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Gianpaolo Ghiani · Gilbert Laporte

A branch-and-cut algorithm for the Undirected Rural Postman Problem

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Abstract. The well-known Undirected Rural Postman Problem is considered and a binary linear problem using new dominance relations is presented. Polyhedral properties are investigated and a branch-and-cut algorithm is developed. Extensive computational results indicate that the algorithm is capable of solving much larger instances than previously reported.

Key words. Arc Routing Problems – Undirected Rural Postman Problem – branch-and-cut – polyhedral analysis – facets

1. Introduction

The purpose of this article is to present a new formulation, polyhedral properties and a branch-and-cut algorithm for the *Undirected Rural Postman Problem* (RPP) defined as follows. Let $G(V, E)$ be an undirected graph, where V is the vertex set, E is the edge set, c_{ij} (≥ 0) is the cost of traversing edge $(v_i, v_j) \in E$, and $R \subseteq E$ is a set of *required* edges. The RPP is to determine a least cost tour traversing each edge of R at least once. Equivalently, solving the RPP is to determine a least cost set of additional edges that, along with the required edges, makes up a Eulerian and connected subgraph.

The RPP is the unconstrained version of more general classes of multi-vehicle Capacitated Arc Routing Problems (CARP) arising, for example, in garbage collection, road gritting, mail delivery, network maintenance, etc. (Eiselt, Gendreau and Laporte, 1995a,b; Assad and Golden, 1995). Applications of the RPP to the control of plotting and drilling machines (Grötschel, Jünger and Reinelt, 1991) and to the optimization of laser-plotter beam movements (Ghiani and Improta, 1997) have been described in recent years. Given a feasible CARP solution, each of the individual vehicle routes can be post-optimized by means of an RPP algorithm.

To our knowledge the only existing exact algorithms for the RPP are those of Christofides, Campos, Corberán and Mota (1981), Corberán and Sanchis (1994), and Letchford (1996). The first describes a branch-and-bound approach based on Lagrangean relaxation. The second presents a cutting plane algorithm founded on a partial description of the rural postman polyhedron in which the separation problems are solved visually. The two algorithms were tested on a set of 24 randomly generated instances with

G. Ghiani, G. Laporte: GERAD, École des Hautes Études Commerciales, 3000 chemin de la Côte-Sainte-Catherine, Montréal, Canada H3T 2A7, e-mail: {ghiani,gilbert}@crt.umontreal.ca

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$9 \leq |V| \leq 84$, $13 \leq |E| \leq 184$ and $4 \leq |R| \leq 78$, before any graph reduction. All 24 instances were solved optimally by branch-and-bound (Christofides *et al.*, 1981), and all but one were solved without branching by Corberán and Sanchis (1994). These authors have also solved two real-life examples with $|V| = 113$ and $|E| = 171$ from the town of Albaida, Spain. Letchford introduced new classes of valid inequalities (path-bridge inequalities) to the formulation of Corberán and Sanchis and solved to optimality, without any branching, all but one of the Corberán and Sanchis instances. In addition, approximate algorithms for the RPP have been presented by Frederickson (1979), Pearn and Wu (1995), and recently Hertz, Laporte and Nanchen-Hugo (1999).

We derive a formulation for the RPP, including a new class of valid inequalities, and we study polyhedral properties of the constraints contained in the formulation. We also develop a highly effective fully automated branch-and-cut algorithm associated with the new formulation.

The remainder of this article is organized as follows. In Sect. 2, we provide two known formulations for the RPP, as well as some old and new dominance relations. A new formulation is presented in Sect. 3 and polyhedral properties are analyzed in Sect. 4. The branch-and-cut algorithm is described in Sect. 5, followed by computational results in Sect. 6, and by the conclusions in Sect. 7.

2. Dominance relations

We first recall the formulations of Christofides *et al.* (1981) and of Corberán and Sanchis (1994). Let C_i $(i = 1, \ldots, p)$ be the i^{th} connected component of the subgraph induced by R, V_R the set of vertices v_i such that an edge (v_i, v_j) exists in R, and $V_k \subseteq V_R$ $(i = 1, \ldots, p)$ the set of vertices of the i^{th} connected component of R. Christofides *et al.* (1981) work on a transformed graph in which only vertices in V_R appear:

Step 1. Add to $G_R(V_R, R)$ an edge between every pair of vertices of V_R having a cost equal to the shortest path length on G;

Step 2. Simplify this graph by deleting: *a)* one of the two edges in parallel if they have the same cost; *b)* all edges $e = (v_i, v_j) \notin R$ such that $c_{ij} = c_{ik} + c_{kj}$ for some v_k.

In the formulation proposed by Christofides *et al.* (1981), given $S \subset V$, $\delta(S)$ is the set of edges of E with one extremity in S and one extremity in $V \setminus S$. If $S = \{v\}$, we simply write $\delta(v)$ instead of $\delta(\{v\})$. The variable x_{ij} is the number of additional copies of edge (v_i, v_j) that must be introduced into G to make it Eulerian, and $2z_i$ is the degree of vertex v_i. The formulation is then:

$$\text{RPP}^{(\text{CHR})} \qquad \text{minimize} \sum_{e \in R} c_e(1 + x_e) + \sum_{e \in E \setminus R} c_e x_e$$

subject to

$$\sum_{e \in \delta(v) \cap R} (1 + x_e) + \sum_{e \in \delta(v) \cap (E \setminus R)} x_e = 2z_i \ (v_i \in V_R) \tag{1}$$

$$\sum_{e \in \delta(S)} x_e \geq 2 \qquad (S = \cup_{k \in P} V_k,\ P \subset \{1, \ldots, p\},\ P \neq \emptyset) \tag{2}$$

$$x_e \geq 0 \text{ and integer} \qquad (e \in E) \tag{3}$$

$$z_i \geq 0 \text{ and integer} \qquad (v_i \in V). \tag{4}$$

In this formulation, degree constraints (1), (3) and (4) stipulate that each vertex of V_R must have an even degree, while constraints (2) force the solution graph to be connected. The resulting graph will therefore be Eulerian.

In the next formulation, due to Corberán and Sanchis (1994), a vertex $v_i \in V_R$ is said to be R-odd (R-even) if and only if an odd (even) number of required edges are incident to v_i. Here, x_e represents again the number of deadheadings of edge e.

$$\text{RPP}^{(\text{CORB})} \qquad \text{minimize} \sum_{e \in E} c_e x_e$$

subject to

$$\sum_{e \in \delta(v)} x_e = 0 \mod (2) \quad (\text{if } v \in V_R \text{ is } R\text{-even}) \tag{5}$$

$$\sum_{e \in \delta(v)} x_e = 1 \mod (2) \quad (\text{if } v \in V_R \text{ is } R\text{-odd}) \tag{6}$$

$$\sum_{e \in \delta(S)} x_e \geq 2 \quad (S = \cup_{k \in P} V_k,\ P \subset \{1, \ldots, p\},\ P \neq \emptyset) \tag{7}$$

$$x_e \geq 0 \text{ and integer} \qquad (e \in E). \tag{8}$$

As in RPP$^{(\text{CHR})}$, the constraints of this model force each vertex to have an even degree and the graph to be connected. The authors show that the convex hull of incidence vectors of this formulation is an unbounded polyhedron.

The main difficulty with RPP$^{(\text{CORB})}$ lies with the non-linear degree constraints (5) and (6). In the following we will develop new dominance relations that will help formulate the RPP using edge associated variables only (and no vertex variables z_i as in RPP$^{(\text{CHR})}$). Broadly speaking, dominance relations are equalities or inequalities that reduce the set of feasible solutions to a smaller set which surely contains an optimal solution. Hence, a dominance relation is satisfied by at least one optimal solution of the problem but not necessarily by all feasible solutions. The only dominance relations for the RPP known until now are due to Christofides *et al.* (1981) and Corberán and Sanchis (1994).

Dominance Relation 1. (Christofides *et al.*, 1981). Every optimal solution satisfies the following relations:

$$x_{ij} \leq 1 \qquad \text{if } (v_i, v_j) \in R, \tag{9}$$

$$x_{ij} \leq 2 \qquad \text{if } (v_i, v_j) \in E \setminus R. \tag{10}$$

Dominance Relation 2. (Corberán and Sanchis, 1994). Let $e = (v_i, v_j)$ be an edge such that v_i and v_j belong to the same component C_h. Then, in every optimal solution, x_e is equal to 0 or 1.

It should be noted that when $e \in R$, then Dominance Relation 2 is equivalent to (9). New dominance relations can be stated as follows:

Dominance Relation 3. Let $x(e^{(1)}), x(e^{(2)}), \ldots, x(e^{(\ell)})$ be the variables associated with edges $e^{(1)}, e^{(2)}, \ldots, e^{(\ell)}$ having exactly one vertex in a given component C_i and one vertex in another given component C_j $(i, j \in \{1, \ldots, p\},\ i \neq j)$. In an optimal solution, only the variable $x(e^{(r)})$ such that $c(e^{(r)}) = \min\left\{c(e^{(1)}), c(e^{(2)}), \ldots, c(e^{(\ell)})\right\}$ can be equal to 2.

Proof. Suppose an optimal solution contained an edge $e^{(r')}$ linking C_i and C_j with $c(e^{(r')}) > c(e^{(r)})$ and $x(e^{(r')}) = 2$. Then removing $e^{(r')}$ would disconnect the solution and replacing it with two copies of $e^{(r)}$ would yield a feasible solution of lesser cost.

□

Consequently no more than $p(p-1)/2$ edges can appear twice in an optimal solution. This condition can be strengthened as follows.

Proposition 1. *There exists an optimal solution to the RPP in which at most $p - 1$ variables are equal to 2.*

Proof. From Dominance Relation 2, if s variables are equal to 2 and $s > p - 1$, the corresponding edges create at least one loop among some components. In this situation $s - p + 1$ of these edges can be eliminated without disconnecting the components or changing the even degree of a node, and the cost does not increase (see Fig. 1).

□

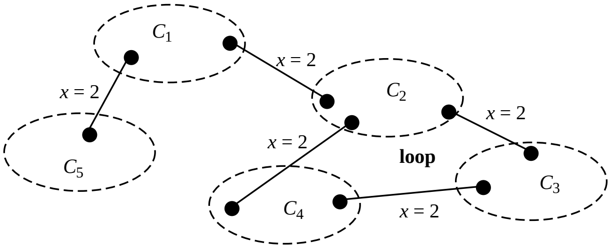

Fig. 1. In an optimal solution no more than $p - 1$ variables can be equal to 2

We now show how some edges can be bounded *a priori*, i.e., before the problem is solved.

Definition. A $0/1/2$ edge is an edge whose associated variable can be equal to 2 in an optimal solution. The remaining edges are said to be $0/1$. Furthermore, the set of $0/1/2$ edges (0/1 edges) will be denoted as E_{012} (E_{01}).

Dominance Relation 4. Let $G_C^*(V_C, E_C)$ be an auxiliary graph having, for every component C_i, a vertex v_i' and, for any pair of components C_i and C_j, an edge $e = (v_i', v_j')$ corresponding to a least cost edge between C_i and C_j. The $0/1/2$ edges belong to a Minimum Spanning Tree on G_C^* (MST_C).

Proof. If, for some edge e between components C_i and C_j, variable $x_e = 2$ is in an optimal solution, then edge e is a *bridge*, i.e., its removal splits up the solution into two Eulerian subtours containing C_i and C_j, respectively. Let $F \subseteq E$ be the edge-cutset induced by this removal (Fig. 2a). If edge e is not on a given Minimum Spanning Tree of G_C (Fig. 2b) there exists a unique path on the MST_C from C_i to C_j (Fig. 2c). This path has at least one edge e^* traversing the edge-cutset F (Fig. 2d). From the *path optimality condition* of the Minimum Spanning Tree (Ahuja, Magnanti and Orlin, 1993), $c(e) \geq c(e^*)$. Consequently, the solution obtained by substituting edge e with edge e^* $(x(e) = 0,\ x(e^*) = 2)$ has a cost no greater than the cost of the previous solution. But, as the initial solution was optimal, $c(e) \leq c(e^*)$, so $c(e) = c(e^*)$. Therefore, the tree obtained by replacing edge e^* with edge e is also an MST_C. The thesis follows by repeating this reasoning for each edge e not in the MST_C and such that $x(e) = 2$.

□

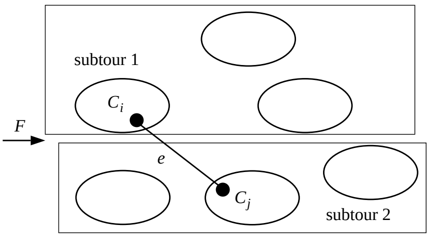

Fig. 2a. Edge-cutset F

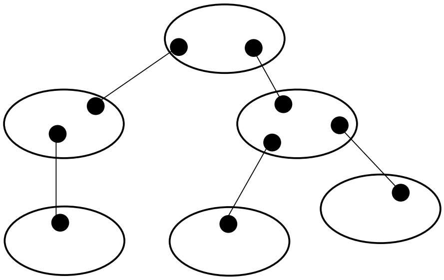

Fig. 2b. Minimum Spanning Tree on G_C

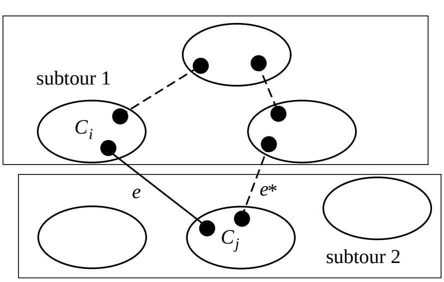

Fig. 2c. Path from C_i to C_j

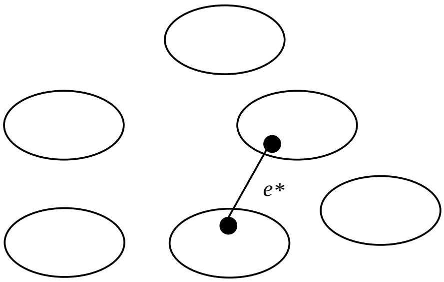

Fig. 2d. Edge substitution

3. New formulations

The previous results allow us to reformulate the RPP as a pure binary integer problem, substituting for each edge $e \in E_{012}$, the integer variable x_e with two binary variables x'_e and x''_e. This can be done in a number of ways. In all, we experimented with three formulations (see Ghiani and Laporte, 1996) and we made preliminary tests on 100 randomly generated instances. The best formulation turned out to be the following "*Twin edges formulation*", where each edge belonging to a given MST_C is replaced by a pair of parallel $0/1$ edges e' and e'':

$$x_e = x'_e + x''_e \quad (e \in E_{012}), \tag{11}$$

$$x'_e, x''_e = 0 \text{ or } 1. \tag{12}$$

Let $E'(E'')$ be the set of edges $e'(e'')$, and let $\overline{E} = E_{01} \cup E' \cup E''$. Substituting (11) and (12) in $\mathrm{RPP}^{(\mathrm{CORB})}$ we obtain the formulation:

$$\mathrm{RPP}^{(01-1)} \qquad \text{minimize} \sum_{e \in \overline{E}} c_e x_e$$

subject to

$$\sum_{e \in \delta(v)} x_e = 0 \mod (2) \quad (\text{if } v \in V_R \text{ is } R\text{-even}) \tag{13}$$

$$\sum_{e \in \delta(v)} x_e = 1 \mod (2) \quad (\text{if } v \in V_R \text{ is } R\text{-odd}) \tag{14}$$

$$\sum_{e \in \delta(S)} x_e \geq 2 \quad (S = \cup_{k \in P} V_k,\ P \subset \{1, \ldots, p\},\ P \neq \varnothing) \tag{15}$$

$$x_e = 0 \text{ or } 1 \quad (e \in \overline{E}). \tag{16}$$

This formulation is equivalent to $\mathrm{RPP}^{(\mathrm{CORB})}$. Here we use $\overline{E}$ instead of E, and all variables are binary. The convex hull of incidence vectors of $\mathrm{RPP}^{(01-1)}$ is obviously a polytope.

We now present a new formulation that does away with the modulo relations:

$$\mathrm{RPP}^{(01-2)} \qquad \text{minimize} \sum_{e \in \overline{E}} c_e x_e$$

subject to

$$\sum_{e \in \delta(v) \setminus F} x_e \geq \sum_{e \in F} x_e - |F| + 1$$
$$(v \in V,\ F \subseteq \delta(v),\ |F| \text{ is odd if } v \text{ is } R\text{-even},$$
$$|F| \text{ is even if } v \text{ is } R\text{-odd}) \tag{17}$$

$$\sum_{e \in \delta(S)} x_e \geq 2 \qquad (S = \cup_{i \in P} V_i,\ P \subset \{1, \ldots, p\},\ P \neq \varnothing) \tag{18}$$

$$x_e = 0 \text{ or } 1 \quad (e \in \overline{E}). \tag{19}$$

Constraints (17) are referred to as *cocircuit inequalities* by Barahona and Grötschel (1986). They state that an even (odd) number of edges are incident to each R-even (R-odd) vertex $v \in V$. Put differently, constraints (17) mean that if an odd (even) number of edges F are incident to a R-even (R-odd) vertex v, then at least another edge has to be incident to v. Constraints (18) are the usual connectivity inequalities.

It is worth noting that RPP$^{(01-2)}$ has the same integer solutions as RPP$^{(01-1)}$. This is a new result. The linear relaxation proposed by Corberán *et al.* (1996) does not verify this property (Letchford, 1996), nor does the stronger linear relaxation introduced by Letchford.

Cocircuit inequalities (17) can be generalized to any non-empty subset S of V:

$$\sum_{e\in\delta(S)\setminus F} x_e \geq \sum_{e\in F} x_e - |F| + 1$$
$$(F \subseteq \delta(S), |F| \text{ is odd if } S \text{ is } R\text{-even},$$
$$|F| \text{ is even if } S \text{ is } R\text{-odd}). \tag{20}$$

Constraints (20) are easily proved to be valid inequalities for RPP$^{(01-2)}$. They state that if an odd (even) number of edges F are incident to an R-even (R-odd) set of vertices S, then at least another edge has to be incident to S. While constraints (20) are new in the context of the RPP, they are formally the same as the "2-matching constraints" if $|F|$ is odd (Edmonds, 1965; Grötschel and Holland, 1987). Also note that, if S is R-odd and F is empty, constraints (20) are the "R-odd inequalities" (Corberán and Sanchis, 1994)

$$\sum_{e\in\delta(S)} x_e \geq 1. \tag{21}$$

If S is R-even and $F = \{e_b\}$, constraints (20) reduce to the new "R-even inequalities"

$$\sum_{e\in\delta(S)\setminus\{e_b\}} x_e \geq x_{e_b}. \tag{22}$$

In most practical cases, constraints (18), (19), (21), and (22) are sufficient to identify a feasible RPP solution. Otherwise a cocircuit inequality (17) can be generated for every vertex v that violates the degree constraint.

4. Polyhedral properties

To study the polyhedral properties of conv(RPP$^{(01-2)}$) , we need to recall some concepts about the polytope of the minimum cost T-join problem (Pulleyblank, 1995).

Definition. Let $G^*(V^*, E^*)$ be an undirected graph. Given a vertex set $T \subseteq V^*$ with $|T|$ even, a T-join is a set $L \subseteq E^*$ such that $|L \cap \delta(v)|$ is odd for every $v \in T$ and even for every $v \in V^* \setminus T$.

The set of incidence vectors of the T-joins is (Schrijver, 1983):

$$J = \Big\{ y \in \{0,1\}^{|E^*|} \Big| \sum_{e \in \delta(V)} y_e = 1 \ (\mathrm{mod}\ 2) \text{ if } v \in T,$$
$$\sum_{e \in \delta(V)} y_e = 0 \ (\mathrm{mod}\ 2) \text{ if } v \in V^* \setminus T \Big\}.$$

If T represents the set of R-odd vertices of a postman tour, J can be expressed as follows:

$$J = \Big\{ y \in \{0,1\}^{|E^*|} \Big| \sum_{e \in \delta(v)} y_e = \delta_R(v) \ (\mathrm{mod}\ 2),\ v \in V^* \Big\}$$
$$= \Big\{ y \in \{0,1\}^{|E^*|} \Big| \sum_{e \in \delta(S)} y_e = \delta_R(S) \ (\mathrm{mod}\ 2),\ S \subset V^*,\ S \neq \varnothing \Big\},$$

where $\delta_R(S)$ is the number of required edges traversing $\delta(S)$.

Polytope J is a special case of the cyclic polytope of a binary matroid (Barahona and Grötschel, 1986):

$$P(M,b) = \Big\{ y \in \{0,1\}^{|E^*|} \Big| My = b \ (\mathrm{mod}\ 2) \Big\},$$

where M is the matrix whose rows are the incidence vectors of all cuts of G, and b is a vector whose element b_S is equal to $\delta_R(S)$.

Barahona and Grötschel (1986) show that:

1) conv(J) is full-dimensional, i.e., $\dim(\mathrm{conv}(J)) = |E^*|$, if and only if each cut $\delta(S)$ of G has at least three edges.

If conv(J) is full-dimensional,

2) $y_e \geq 0$ is a facet of conv(J) if and only if each cut edge-set containing e has at least four edges;
3) $y_e \leq 1$ is a facet of conv(J) if and only if each cut edge-set containing e has at least four edges;
4) $\sum_{e \in \delta(S) \setminus F} y_e \geq \sum_{e \in F} y_e - |F| + 1 \quad (S \subset V,\ F \subseteq \delta(S), |F|$ odd is $\delta(S)$ is R-even or $|F|$ even if $\delta(S)$ is R-odd) is a facet if and only if for every subset $S' \subset S$

$$|\delta(S') \setminus \delta(S)| \geq 2$$

and for every subset $S'' \subset V \setminus S$

$$|\delta(S'') \setminus \delta(S)| \geq 2.$$

These results will be used to study the polytope conv(RPP$^{(01-2)}$) .

Proposition 2. *The convex hull of incidence vectors of RPP$^{(01-2)}$ is full-dimensional if and only if every cut-edge set $\delta(S)$ has at least three edges and every cut-edge set $\delta(S)$ such that $S = \cup_{i \in P} V_i$ $(P \subset \{1,\ldots,p\},\ P \neq \varnothing)$ has at least four edges, where edges $e \in E_{012}$ are counted as two distinct edges ($e \in E'$ and $e \in E''$).*

Proof. The condition is necessary. If there is a cut edge-set having only one edge, then e should be a required edge and $x_e = 1$. Hence, $\text{conv(RPP}^{(01-2)}) \subset \{x : x_e = 1\}$. Assume now there exists a subset $S \subset V$, with $\delta(S) = \{e^{(1)}, e^{(2)}\}$. If $S = \cup_{i \in P} V_i$ $(P \subset \{1, \ldots, p\},\ P \neq \emptyset)$, then $\text{conv(RPP}^{(01-2)}) \subset \{x : x_{e^{(1)}} = 1$ and $x_{e^{(2)}} = 1\}$. Otherwise, if $\delta(S)$ is R-even, $\text{conv(RPP}^{(01-2)}) \subset \{x : x_{e^{(1)}} = x_{e^{(2)}}\}$ and if $\delta(S)$ is R-odd, $\text{conv(RPP}^{(01-2)}) \subset \{x : x_{e^{(1)}} + x_{e^{(2)}} = 1\}$. Finally, if $S = \cup_{i \in P} V_i$ $(P \subset \{1, \ldots, p\},\ P \neq \emptyset)$ or and $\delta(S) = \{e^{(1)}, e^{(2)}, e^{(3)}\}$, then $\text{conv(RPP}^{(01-2)}) \subset \{x : x_{e^{(1)}} + x_{e^{(2)}} + x_{e^{(3)}} = 2\}$. *The condition is sufficient.* Under the hypotheses of the proposition, it is easy to find $|\overline{E}| + 1$ affinely independent T-joins satisfying the connectivity inequalities.

□

In what follows the polytope $\text{conv(RPP}^{(01-2)})$ is studied under the hypotheses of Proposition 2, i.e., $\dim(\text{conv(RPP}^{(01-2)})) = |\overline{\text{E}}|$.

Proposition 3. *The inequality $x_e \geq 0$ induces a facet of* $\text{conv(RPP}^{(01-2)})$ *if and only if every cut-edge set $\delta(S)$ containing e has at least four edges and every cut-edge set $\delta(S)$ containing e such that $S = \cup_{i \in P} V_i$ $(P \subset \{1, \ldots, p\},\ P \neq \emptyset)$ has at least five edges.*

Proof. The face $\{x \in \text{conv(RPP}^{(01-2)}) : x_e = 0\}$ has the same dimension as the polytope associated to the RPP problem obtained by removing edge e from G.

□

Proposition 4. *The inequality $x_e \leq 1$ defines a facet of* $\text{conv(RPP}^{(01-2)})$ *if and only if every cut-edge set $\delta(S)$ containing e has at least four edges.*

Proof. The condition is necessary. If there exists a cut-edge set $\delta(S) = \{e, f, g\}$ then $\{x \in \text{conv(RPP}^{(01-2)}) : x_e = 1\} \subset \{x \in \text{conv(RPP}^{(01-2)}) : x_e = 1,\ x_f + x_g = 1\}$ if $\delta(S)$ is R-even, or $\{x \in \text{conv(RPP}^{(01-2)}) : x_e = 1\} \subset \{x \in \text{conv(RPP}^{(01-2)}) : x_e = 1, x_f - x_g = 0\}$ otherwise. *The condition is sufficient.* It is easy to show that, under the hypotheses of the proposition, there exists $|\overline{E}|$ feasible and affinely independent points lying on the hyperplane $x_e = 1$.

□

Proposition 5. *The connectivity inequality (18) induces a facet of* $\text{conv(RPP}^{(01-2)})$ *if and only if both $G(S)$ and $G(V \setminus S)$ are connected and for every subset of components S' included in S (or in $V \setminus S$), $|\delta(S') \setminus \delta(S)| \geq 2$.*

Proof. The condition is necessary. First, suppose $G(S)$ is not connected (if $G(V \setminus S)$ is not connected, a similar reasoning can be made). Let S_1 be a component of $G(S)$. Then inequality (18) is dominated by the connectivity inequality corresponding to $G(S_1)$, $\sum_{e \in \delta(S_1)} x_e \geq 2$. Suppose now there is a subset of components S' in, for example, S such that there is only one edge between S' and $S \setminus S'$. Then connectivity constraint (18). associated with S is dominated by the sum of connectivity constraints (18) associated with S' and $S \setminus S'$. *The condition is sufficient.* It is easy to show that, under the hypotheses of the proposition, there exist $|\overline{E}|$ feasible and affinely independent points lying on the hyperplane $\sum_{e \in \delta(S)} x_e = 2$.

□

Proposition 6. *Inequality (20) defines a facet of* conv($RPP^{(01-2)}$) *if and only if*

a) for every subset $S' \subset S$ *(and for every subset* $S' \subset V \setminus S$*)*

$$|\delta(S') \setminus \delta(S)| \geq 2$$

b) if $|F| = 1$*, S is not a set of components, i.e., S cannot be expressed as* $\cup_{i \in P} V_i$ $(P \subset \{1, \ldots, p\},\ P \neq \varnothing)$.

Proof. The proof is the same as that of Corollary 4.23 of Barahona and Grötschel (1986), except for the fact that if S is a set of components and $F = \{e_b\}$, then inequality (20) is dominated by the connectivity inequality (18) associated with S. □

5. Branch-and-cut algorithm

In this section, we describe a branch-and-cut algorithm based on the $RPP^{(01-2)}$ formulation introduced in Sect. 3.

Step 1 (Upper bound). *Compute an upper bound* $\overline{z}$ *on the optimal solution value* z^* *using Frederickson's (1979) heuristic.*

Step 2 (First node of the tree). *Define a first subproblem by the linear program containing a connectivity inequality for each single component and a cocircuit inequality with* $F = \varnothing$ *for each R-odd vertex. Insert this problem in a list.*

Step 3 (Termination test). *If the list is empty, stop. Otherwise select a subproblem from the list according to the best lower bound strategy.*

Step 4 (Subproblem solution). *Solve the subproblem using CPLEX; let z be the solution value. If* $z \geq \overline{z}$*, go to Step 3. If the solution is feasible for the RPP, set* $\overline{z} := z$ *and go to Step 3.*

Step 5 (Constraint elimination). *Among all constraints, eliminate those that have been ineffective for 20 executions of Step 4.*

Step 6 (Constraint generation). *Identify up to 60 violated inequalities considering, in this order: (18), (21), (22), and (17). If no inequality can be generated, go to Step 7. Otherwise, generate the most violated inequalities, up to 40, add them to the current subproblem, and go to Step 4.*

Step 7 (Branching). *Branch on the fractional variable* x_e *nearest to 0.5; if the variable corresponds to an edge* $e \in E_{01}$*, two son subproblems are generated setting* $x_e = 0$ *and* $x_e = 1$*; if the variable corresponds to an edge* $e \in E_{012}$*, three son subproblems are generated setting* $x_{e'} = 0$ *and* $x_{e''} = 0$*;* $x_{e'} = 1$ *and* $x_{e''} = 0$*;* $x_{e'} = 1$ *and* $x_{e''} = 1$*. (The subproblem* $x_{e'} = 0$ *and* $x_{e''} = 1$ *can obviously be omitted.) Insert the subproblems in a list and go to Step 3.*

The separation problems for constraints (17), (18), (21) and (22) in Step 6 are solved heuristically as follows.

Cocircuit inequalities (17). Suppose v is R-even so that $|F|$ must be odd. We can express the slack of (17) as $\sum_{e \in \delta(v) \setminus F} x_e + \sum_{e \in F} (1 - x_e) - 1$. To maximize this slack, include in F every edge e such that $x_e \leq 0.5$ and include in $\delta(v) \setminus F$ every edge e such that $x_e > 0.5$. A similar reasoning holds if v is R-odd.

Connectivity inequalities (18). Although this separation problem is well known to be solvable in $O(|V|^3)$ time, we use the heuristic procedure proposed by Fischetti, Salazar and Toth (1997). A maximum spanning tree is built in an auxiliary graph $\overline{G}$ where each connected component C_h of R, is represented by a vertex v'_h and edges $e' = (v'_h, v'_k)$ have cost equal to the sum of variables x_e corresponding to edges $e = (v_i, v_j)$ such that $v_i \in C_h$ and $v_j \in C_k$. At any stage of the construction of this tree, let S be the set of connected components of R corresponding to vertices of the partial tree. If S yields a violated connectivity constraint, this constraint is generated. Once the spanning tree is complete, another check for violated connectivity constraints is made by removing in turn each edge of the tree.

***R*-odd cut inequalities (21).** The separation problem for the R-odd cut inequalities was proved to be solvable in $O(|V|^4)$ time by Padberg and Rao (1982). However, as the computational effort of this procedure is quite heavy, we use a heuristic inspired by a procedure developed by Grötschel and Win (1992) for the Windy Postman Problem. Let $\overline{G}(\lambda)$ be an auxiliary graph where edge e exists if and only if $x_e \geq \lambda$ if $e \in E_{01}$, or $x_{e'} + x_{e''} \geq \lambda$ if $e \in E_{012}$, where $\lambda \in (0, 1)$ is a given parameter. The violation of R-odd cut inequalities is checked for each connected component of $\overline{G}(\lambda)$. We first try $\lambda = 0.5$; if no violated inequality is detected, we try $\lambda = 0.3$; finally, if this does not succeed, we try $\lambda = 0.1$.

***R*-even cut inequalities (22).** The R-even cut inequalities are new in the context of the RPP, but can be viewed as a special case of two-matching constraints. The exact separation algorithm of Padberg and Rao (1982) could be used to identify violated inequalities, but we have developed a faster heuristic procedure that detects several violations at a time instead of only one. Let $\overline{G}(\lambda)$ be an auxiliary graph where edge e exists if and only if $x_e \geq \lambda$ if $e \in E_{01}$, or $x_{e'} + x_{e''} \geq \lambda$ if $e \in E_{012}$. Edge e has a cost $\overline{c}_e$ equal to x_e if $e \in E_{01}$, or $x_{e'} + x_{e''}$ if $e \in E_{012}$, where $\lambda \in (0, 1)$ is a given parameter. For each R-even component of $\overline{G}(\lambda)$, the maximum spanning tree is constructed; if the removal of a tree edge e divides a R-even component $C \subset V$ into two R-even subcomponents C' and C'' (Figure 3), a double check for R-even inequalities violation is made considering $S = C',\ e_b = e$ and $S = C'',\ e_b = e$.

6. Computational results

The algorithm was coded in C using the Watcom Integrated Development Environment and the CPLEX library. The code ran on a PC with a Pentium processor at 90 Mhz with 16 Mbytes RAM. The algorithm was tested on one real-world and several randomly generated instances.

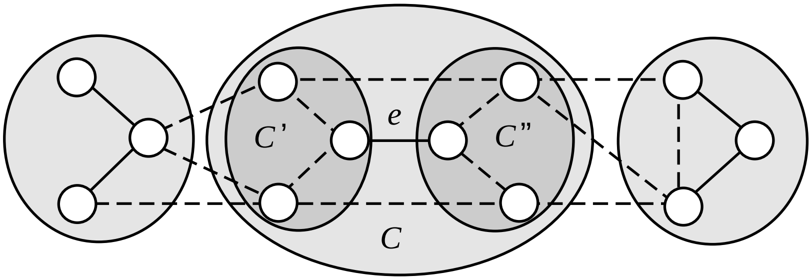

Fig. 3. Separation heuristic for the R-even cut inequalities

The real-world example is the Albaida1 graph studied by Corberán and Sanchis (1994). This instance contains 113 vertices, 171 edges and ten connected components. To eliminate some inconsistencies in this graph, Hertz, Laporte and Nanchen-Hugo (1999) propose making the four edges (17,21), (25,26), (26,33) and (29,33) required. We also adopt this convention. This problem was solved to optimality at the root of the search tree in 14 seconds, using ten connectivity constraints, 153 R-odd cut inequalities, and one R-even cut inequality. Using a "visual heuristic" to generate violated inequalities, Corberán and Sanchis (1994) were able to solve the unmodified instance at the root of the search tree using one connectivity constraint and 36 R-odd cut inequalities. The same problem was solved in 89 seconds on a faster computer by Jünger, Reinelt and Rinaldi (1995), using a transformation of the RPP into a Traveling Salesman Problem (TSP), and a state-of-the-art branch-and-cut TSP code. No information on the number of nodes in the search tree is provided.

We also generated Type 1, 2 and 3 random graphs as in Hertz, Laporte and Nanchen-Hugo (1999). Type 1 graphs are graphs with vertices randomly generated in a plane with a test to ensure that they are connected. In practice, R is always disconnected in these graphs. Type 2 graphs are generated on a grid with disconnected required edge sets. Type 3 graphs are grid graphs with vertex degrees equal to 4, and disconnected required edge sets.

Five graphs of Type 1 were considered for $|V| = 50, 100, 150, 200, 250, 300, 350$; five graphs of Type 2 were considered for each combinations of $|V| = 7 \times 7$, 10×10, 12×12, 15×15, 17×17 and expected density $\pi = 0.30$, 0.50 and 0.70; five graphs of Type 3 were considered for each combinations of $|V| = 50, 100, 150, 200, 250, 300$ and $\pi = 0.30$, 0.50 and 0.70.

An instance was deemed successful if it could be solved within 10,000 seconds. Average statistics over all successful instances are reported in Tables 1, 2 and 3. The column headings are defined as follows:

$\|V\|$	:	number of vertices of the original graph used to generate the test network (see Hertz, Laporte and Nanchen-Hugo, 1999);
π	:	for Type 2 and 3 graphs, probability that an edge is required (see Hertz, Laporte and Nanchen-Hugo, 1999);
p	:	number of connected components induced by the required edges;
Succ	:	number of instances solved successfully;

Root	:	number of instances solved to optimality at the root node of the search tree, without branching;
Connect	:	number of connectivity inequalities;
R-odd	:	number of R-odd cut inequalities;
R-even	:	number of R-even cut inequalities;
LB/z^*	:	lower bound at the root node of the search tree, divided by the optimal solution value;
Nodes	:	number of nodes in the search tree;
Seconds	:	CPU time in seconds.

Results presented on Table 1 to 3 indicate that the proposed algorithm can easily solve instances involving up to 300 or 350 vertices. Even if computation times are still rather modest for these sizes, memory then becomes the limiting factor. Our results show that when constraints (21) and (22) are generated, it is never necessary to generate any cocircuit constraint (17). As a rule, unstructured instances (Table 1) are easier then those generated on grids (Tables 2 and 3), and type 2 instances tend to be the most difficult. The value of the continuous relaxation at the root of the search tree is always very close to that of the optimum. In our test problems, the average LB/z^* ratio almost always exceeds 0.997. This performance can be explained to a large extent by the R-even cut inequalities (22) which were introduced in this article. The number of nodes in the search tree tends to be very low. For example, for the randomly generated instances of Table 1, the average number of nodes over the successful instances attains only 22.4 when $|V| = 350$. This figure can be compared with the tree sizes obtained by Christofides *et al.* (1981) on similarly generated instances (with $|V| \leq 84$) where the size of the search tree varied between 1 and 14791 nodes, with an average of 1942.8. Using connectivity constraints, R-odd cut inequalities and path-bridge inequalities, Letchford (1996) obtained an average LB/z^* ratio of 99.72% at the root of the search tree for 11 instances containing 50 vertices or less.

Table 1. Computational results for Type 1 graphs

$\lvert V\rvert$	Succ	p	Root	Connect	R-odd	R-even	LB/z^*	Nodes	Seconds
50	5	7.4	4	9.6	27.0	1.0	0.999	1.4	0.4
100	5	14.0	3	19.4	59.4	10.0	0.999	2.2	1.6
150	5	19.6	2	27.2	128.8	18.0	0.998	4.6	14.2
200	5	23.8	3	28.2	125.4	6.2	0.998	1.8	24.2
250	5	36.2	2	54.2	190.8	91.0	0.998	3.8	62.4
300	5	46.8	1	67.8	276.8	133.3	0.997	11.5	133.0
350	3	53.6	0	75.0	374.2	154.3	0.998	22.4	332.5

7. Conclusions

The Undirected RPP is a hard combinatorial optimization problem with several applications in the fields of manufacturing and distribution management. We have provided new properties and a compact binary linear formulation that provides a full description of the RPP in terms of binary edge variables only. To our knowledge, this is the first such formulation. Polyhedral properties of the formulation were investigated and a fully

Table 2. Computational results for Type 2 graphs

$\|V\|$	π	p	Succ	Root	Connect	R-odd	R-even	LB/z^*	Nodes	Seconds
7×7	0.30	9.6	5	5	12.0	33.4	5.6	1.000	1	0.8
	0.50	8.4	5	3	10.4	40.8	26.6	0.995	1.8	0.8
	0.70	6.0	5	4	8.0	39.2	28.8	0.996	3.0	1.2
10×10	0.30	20.2	5	1	39.4	187.8	635.2	0.986	49.4	23.4
	0.50	11.4	5	1	19.6	246.8	1399.2	0.997	33.8	56.4
	0.70	6.0	5	3	7.8	114.4	105.8	0.999	5.8	17.8
12×12	0.30	26.8	5	1	55.6	435.4	1480.8	0.990	96.6	88.6
	0.50	16.8	5	1	19.2	116.0	23.8	0.999	2.6	95.0
	0.70	5.0	5	3	8.6	379.6	680.2	0.998	27.4	122.6
15×15	0.30	41.8	5	0	50.2	179.8	221.2	0.997	7.0	110.4
	0.50	22.0	4	0	54.0	712.5	3684.7	0.995	113.5	580.7
	0.70	7.2	5	0	8.0	312.6	564.8	0.999	11.0	629.4
17×17	0.30	55.8	5	1	189.0	1114.4	5587.6	0.997	199.8	820.8
	0.50	26.4	5	0	33.0	441.8	969.8	0.998	21.4	964.6
	0.70	6.6	4	0	6.8	522.5	590.7	0.999	12.0	1368.0

Table 3. Computational results for Type 3 graphs

$\|V\|$	π	p	Succ	Root	Connect	R-odd	R-even	LB/z^*	Nodes	Seconds
50	0.30	8.4	5	5	10.4	32.0	0.5	1.000	1.0	0.8
	0.50	8.0	5	3	11.2	55.0	21.4	0.999	4.2	1.6
	0.70	6.6	5	3	9.2	62.8	63.8	0.996	7.8	2.2
100	0.30	19.0	5	3	23.0	92.5	12.0	0.998	1.8	4.25
	0.50	14.8	5	3	18.4	122.6	66.2	0.999	5.8	9.0
	0.70	6.0	5	4	11.8	171.4	199.0	0.999	11.4	16.8
150	0.30	29.0	5	1	44.3	182.3	351.5	0.995	13.5	29.0
	0.50	19.6	5	2	26.2	226.6	144.0	0.996	8.2	50.2
	0.70	8.6	5	1	21.6	344.0	245.0	0.998	13.4	85.8
200	0.30	38.0	5	1	47.0	241.7	175.3	0.997	6.5	71.0
	0.50	22.0	5	2	30.5	289.3	310.3	0.998	8.5	181.8
	0.70	9.6	5	2	11.6	342.2	344.2	0.999	6.6	241.8
250	0.30	49.6	5	0	70.5	244.3	335.8	0.997	18.0	194.0
	0.50	30.2	5	2	34.6	295.0	111.6	0.998	7.0	420.0
	0.70	9.8	5	1	10.8	350.0	79.3	0.999	2.5	563.5
300	0.30	57.6	4	0	82.3	302.8	280.2	0.997	23.8	320.8
	0.50	32.4	5	0	52.1	325.8	420.5	0.998	20.2	410.2
	0.70	9.6	4	0	26.6	341.9	370.8	0.998	21.9	497.8

automated branch-and-cut algorithm was developed and implemented. This algorithm is capable of solving to optimality, within moderate solution times, instances involving up to 350 randomly generated vertices. The largest problem size attained exceeds by far the size of the largest instances reported solved in the literature.

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