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SERIAL RINGS AND DIRECT DECOMPOSITIONS

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We have studied some relationships between serial rings and the lifting (extending) property of simple modules in [5], [6] and [9]. We shall continue to study the similar problem in this note.

In the second section we shall deal with more general rings than serial rings defined by Nakayama [10]. He showed that if a ring R is a serial ring, then every finitely generated module is a direct sum of serial modules. In this paper we shall give a characterization for a semi-perfect ring to satisfy the above decomposition for modules of finite length (Theorem 4).

We refer to [5] and [9] for the definition of the lifting (extending) property of simple modules.

1. Definitions

Let R be a ring with identity. Every module in this paper is a unitary right R-module. For an R-mdule M, $|M|$ means the length of the composition series. We shall denote the Jacobson radical and the socle of M by $J(M)$ and $S(M)$, respectively. Put $J^n(M)=J(J^{n-1}(M))$ and $S_n(M)/S_{n-1}(M)=S(M/S_{n-1}(M))$ inductively. Then $M\supseteq J(M)\supseteq J^2(M)\supseteq\cdots$ and $0\subseteq S_1(M)\subseteq S_2(M)\cdots$ are called the *upper Loewy series* and the *lower Loewy series*, respectively. If each factor module $J^n(M)/J^{n+1}(M)$ $(S_{n+1}(M)/S_n(M))$ is simple or zero, this upper (lower) Loewy series is a unique composition series such that $|M/J^n(M)|=n$ $(|S_n(M)|=n)$ and if $|M/N|=m<\infty$ $(|N|=m<\infty)$ for some submodule N, $N=J^m(M)$ $(N=S_m(M))$, provided that $J^k(M)\neq J^{k+1}(M)$ $(S_k(M)\neq S_{k+1}(M))$ for all $k\leq m-1$. If M has the unique chain as above, we call M an *upper* (*lower*) *serial module*. An upper (lower) serial module M with $J^t(M)=0$ $(S_t(M)=M)$ for some t is called a *serial module* and in this case $S_r(M)=J^{t-r}(M)$. It may happen, by the definition, for upper (lower) serial modules that $J^{n-1}(M)\supsetneq J^n(M)=J^{n+1}(M)$ $(S_{n-1}(M)\subsetneq S_n(M)=S_{n+1}(M))$. Which means that $M/J^n(M)\supset J(M)/J^n(M)\supset\cdots\supset 0=J^n(M)/J^n(M)$

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($0 \subset S_1(M) \subset S_2(M) \subset \cdots \subset S_n(M)$) is a unique composition series of $M/J^n(M)$ ($S_n(M)$) and that $J^n(M)$ contains no maximal submodules ($M/S_n(M)$ contains no simple submodules).

If there exists a non-zero cyclic hollow and projective module, and every cyclic hollow projective module is an upper serial module, then we call R a *right upper serial ring* (cf. [15]). Such a projective module is isomorphic to eR, where e is a primitive idempotent by [3], Proposition 1. As the concept dual to the above, if every indecomposable injective module with non-zero socle is a lower serial module, we call R a *right lower coserial ring* (cf. [6]). Further, if R is a right artinian ring in the above, we say that R is a *right serial* (*coserial*) *ring*, following Nakayama [10]. If we say that R is artinian or serial, etc., then we mean that R is left and right artinian or serial, etc.

2. Upper serial rings

We shall give a characterization for a semi-perfect ring R to be upper serial (lower coserial).

Lemma 1. *Let R be a perfect ring. Then every upper (lower) serial module has finite length and is a serial module.*

Proof. Let $M \supset J(M) \supset J^2(M) \supset \cdots$ be the upper Loewy series, which is serial. Put $M' = \bigcap_n J^n(M)$. Since R is perfect, M/M' has a simple submodule T/M', and $|M/T| < \infty$ from the definition. Hence $T = J^m(M)$ for some m, and so $J^{m+1}(M) = J^{m+2}(M)$. Therefore $J^{m+1}(M) = 0$. Let $0 \subset S_1(M) \subset S_2(M) \subset \cdots$ be the lower Loewy series, which is serial and put $M' = \bigcup_n S_n(M)$. Then $M' \neq M'J$. Hence $|M'J| < \infty$, and so we know similarly that $S_m(M) = S_{m+1}(M)$ for some m. Since R is semi-artinian, $M = S_m(M)$.

We know from this lemma that a perfect and right upper serial ring is a right serial ring. We shall give some examples of non-artinian serial rings in the last part of this section.

Consider an exact sequence

$$(*) \qquad 0 \to A \to \bigoplus_{i=1}^{n} P_i \to B \to 0,$$

where the P_i are serial modules of finite length.

Theorem 2. (1) *Every B in any exact sequence $(*)$ is a direct sum of serial modules if and only if R is a right lower coserial ring.*

(2) *Assume that R is a semi-perfect ring. Every A in any exact sequence $(*)$ is a direct sum of serial modules if and only if R is a right upper serial ring.*

Proof. (1) Assume that R is a right lower coserial ring. Put $P=\bigoplus_{i=1}^n P_i$ and let S be a simple submodule of A. Since P has the extending property of simple modules by [6], Theorem 6, we have a direct decomposition $P=\bigoplus_{i=1}^n P_i'$ such that $S\subseteq P_1'$ and each P_i' is isomorphic to some $P_{i'}$. Then $B\approx P/A\approx(P/S)/(A/S)\approx(P_1'/S\oplus\bigoplus_{i\geq 2}P_i')/(A/S)$. We note that P_1'/S is also serial. Hence, repeating this manner, we know that $B\approx\bigoplus_{i=1}^n P_i''/A_i$ and P_i''/A_i are serial. Conversely, we assume that B is a direct sum of serial modules. let E be an indecomposable and injective module with simple socle. Assume that $0\subset S_1(E)\subset S_2(E)\subset\cdots\subset S_n(E)$ is the lower Loewy series with $S_i(E)/S_{i-1}(E)$ simple. Let C_i be a submodule of $S_{n+1}(E)$ such that $C_i/S_n(E)$ is simple for $i=1,2$. Then C_i is serial. Put $D=C_1\oplus C_2$. Then we have the natural epimorphism $: D\to C_1+C_2\subseteq E$. Since C_1+C_2 is uniform, C_1+C_2 is serial by assumption. Hence $C_1=C_2$.

(2) Assume that R is a right upper serial and semi-perfect ring. Then $R=\bigoplus_{i=1}^t e_iR$, where the e_i are primitive idempotents. Let P be a serial module of finite length. Since $P=\sum_{p\in P}pe_iR$ and P is hollow, $P\approx e_iR/e_iJ^{s_i}$ for some integer s_i. Hence we may assume that $P=\bigoplus_{i=1}^n e_iR/e_iJ^{s_i}\supset A$. Let N be a maximal submodule containing A. Since P has the lifting property of simple modules modulo the radical by [6], Theorem 5, there exists a direct decomposition $P=\bigoplus_{i=1}^n P_i'$ such that $N=J(P_1')\oplus\bigoplus_{i\geq 2}P_i'$ by [7], Theorem 1, where $P_i'\approx e_i'R/e_i'J^{s_i'}$. Since P_1' is serial, $J(P_1')$ is serial. Hence we know by induction that $A\approx\bigoplus e_iR/e_iJ^{n_i}$. Conversely, we assume that A is a direct sum of serial modules. Let e be a primitive idempotent. Assume that $eR\supset eJ\supset eJ^2\supset\cdots\supset eJ^n$ is the upper Loewy series with eJ^i/eJ^{i+1} simple. Let C_i be a submodule of eJ^n such that eJ^n/C_i is simple and $C_i\supseteq eJ^{n+1}$ for $i=1,2$. Then we have the natural monomorphism $: eR/(C_1\cap C_2)\to eR/C_1\oplus eR/C_2$. Hence $eR/(C_1\cap C_2)$ is serial by assumption, and so $C_1=C_2$. Therefore eJ^n/eJ^{n+1} is either simple or zero.

Let R be a semi-perfect ring. If eR contains the simple socle essential in eR for every primitive idempotent e, R is called a *right QF-2 ring* [14]. If every indecomposable injective module is a cyclic hollow module, R is called a *right QF-2** ring [4].

Corollary 3. *Let R be a right artinian ring.*

(1) *R is a right coserial ring if and only if every B in (*) is a direct sum of uniform modules, provided that the P_i in (*) are always uniform modules, and R is right QF-2*.*

(2) *R is a right serial ring if and only if A in (*) is a direct sum of hollow modules, provided that the P_i are always hollow modules, and R is a right QF-2 ring.*

Proof. (1) Assume that R is a right QF-2* ring and that B is a direct sum of uniform modules. Let E be an indecomposable injective module. Then $J(E)$ is a unique maximal submodule of E. Let N be a proper submodule of E. Then E/N is indecomposable and uniform by assumption. Hence E is a (lower) serial module. The converse is clear from lemma 1 and Theorem 2.

(2) This is dual to (1).

Theorem 4. *Let R be a semi-perfect ring. Then the following condition are equivalent:*

(1) *Every module of finite length is a direct sum of serial modules.*

(2) *R is a right lower coserial and right upper serial ring.*

Proof. (1)⇒(2). This is an immediate consequence of Theorem 2.

(2)⇒(1). Let M be an R-module of finite length. Then M is a sum of cyclic hollow submodules isomorphic to some e_iR/e_iA_i for R is semi-perfect, where the e_i are primitive idempotents. Since the e_iR/e_iA are serial by (2), M is a direct sum of serial modules by Theorem 2.

Corollary 5. *Let R be a commutative and semi-perfect ring. Then the following conditions are equivalent:*

(1) *R is an upper serial ring.*

(2) *R is a lower coserial ring.*

(3) *Every module of finite length is a direct sum of serial modules.*

Proof. (1)⇔(2). R is a direct sum of local rings. Hence we may assume that R is a local ring with J maximal ideal. Let $E=E(R/J)$ the injective hull of R/J. Then

$$\mathrm{Hom}_R(J^n/J^{n+1},E)=\mathrm{Hom}_R(J^n/J^{n+1},R/J)\approx S_{n+1}(E)/S_n(E)$$

by [13], Lemma 1. Hence, if R is an upper serial ring, R is a lower coserial ring. Furthermore, E is a cogenerator. Hence $r_R(l_E(A))=A$ for every ideal A of R, where

$$l_E(A)=\{x\in E \mid xA=0\} \quad \text{and} \quad r_R(X)=\{r\in R \mid Xr=0\}$$

for a subset X of E. Let $S=\mathrm{End}_R(E)$. Then S contains the naturally homomorphic image of R and the $S_n(E)$ are S-modules. If $S_{n+1}(E)/S_n(E)$ is a simple R-module (and hence simple S-module), J^n/J^{n+1} is simple, since $J^n=r_R(l_E(J^n))=r_R(S_n(E))$, and so if R is a lower coserial ring, then R is an upper serial ring.

(1)⇔(3). This is clear by Theorem 4.

Examples. (1) Let Z be the ring of integers and p a prime. Then Z is a lower coserial ring, but not an upper serial ring, for there exist no cyclic hollow projectives. Put $R_1=Z_p\oplus E(Z/p)$ and $R_2=Z_{(p)}\oplus E(Z/p)$ trivial extensions, where $Z_{(p)}=\mathrm{End}_Z(E(Z/p))$ [11]. Then every ideal of R_i is of the form $(p^n)\oplus E(Z/p)$ or $0\oplus(a/p^n)$. Hence R_1 and R_2 are upper serial and lower coserial rings by Corollary 5. Since R_2 is a self-injective ring and is an essential extension of R_1 as an R_1-module, R_2 is an R_1-injective. We note that

$$\bigcup_n S_n(R_i)\neq R_i \quad \text{and} \quad \bigcap_n ((p^n)\oplus E(Z/p))\neq 0.$$

$\prod_p Z_p$ is an upper serial ring which is not semi-perfect ring.

Every local ring R with $J^2=J$ is an upper serial and lower coserial ring. In this case the modules with finite length are semi-simple.

(2) Let R be a maximal (hereditary) order over a local Dedekind domain K in a simple algebra over the quotient field of K. Then R is an upper serial and lower coserial ring by Theorem 4 and [15], Theorem 3.1.

3. Serial rings

In this section we shall study some characterizations of (artinian) serial rings. First we shall give an alternative proof for the essential part (d) $\Rightarrow$ (a) of [2], Theorem 5.4.

Theorem 6 (Fuller). *Let R be a perfect ring. If R is a right lower coserial and right QF-2 ring (if R is a right upper serial and right QF-2* ring), then every finitely generated module is a direct sum of serial modules. Hence R is a serial ring* (*cf.* [1]).

Proof. If R is a right upper serial ring and a right QF-2* ring, then R is a right artinian ring by Lemma 1 and every indecomposable injective module is of the form eR/eA. Hence R is a right lower coserial and right QF-2 ring. Therefore every finitely generated module is a sum of serial submodules and so is a direct sum of serial modules by Theorem 2. Accordingly, R is a left serial ring by [6], Theorem 6 and [12], Corollary 4.4.

Theorem 7. *Let R be a perfect ring. Then the following conditions are equivalent:*

(1) *R is an artinian serial ring.*

(2) *Every module (generated by two elements) has the lifting property of simple modules modulo the radical.*

(3) *Every module (generated by two elements) has the extending property of simple modules.*

Proof. (2) $\Rightarrow$ (1). Assume that every module generated by two elements has the lifting property of simple modules modulo the radical. Then R is a right artinian and right serial ring by [5], Theorem 2 and [6], Theorem 5. Further every indecomposable module generated by two elements is hollow by (2). Let e and f be primitive idempotents. Consider the module $A = eR/eJ^t \oplus fR/fJ^{t'}$ and A/S, where

$$S = \{(x, g(x)) \mid x \in eR/eJ^t, g \in \mathrm{Hom}_R(S(eR/eJ^t), S(fR/fJ^{t'}))\}.$$

If neither eR/eJ^t nor fR/fJ^t is simple, A/S is not indecomposable by the above remark. Hence R is left serial by [12], Lemmas 2.1 and 4.3.

(3) $\Rightarrow$ (1). Since eR/eA is indecomposable for any primitive idempotent e and any right ideal eA, eR/eA is uniform by (3). Hence eR is a right artinian and serial module by Lemma 1. Let E be and indecomposable and injective module. Assume $S_i(E)/S_{i-1}(E)$ is simple for $i \leq t$. Let A_1 and A_2 be submodules of $S_{t+1}(E)$ such that $A_i \supset S_t(E)$ and $A_i/S_t(E)$ is simple for $i = 1, 2$. Then it is clear from the construction

of $S_i(E)$ that $J(A_i) = S_t(E)$ and hence A_i is hollow. Put $B = A_1 \oplus A_2$. Since B is generated by two elements, B has the extending property of simple modules by (3). Hence the identity mapping of $S_1(E)$ is extendable to an element g in $\mathrm{Hom}_R(A_1, A_2)$ by [9], Corollary 8. g is also extendable to an element in $\mathrm{End}_R(E)$. Hence $A_1 = A_2$ by the proof of [6], Theorem 6. Therefore R is a right lower serial ring, and so R is serial by Theorem 6.

(1)$\Rightarrow$(2),(3). This is clear by [5], Theorem 4, [9], Theorem 36 and [10].

If R is a right serial and local ring, every cyclic module has the lifting property and the extending property of simple modules. However, R is not a serial ring in general. In the above proof, we have utilized the lifting and extending property of simple modules for modules of the form $eR/eA \oplus fR/fB$, where e and f are primitive idempotents. If $R/J \approx \oplus_i (\Delta_i)_{n_i}$ and $n_i \geq 2$ for all i, $eR \oplus fR$ is always a cyclic module, where the Δ_i are division rings. Thus we have immediately the following theorems.

Theorem 8. *Let R be a perfect ring. Assume that $R/J = \oplus_i (\Delta_i)_{n_i}$ and $n_i \geq 2$ for all i. Then the following conditions are equivalent:*

(1) *R is a serial ring.*

(2) *Every cyclic module has the lifting property of simple modules modulo the radical.*

(3) *Every cyclic module has the extending property of simple modules.*

Theorem 9. *Let R be a perfect ring. Then the following conditions are equivalent:*

(1) *R is a serial ring.*

(2) *Every cyclic R_n-module has the lifting property of simple modules modulo the radical ($n \geq 2$).*

(3) *Every cyclic R_2-module has the above property.*

We have a similar statement for the extending property of simple modules.

Let A be an algebra over a field K of finite dimension. Then A is right coserial if and only if A is left serial. However, the example in [13] shows that this fact is not true for a left artinian ring. Assume that A is a local and right serial algebra. Then A is a quasi-Frobenius algebra by [8], Theorem 2. Hence A is a right coserial algebra and so A is serial. Dually, if A is a right serial and local algebra, then A is a serial algebra.

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