $F(M)_S$. We can easily check that $\bar{\tau}_2$-length $F(M)_S=\tau_2$-length $F(M)_T$. Hence $F(M)_T$ is τ_2-finitely generated and is contained in $\mathscr{C}_2$, because $F(M)_T$ is embedded in a direct product of copies of $\bar{U}_T$ as a τ_2-closed submodule from Lemma 1.5. Moreover, τ_1-length ${}_RM=\tau_1$-length ${}_RGF(M)$, since ${}_R\mathrm{Hom}(F(M)_S,\bar{U}_S)$ $(={}_RGF(M))$ and $F(M)_S$ form an orthogonal pair with respect to ${}_R\bar{U}_S$. This implies that the τ_1-quotient $\bar{U}$-torsionless module M is embedded in the τ_1-torsion free module $GF(M)$ as a τ_1-dense submodule and hence ${}_RM\cong{}_RGF(M)$ canonically. By the left right symmetry we have that for every $N\in\mathscr{C}_2$, $G(N)\in\mathscr{C}_1$ and $N_T\cong FG(N)_T$ canonically. This completes the proof.

Now, in the following a submodule M of a left R-module N is said to be ${}_RR$*-rationally closed* in N, provided N/M is $I({}_RR)$-torsionless. On the other hand, M is called ${}_RR$*-dense* in N, if $\mathrm{Hom}({}_RN/M,{}_RI({}_RR))=0$. If M is embedded in a direct product of copies of $I({}_RR)$ as an ${}_RR$-rationally closed submodule, we shall say $I({}_RR)$*-dominant dimension* $M\geq 2$. Let $\mathscr{D}_1=\{$left R-module X; $I({}_RR)$-dominant dimension $X\geq 2$ and X has a finitely generated ${}_RR$-dense submodule$\}$ and $\mathscr{D}_2=\{$right R-module Y; $I(R_R)$-dominant dimension $Y\geq 2$ and Y has a finitely generated R_R-dense submodule$\}$.

Corollary 2.4. *The following conditions are equivalent for a ring R:*

(i) *R has a maximal two-sided quotient ring Q such that Q has DCC on both annihilator left ideals and annihilator right ideals, and there exists a duality between $\mathscr{D}_1$ and $\mathscr{D}_2$ via functors*

$$\mathrm{Hom}(-, {}_RQ): \mathscr{D}_1 \to \mathscr{D}_2 \quad \textit{and} \quad \mathrm{Hom}(-, Q_R): \mathscr{D}_2 \to \mathscr{D}_1.$$

(ii) *R has DCC on annihilator left ideals and every finitely generated submodule of $I({}_RR)$ is torsionless.*

Proof. Clearly, R has DCC on annihilator left ideals (resp. ${}_RR$-rationally closed left ideals), if and only if so does its maximal left quotient ring.

(ii)⇒(i) This is immediate from Theorem 2.1 and 2.3, since every rationally closed left ideal coincides with an annihilator left ideal (cf. [16]).

(i)⇒(ii) Let K be a ${}_QQ$-rationally closed left ideal of Q. Since Q/K is a submodule of an R-module contained in $\mathscr{D}_1$, it is Q-torsionless and hence K is an annihilator left ideal of Q. So R has DCC on ${}_RR$-rationally closed left ideals.

Remark 2. Let R be a left QF-3 ring with DCC on annihilator left ideals and Re an injective faithful left ideal which is embedded in every faithful left R-module, where e is an idempotent. As ${}_RRe$ is a dominant module (see [6]), $I(Re_{eRe})$ is a cogenerator. If we put ${}_RU_T = {}_RRe_{eRe}$, the category $\mathscr{A}$ of all finitely generated right eRe-modules coincides with $\mathscr{C}_2$ (in Theorem 2.3). Hence there exists a duality between $\mathscr{D}_1$ and $\mathscr{A}$.

3. Reflexive modules

Let M be a left R-module. Write $M^* = \mathrm{Hom}({}_RM, {}_RR)$ and $M^{**} = \mathrm{Hom}(M_R^*, R_R)$. If ${}_RM \cong {}_RM^{**}$ canonically, M is called *reflexive*.

Theorem 3.1. *Assume R is a ring with DCC on annihilator left ideals and every finitely generated submodule of $I({}_RR)$ is torsionless. Then, a finitely generated left R-module X is reflexive, if and only if X is embedded in a direct product of copies of ${}_RR$ as an ${}_RR$-rationally closed submodule.*

Proof. The 'only if' part is an immediate consequence of Lemma 1.5.

Conversely, assume there exists an R-monomorphism $\phi: X \to \prod_{i\in I} R^{(i)}$ such that $\mathrm{Im}\,\phi$ is ${}_RR$-rationally closed submodule of $\prod_{i\in I} R^{(i)}$, where $R^{(i)}$ is a copy of ${}_RR$. If we put ${}_RU_T = {}_RR_R$, by Proposition 1.4 τ_1-length ${}_RX = \tau_2$-length $X_R^* = \tau_1$-length ${}_RX^{**}$. Let $\sigma: X \to X^{**}$ be the canonical R-monomorphism. Then, $\mathrm{Im}\,\sigma$ is ${}_RR$-dense in X^{**}. Let p_j be the projection

$$\prod_{i\in I} R^{(i)} \to R^{(j)}, \quad j\in I.$$

Since $\phi\cdot p_j\in X^*$, we can define an R-homomorphism $\theta: X^{**}\to\prod_{i\in I}R^{(i)}$ by $(\alpha)\theta\cdot p_j=\alpha(\phi\cdot p_j)$, $\alpha\in X^{**}$ and $j\in I$. Then, $\operatorname{Ker}\theta\cap\operatorname{Im}\sigma=0$ and clearly $\operatorname{Im}\sigma$ is an essential submodule of ${}_RX^{**}$. So we have θ is a monomorphism. It is evident that $\operatorname{Im}\phi=(\operatorname{Im}\sigma)\theta\subseteq\operatorname{Im}\theta\subseteq\prod_{i\in I}R^{(i)}$ and $(\operatorname{Im}\sigma)\theta$ is an ${}_RR$-dense submodule of $\operatorname{Im}\theta$. It follows that $(\operatorname{Im}\sigma)\theta=\operatorname{Im}\theta$ and hence $\operatorname{Im}\sigma=X^{**}$, since $(\operatorname{Im}\sigma)\theta$ is ${}_RR$-rationally closed in $\prod_{i\in I}R^{(i)}$.

In [2] H. Bass has proved that a left R-module K can be embedded in $\bigoplus_{i=1}^n R^{(i)}$ and $\bigoplus_{i=1}^n R^{(i)}/K$ is torsionless, if and only if $K\cong\operatorname{Hom}(B_R, R_R)$, where B is a right R-module generated by n elements. Therefore, every finitely generated reflexive left module over a right Noetherian ring can be embedded in a finitely generated free R-module by this way.

As a generalization of a result obtained by Morita [12] we have:

Corollary 3.2. *Let R be a right Noetherian ring such that every finitely generated submodule of $I({}_RR)$ is torsionless. Then, a finitely generated left R-module X is reflexive, if (and only if) X is embedded in a finitely generated free left R-module F and F/X is torsionless.*

Remark 3. A submodule M of a left R-module N is said to be *closed*, if M has no proper essential extension in N. Let R be a semi-prime (two-sided) Noetherian ring. Then, R has a classical two-sided quotient ring Q_{cl} such that $Q_{cl}=I({}_RR)=I(R_R)$. So every finitely generated left R-module X is reflexive, if and only if X is embedded in a finitely generated free left R-module as a closed submodule.

Now, in the following we shall study reflexive modules over QF-3 ring without assuming 'finitely generated'.

Proposition 3.3. *Let M be a reflexive left module over a left* QF-3 *ring R. Then, every ${}_RR$-rationally closed submodule of M is reflexive.*

Proof. Let L be an ${}_RR$-rationally closed submodule of M and $\alpha\in L^{**}$. We can define $\bar\alpha: M_R^*\to R_R$ by $\bar\alpha(f)=\alpha(f\,|\,L)$, $f\in M^*$, where $f\,|\,L$ is the restriction of f. Since ${}_RM$ is reflexive, there exists $k\in M$ such that $\alpha(f\,|\,L)=(k)f$ for each $f\in M^*$. Suppose $k\notin L$. As M/L is $I({}_RR)$-torsionless and hence torsionless, there exists an R-homomorphism $\theta: M/L\to R$ such that $(k+L)\theta\neq0$. Let $\pi: M\to M/L$ be the canonical mapping. Then, we have a contradiction, since $0\neq(k)\pi\cdot\theta=\alpha(\pi\cdot\theta\,|\,L)=0$. This implies $k\in L$. As L is torsionless, there exists a canonical R-monomorphism $\sigma: L\to L^{**}$. Suppose $\alpha\neq(k)\sigma$. There exists $g\in L^*$ such that $[\alpha-(k)\sigma](g)\neq0$. Let Re be an injective faithful left ideal, where e is an idempotent. We may assume $g\in\operatorname{Hom}({}_RL, {}_RRe)$. Because, there exists $re\in Re$ such that

$[\alpha-(k)\sigma](g)\cdot re\neq 0$ and hence $0\neq g\cdot re\in \mathrm{Hom}({}_RL, {}_RRe)$. Let $\bar{g}: {}_RM\to {}_RRe$ be an extension of g. Then, we have a contradiction, since $[\alpha-(k)\sigma](g)=(k)\bar{g}-(k)g=0$. Thus $\alpha=(k)\sigma$ and hence ${}_RL$ is reflexive.

Lemma 3.4. *Assume R is a ring with a maximal two-sided quotient ring Q and R has a minimal ${}_RR$-dense left ideal and a minimal R_R-dense right ideal. If M is a reflexive left R-module and $\bar{M}=\{x\in I({}_RM)$; there exists an ${}_RR$-dense left ideal J such that $Jx\subseteq M\}$, then $\bar{M}$ is a reflexive left Q-module.*

Proof. It is well known that $\bar{M}$ becomes a left Q-module. M^* is embedded in $\mathrm{Hom}({}_Q\bar{M}, {}_QQ)$ canonically. Let f be an element in $\mathrm{Hom}({}_Q\bar{M}, {}_QQ)$ and D the minimal R_R-dense right ideal of R. Since $(M)f\cdot D\subseteq R$, i.e., $fD\subseteq M^*$, M_R^* is an R_R-dense submodule of $\mathrm{Hom}({}_QM, {}_QQ)_R$. By a same argument ${}_RM^{**}$ and hence ${}_RM$ are embedded in ${}_R[\mathrm{Hom}(\mathrm{Hom}({}_Q\bar{M}, {}_QQ)_Q, Q_Q)]$ as an ${}_RR$-dense submodule. The fact that ${}_RM$ is torsionless implies ${}_Q\bar{M}$ is torsionless. So $M\subseteq\bar{M}\subseteq \mathrm{Hom}(\mathrm{Hom}({}_Q\bar{M}, {}_QQ)_Q, Q_Q)$ and then we can conclude that $\bar{M}$ is a reflexive left Q-module.

Now, it is well known that when R is a quasi-Frobenius ring, a left R-module X is reflexive, if and only if X is finitely generated. This result does not holds in the case where R is QF-3. However, extending this result we have:

Theorem 3.5. *Assume R is a* QF-3 *ring with ACC on annihilator left (or right) ideals. Then, a left R-module X is reflexive, if and only if X has DCC on ${}_RR$-rationally closed submodules and X is embedded in a direct product of copies of ${}_RR$ as an ${}_RR$-rationally closed submodule.*

Proof. QF-3 rings with ACC on annihilator left ideals have also DCC on annihilator left ideals [14]. The 'if' part is evident from the proof of Theorem 3.1.

Conversely, assume X is a reflexive left R-module. If we show that ${}_RX$ is contained in a finitely generated free left R-module, the proof of this theorem will complete. Suppose X has a submodule N such that $N=\bigoplus_{i\in I}N_i$, an infinite direct sum of its submodules. Let $\bar{N}$ and $\bar{N}_i$ be the same as in Lemma 3.4. Since R has ACC on ${}_RR$-rationally closed left ideals, by [4, Proposition 14.1 and Proposition 14.10] we have $\bar{N}=\bigoplus_{i\in I}\bar{N}_i$. Clearly, $\bar{N}\cap X$ is ${}_RR$-rationally closed in X. So it is a reflexive left R-module by Proposition 3.3. As R is QF-3, R has a maximal two-sided quotient ring Q (cf. [8]) and has a minimal ${}_RR$-dense left ideal and a minimal R_R-dense right ideal by [13]. Since it is evident that $\bar{N}=\overline{\bar{N}\cap X}$, by Lemma 3.4 $\bar{N}$ is a reflexive left Q-module. Set

$$S_j=\bigoplus_{i\in I\setminus\{j\}}\bar{N}_i \quad\text{and}\quad A_j=\{f\in \mathrm{Hom}({}_Q\bar{N}, {}_QQ);\ (S_j)f=0\}$$

for each $j\in I$. Assume that $\sum_{j\in I}A_j$ is not ${}_QQ$-rationally closed in $\mathrm{Hom}({}_Q\bar{N}, {}_QQ)_Q$.

Since $I(Q_Q)$ is torsionless, there exists a non-zero element ϕ in

$$\mathrm{Hom}([\mathrm{Hom}({}_Q\bar{N}, {}_QQ)/\textstyle\sum_{j\in I} A_j]_Q, Q_Q).$$

It follows that there exists a non-zero element $n\in\bar{N}$ such that $\phi(g)=(n)g=0$ for every $g\in\sum_{j\in I} A_j$. Put $n=n_{i_1}+\cdots+n_{i_t}$, where $\{i_1,\dots,i_t\}\subset I$ and $n_{i_k}\in\bar{N}_{i_k}$, $k=1,\dots,t$. Since ${}_Q\bar{N}$ is torsionless, there exists $h\in\mathrm{Hom}({}_Q\bar{N}, {}_QQ)$ such that $(n_{i_1})h\neq 0$ and $(S_{i_1})h=0$. This is a contradiction, since $h\in A_{i_1}$ and $(n)h=(n_{i_1})h\neq 0$. Hence $\sum_{j\in I} A_j$ is a Q_Q-rationally closed submodule of the right Q-module $\mathrm{Hom}({}_Q\bar{N}, {}_QQ)$. One can see that there exists $f\in\mathrm{Hom}({}_Q\bar{N}, {}_QQ)$ such that $(\bar{N}_i)f\neq 0$ for each $i\in I$, since ${}_Q\bar{N}$ is torsionless. Further there exists a Q_Q-dense right ideal Q of Q such that $0\neq fD\subsetneq \sum_{j\in I} A_j$. As Q has ACC on Q_Q-rationally closed right ideals, we may assume that D is finitely generated from [4, 14.9]. Put $D=q_1Q+\cdots+q_mQ$, $q_i\in Q$. There exists a subset $\{j_1,\dots,j_r\}$ of I such that $fq_k\in A_{j_1}+\cdots+A_{j_r}$, $k=1,\dots,m$. Let $j\in I\setminus\{j_1,\dots,j_r\}$. Since $\bar{N}_j\subseteq S_{j_i}$ $(i=1,\dots,r)$, we have $(\bar{N}_j)fq_k=0$ $(k=1,\dots,m)$. It follows $(\bar{N}_j)fD=0$ and this is a contradiction, as D is a Q_Q-dense right ideal. Thus, we see that ${}_RX$ is finite Goldie dimensional. Put $I({}_RX)=U_1\oplus\cdots\oplus U_n$, where U_i is an injective indecomposable submodule. Since $I({}_RX)$ is torsionless and hence projective by [10], U_i is embedded in R. This completes the proof.

From this proof we have:

Theorem 3.6. *Every reflexive left module over a left Artinian* QF-3 *ring is finitely generated.*

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