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ON THE GLOBAL DIMENSION OF THE FUNCTOR CATEGORY ((mod R)^{op}, Ab) AND A THEOREM OF KULIKOV

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1. Introduction

In representation theory as well as in other topics of ring theory it is convenient to consider for a ring R the categories (mod R, Ab) =: $D(R)$ and ((mod $R)^{op}$, Ab) =: $L(R)$ of additive covariant resp. contravariant abelian group-valued functors on the category mod R of finitely presented right R-modules. In particular the close connection of homological properties of these functor categories with decomposition properties of R-modules has attracted much attention (see e.g. [2], [10], [16], [17]).

Jensen shows as a main result of [10] that the commutative noetherian rings R such that the global dimension of $D(R)$ is ≤ 2 are precisely the commutative rings of finite representation type (i.e. the *artinian principal ideal – or Köthe rings*). In the present paper we show (Theorem 4.3) that the commutative noetherian rings R with glob.dim. $L(R) \leq 2$ are just the finite products $R = R_1 \times \cdots \times R_n$ of rings, where each R_i is a Köthe ring or a Dedekind domain. Together with Corollary 1 of [10] this result yields for example gl.dim. $D(\mathbb{Z}) = 3$ and gl.dim. $L(\mathbb{Z}) = 2$. Observe that in contrast to this example one can show for an Artin algebra R:

gl.dim. $D(R) = 2$ if and only if
gl.dim. $L(R) = 2$ if and only if
R is non-semisimple of finite representation type (cf. [2]).

So one may ask in general for the meaning of "gl.dim. $L(R) \leq 2$". We show that the right noetherian rings R of this property are precisely those who admit the following version of *Kulikov's theorem* ([6], Thm. 18.1): Every submodule of a pure projective right R-module is pure projective. (Recall that a module is *pure projective* if it is a direct summand of a direct sum of finitely presented modules.) Another characterizing property of those rings is the right noetherianess of the ringoid mod R (Theorem 2.1).

Since the *Kulikov property* (i.e. the heredity of pure projectivity) for a module category is something 'between' pure global dimension 0 and 1 it seems to be unnatural to handle it in terms of pure homological algebra. So theorem 2.1 below may be viewed as another homological approach to this aspect of representation theory. It's relative version (Theorem 3.2) seems to be useful also in the investigation of hereditary *algebras of tame representation type* (cf. [3], [14]). For artinian rings some consequences of the results below are already presented in [4].

2. Heredity of pure projectivity

In order to make – once for all – some basic arguments available let us work within a more general setting:

Denote by $\mathcal{A}$ an arbitrary *locally finitely presented category*, i.e. a Grothendieck category with a generating set of finitely presented objects (cf. [16]). Recall that an object $P \in \mathcal{A}$ is *finitely presented* provided the functor $\mathrm{Hom}(P,-): \mathcal{A} \to \mathrm{Ab}$ preserves directed colimits. The full subcategory $\mathcal{C}$ of finitely presented objects of $\mathcal{A}$ is skeletally small and we consider the category $(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$ of additive contravariant abelian group-valued functors on $\mathcal{C}$. $(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$ is also called the category Mod-$\mathcal{C}$ of *right-modules* over the ringoid (small additive category) $\mathcal{C}$. In the sense of Mitchell's several object version of ring theory (cf. [13], [11]), Mod-$\mathcal{C}$ may be treated just like a usual module category (for any ringoid $\mathcal{C}$). In particular, the representable functors of $\mathcal{C}^{\mathrm{op}}$ form a generating set of finitely generated projectives for Mod-$\mathcal{C}$ whose directed colimits are just the flat right $\mathcal{C}$-modules. If R is a ring and $\mathcal{A} := \text{Mod-}R$, then, clearly, $\mathcal{C} = \text{mod } R$ and Mod-$\mathcal{C} = L(R)$ (cf. the introduction).

Now we aim to prove the following theorem.

Theorem 2.1. *Let $\mathcal{A}$ be locally noetherian (i.e. the notions 'finitely presented' and 'finitely generated' coincide for objects of $\mathcal{A}$) and let $\mathcal{C}$ be the full subcategory of finitely generated objects of $\mathcal{A}$. The following statements are equivalent:*

(a) right glob.dim. $\mathcal{C} \leq 2$.

(b) $\mathcal{C}$ *is right noetherian* (*i.e.* Mod-$\mathcal{C}$ *is locally noetherian*).

(c) *Every submodule of a flat right $\mathcal{C}$-module has projective dimension* ≤ 1.

(d) *In $\mathcal{A}$ every subobject of a pure projective object is pure projective* (*Kulikov property*).

We begin with an easy but somehow surprising property of locally noetherian Grothendieck categories.

Lemma 2.2. *Let $\mathcal{A}$ be locally noetherian and $M \in \mathcal{A}$. Suppose $U = \bigoplus_{i \in I} U_i$ is a subobject of M such that M/U is finitely generated. Then, for some cofinite subset of J of I, the sum $\bigoplus_{i \in J} U_i$ is a direct subobject of M.*

Proof. First assume that U is essential in M. Since $\mathcal{M}$ is locally noetherian the formation of injective hulls commutes with direct sums [18]. So we get $\bigoplus H(U_i) = H(U) = H(M)$ for the respective injective hulls; moreover $M = U + V$ for some finitely generated $V \subset M$. Then $V \subset \bigoplus_{k \in K} H(U_k)$, for some finite $K \subset I$, and obviously $M = (\bigoplus_{i \in J} U_i) \oplus (V + \bigoplus_{k \in K} U_k)$, where $J := I \setminus K$. If U is arbitrary, then (by Zorn's lemma) there is some subobject W of M such that $W \oplus U$ is essential in M, and again $M/(W \oplus U)$ is finitely generated. So apply the first part of the proof to the direct sum $W \oplus (\bigoplus_{i \in I} U_i)$ to get the desired result.

Let us recall now some more or less well-known facts (cf. [16], [17], [7], [8]). The arbitrary locally finitely presented category $\mathcal{M}$ admits a full, left exact embedding

$$T : \mathcal{M} \to \text{Mod-}\mathcal{C},$$

given by

$$M \mapsto (-, M) := \text{Hom}(-, M)\,|_{\mathcal{C}},$$

which preserves limits and direct colimits. Flat $\mathcal{C}$-modules coincide with left exact functors on $\mathcal{C}^{op}$; denote the full subcategory of these (in Mod-$\mathcal{C}$) by $\mathcal{L}$. Then T induces an equivalence $\mathcal{M} \sim \mathcal{L}$ of categories. In particular, the finitely presented objects of $\mathcal{M}$ correspond (via T) to the finitely generated projective $\mathcal{C}$-modules (i.e. to the representable functors of $\mathcal{C}^{op}$) and the projective $\mathcal{C}$-modules are just the functors of the form $TP = (-, P)$, where $P \in \mathcal{M}$ is pure projective.

$\mathcal{L}$ is closed under kernels in Mod-$\mathcal{C}$, hence

$$\text{flat right glob.dim. } \mathcal{C} \leq 2.$$

Observe, however, that $\mathcal{L}$ is not closed in Mod-$\mathcal{C}$ under the formation of subobjects and cokernels.

Now we are ready to give the proof of Theorem 2.1.

Proof of Theorem 2.1. (a)$\Rightarrow$(b). Let $TC = (-, C)$, with $C \in \mathcal{C}$, be any small projective right $\mathcal{C}$-module and $U \subset TC$ be an arbitrary submodule. We have to show that U is finitely generated. By assumption,

$$\text{proj.dim. } U \leq 1;$$

so there is an exact sequence $0 \to TK \to TP \to TC$ of projective $\mathcal{C}$-modules, with image $(TP \to TC) = U$. This sequence comes from an exact sequence $0 \to K \to P \xrightarrow{f} C$ of pure projective objects in $\mathcal{M}$. We may assume that $K = \bigoplus_{i \in I} C_i$ for suitable finitely generated objects $C_i \in \mathcal{M}$ (eventually after adding to the above sequence an exact sequence of the form $0 \to L = L \to 0$, where $L \oplus K = \bigoplus_{i \in I} C_i$). Since $\mathcal{M}$ is locally noetherian P/K is finitely generated being isomorphic to a subobject of the finitely generated object C of $\mathcal{M}$; thus, for some cofinite subset J of the above index set I, the sum $\bigoplus_{i \in J} C_i =: S$ is a direct subobject of P (Lemma 2.2). Let F be a direct complement of S in P. Then, by construction, F is finitely generated (i.e. $F \in \mathcal{C}$) and $S \subset \ker f = K$. Therefore $f : P \to C$ is described by $(0, f\,|_F)$ with respect to the decomposition $S\oplus F$ of P. Consequently, $U=\mathrm{im}(Tf)=\mathrm{im}(TF\to TC)$ is a finitely generated $\mathscr{C}$-module since so is TF.

(b)$\Rightarrow$(c). If $\mathscr{C}$ is right noetherian, then

$$\text{right glob.dim. } \mathscr{C} = \text{flat right glob.dim. } \mathscr{C} \leq 2.$$

So assertion (c) is true for submodules of projective right $\mathscr{C}$-modules. Now let U be a submodule of an arbitrary flat right $\mathscr{C}$-module F. Then $F=TM$ for some $M\in\mathscr{A}$. By transfinite induction we may write M as a continuous well-ordered union of a chain $(M_\alpha)_{\alpha\in I}$ of subobjects over some ordinal I, such that $M_0=0$ and $M_{\alpha+1}/M_\alpha$ is finitely generated for all $\alpha+1\in I$. Similarly – applying T – F can be written as the continuous well-ordered union of the chain $(TM_\alpha)_{\alpha\in I}$ over I. Put $U_\alpha=U\cap TM_\alpha$ for all $\alpha\in I$. Then the Grothendieck property of Mod-$\mathscr{C}$ finally yields a continuous well-ordered chain $(U_\alpha)_{\alpha\in I}$ over I whose union is U. We further have $U_0=0$,

$$U_{\alpha+1}/U_\alpha=(U\cap TM_{\alpha+1})/((U\cap TM_{\alpha+1})\cap TM_\alpha)$$
$$\cong((U\cap TM_{\alpha+1})+TM_\alpha)/TM_\alpha\subset TM_{\alpha+1}/TM_\alpha$$

and $TM_{\alpha+1}/TM_\alpha\subset T(M_{\alpha+1}/M_\alpha)$ since T is left exact. Hence $U_{\alpha+1}/U_\alpha\subset T(M_{\alpha+1}/M_\alpha)$ for all $\alpha+1\in I$. But $T(M_{\alpha+1}/M_\alpha)$ is a projective $\mathscr{C}$-module since $M_{\alpha+1}/M_\alpha$ is finitely presented in $\mathscr{A}$. Thus, for all $\alpha+1\in I$, pr.dim.$(U_{\alpha+1}/U_\alpha)\leq 1$ by the initial remark. Consequently pr.dim. $U\leq 1$ by an argument of Auslander [1].

(c)$\Rightarrow$(d). Take any exact sequence $0\to U\to P\to Q\to 0$ in $\mathscr{A}$ with pure projective P. We have to show that U is pure projective. Indeed, $0\to TU\to TP\to TQ$ is an exact sequence of flat right $\mathscr{C}$-modules, where TP is projective. By (c) TU is projective, thus U is pure projective.

(d)$\Rightarrow$(a). By means of the functor T assumption (d) can be translated to the statement that every flat submodule of a projective $\mathscr{C}$-module is projective. Thus (a) follows from the fact that weak right glob.dim. $\mathscr{C}\leq 2$. The theorem is proved.

We pause to look at some applications.

First consider the *oriented cycle* Γ_n of length $n\geq 1$ and denote by $k[\Gamma_n]$ the path algebra of this quiver over some field k. Mod-$k[\Gamma_n]$ is isomorphic to the category of k-linear representations of Γ_n. In [3] it is shown that Mod-$k[\Gamma_n]$ admits Kulikov's theorem by pointing out that mod $k[\Gamma_n]$ is right noetherian.

Similarly, let $k[\mathbb{N}]$ be the k-linearization of the *ordered set* $\mathbb{N}$ (considered as a category; cf. [13], [5]). Mod-$k[\mathbb{N}]$ is hereditary and locally artinian being isomorphic to the category of k-linear representations of the ordered set $\mathbb{N}^{\mathrm{op}}$. Here a rather easy factorizing property of diagrams yields the right noetherianess of mod $k[\mathbb{N}]$ as is shown in [4]. Again, by Theorem 2.1, Mod-$k[\mathbb{N}]$ admits Kulikov's theorem.

Consider finally the classical example of a *Dedekind domain*. In this case we want to give a functorial application of Kulikov's theorem. For simplicity of notation let us restrict ourselves to the case $\mathscr{A}=\mathrm{Ab}$, $\mathscr{C}=\mathrm{mod}\,\mathbb{Z}$, Mod-$\mathscr{C}=L(\mathbb{Z})$. We know that the left exact functors on $\mathscr{C}^{\mathrm{op}}$ are simply those of the form $(-,M)=\mathrm{Hom}_{\mathbb{Z}}(-,M)$, $M\in\mathrm{Ab}$. What about right exact functors? With the aid of the theorem it turns out that a $\mathscr{C}$-module is a right exact functor if and only if it is a direct sum of copies of functors of the following four types (for various primes p):

$$\mathrm{Ext}(-,\mathbb{Z}(p^n)),\qquad \mathrm{Ext}(-,\mathbb{Z}_{(p)}),\qquad (-,\mathbb{Z}(p^\infty)),\qquad (-,\mathbb{Q}),$$

all restricted to $\mathscr{C}$; where $\mathbb{Z}(p^n)$ is the cyclic group of order p^n $(n\geq 1)$, $\mathbb{Z}_{(p)}$ the localization at p and $\mathbb{Z}(p^\infty)$ the Prüfer group belonging to p.

The proof of the latter statements may be performed in the following steps: By the original theorem of Kulikov and Theorem 2.1, $\mathscr{C}$ is right noetherian. So right exact functors coincide with *injective $\mathscr{C}$-modules*, moreover, every injective $\mathscr{C}$-module is a unique direct sum of indecomposable ones (Matlis). Further observe that the embedding $T:\mathrm{Ab}\to\mathrm{Mod}\text{-}\mathscr{C}$ (as above) has an exact left adjoint which vanishes on Ext-functors. Using this, together with the observation that $\mathrm{Ext}(-,\mathbb{Z})\cong\bigoplus_p\mathrm{Ext}(-,\mathbb{Z}_{(p)})$ on $\mathscr{C}$ (where p runs through all primes), one shows, that every finitely generated $\mathscr{C}$-Module is contained in a (finite) direct sum of functors of the above four types. Since these are indecomposable – indeed, their endomorphism rings are local – there are no other indecomposable injective $\mathscr{C}$-modules.

Remark. The latter example actually shows that for an arbitrary (reduced) abelian group M the unique decomposition of the injective functor $\mathrm{Ext}(-,M)$ attaches on M two sets of prime and cardinal invariants. These invariants will be described elsewhere.

3. A relative version of Kulikov's theorem

In this section we outline a relative version of Kulikov's theorem which mainly is suggested by the following situation (cf. [14]). Let A be a finite dimensional hereditary algebra of tame representation type over an algebraically closed field. let $\mathscr{R}\subset\mathrm{mod}\,A$ be the full subcategory of regular modules of finite length. An arbitrary module is called regular torsion if it is a directed union of objects of $\mathscr{R}$. Then the ringoid $\mathscr{R}$ is right noetherian and every regular torsion submodule of a direct sum of objects of $\mathscr{R}$ is again of this form.

So let A be an arbitrary ring and $\mathscr{R}\subset\mathrm{Mod}\,A$ be a full subcategory of finitely presented objects which is closed under direct summands and finite direct sums. We call a right A-module *$\mathscr{R}$-regular torsion* if it is a directed colimit of objects of $\mathscr{R}$. Some elementary properties of $\mathscr{R}$-regular torsion modules we summarize as follows.

Proposition 3.1. *Let $\mathscr{M}$ be the full subcategory of $\mathscr{R}$-regular torsion modules of* Mod-A. *Then*

(a) *The functor $T:\mathrm{Mod}\text{-}A\to\mathrm{Mod}\text{-}\mathscr{R}$, given by $M\mapsto(-,M):=\mathrm{Hom}(-,M)|_{\mathscr{R}}$, is left exact, preserves limits and directed colimits and induces an equivalence of $\mathscr{M}$ to the full subcategory of flat right $\mathscr{R}$-modules.*

(b) *The pure projective A-modules belonging to $\mathcal{M}$ are direct summands of direct sums of objects of $\mathcal{R}$ and correspond via T to the projective right $\mathcal{R}$-modules.*

(c) *If $\mathcal{R}$ is abelian, then $\mathcal{M}$ is locally finitely presented (and locally coherent) and $\mathcal{R}$ is the category of finitely presented objects of $\mathcal{M}$.*

(d) *If A is right noetherian and $\mathcal{R}$ an exact subcategory of* Mod-A, *then $\mathcal{M}$ is an exact locally noetherian subcategory of* Mod-A.

The proof of the proposition consists of applications of the Yoneda lemma and of properties of directed colimits and functor categories Mod-$\mathcal{R}$ with abelian $\mathcal{R}$ (cf. [7], [8], [12]). For instance, as to the fullness of T on $\mathcal{M}$, one shows first that

$$T: \mathrm{Hom}(M, N) \to \mathrm{Hom}(TM, TN)$$

is surjective if M and N are pure projective $\mathcal{R}$-regular torsion modules. If M and N are arbitrary $\mathcal{R}$-regular torsion modules, then there are canonical (pure) exact sequences

$$M_1 \to M_0 \to M \to 0, \qquad N_1 \to N_0 \to N \to 0,$$

where M_1, M_0, N_1, N_0 are direct sums of objects of $\mathcal{R}$ – the sequences expressing the fact that M and N are directed colimits of objects of $\mathcal{R}$. These colimit-sequences are preserved by T since the objects of $\mathcal{R}$ are finitely presented. Using now the projectivity of TM_1, TM_0 and the faithfulness of T on $\mathcal{M}$ one gets the fullness of T in the obvious fashion.

Now we can state a relative version of Theorem 2.1.

Theorem 3.2. *Let A be a ring and $\mathcal{R}$ be a full abelian subcategory of the finitely presented right A-modules such that every object of $\mathcal{R}$ has the ascending chain-condition on $\mathcal{R}$-subobjects. Then the full subcategory $\mathcal{M}$ of $\mathcal{R}$-regular torsion modules of* Mod-A *is a locally noetherian Grothendieck category and the following statements are equivalent:*

(a) right glob.dim. $\mathcal{R} \leq 2$.

(b) *$\mathcal{R}$ is right noetherian.*

(c) *If $0 \to F \to M \to P$ is an exact sequence of right A-modules with $M, P \in \mathcal{M}$, F torsion-free and P pure projective, then M is pure projective.*

Here 'torsion-free' is meant with respect to the torsion theory generated by $\mathcal{R}$ (cf. [18]). So the exactness of the above sequence simply means that $M \to P$ is a monomorphism in $\mathcal{M}$. Thus, with the aid of Theorem 2.1 and Proposition 3.1, the proof of the theorem is straightforward. Note that if in addition $\mathcal{R}$ is assumed to be an exact subcategory of Mod-A, then the regular torsion modules are just the directed unions of objects of $\mathcal{R}$ and condition (c) can be restated as follows:

Every regular torsion A-submodule of a pure projective regular torsion module is pure projective. This is the case in the above mentioned example of a finite dimensional hereditary algebra A of tame representation type.

4. Functor categories of small global dimensions

In this section we return to the study of global dimensions of the functor category $L(R) = ((\text{mod } R)^{op}, \text{Ab})$, where R is a ring.

First something about weak dimensions.

Proposition 4.1. *Let R be an arbitrary ring. Then* weak gl.dim. $L(R)$ *is either* 0 *or* 2 (weak gl.dim. $L(R)=1$ *does not occur*). *Moreover,* weak gl.dim. $L(R)=0$ *if and only if R is **von Neumann-regular**.*

Proof. Weak gl.dim. $L(R) \leq 2$, as was already remarked. Assume now weak gl.dim $L(R) \leq 1$. We show weak gl.dim $L(R)=0$. Again consider the full embedding $T: \text{Mod-}R \to L(R)$ we used in Section 2, whose image is the full subcategory of flat objects in $L(R)$. Let X be an arbitrary object of $L(R)$. By assumption we get a flat resolution $0 \to F_1 \to F_0 \to X \to 0$ of X in $L(R)$, which also may be written in the form $0 \to TM_1 \xrightarrow{Tf} TM_0 \to X \to 0$, where $f: M_1 \to M_0$ must be a monomorphism of right R-modules. Let $0 \to M_1 \to M_0 \to Q \to 0$ be exact in Mod-R. Then the left exactness of T yields $X \subset TQ$. Thus X is flat, $T: \text{Mod-}R \to L(R)$ is an equivalence of categories and R is von Neumann-regular.

On the other hand, if R is von Neumann-regular, then mod R coincides with the full subcategory of finitely generated projective R-modules, so that $T: \text{Mod-}R \to L(R)$ is an equivalence [12]. Thus weak gl.dim. $L(R)=0$.

As an immediate consequence we obtain:

Proposition 4.2. *Let R be a ring. Then* glob.dim. $L(R)=0$ *if and only if R is semi-simple artinian;* glob.dim. $L(R) \leq 1$ *if and only if R is hereditary and von Neumann regular.*

Finally we wanted to characterize rings with glob.dim. $L(R)=2$. We succeeded only in the case where R is commutative noetherian. So the following result may be viewed as a counterpart to the theorem in [10].

Theorem 4.3. *For a commutative noetherian ring R the following statements are equivalent.*

(a) glob.dim. $L(R) \leq 2$.

(b) *R admits Kulikov's theorem* (*in the original form*).

(c) $R = R_1 \times \cdots \times R_n$ $(n < \infty)$, *where each R_i is a Köthe ring or a Dedekind domain.*

Proof. (a)$\Leftrightarrow$(b). This is Theorem 2.1 for $\mathcal{M} := \text{Mod-}R$.

(c)$\Rightarrow$(b) is obvious.

(b)$\Rightarrow$(c). Suppose R has the Kulikov property. Then a routine consideration shows that all localizations $R_{\mathfrak{p}}$ at prime ideals $\mathfrak{p}$ as well as all residue-class rings $R/\mathfrak{a}$, for arbitrary ideals $\mathfrak{a}$, inherit the Kulikov property (and the noetherianess) from R.

First consider the case where R even is artinian. Then every indecomposable injective R-module is finitely generated (R being commutative), thus every injective R-module is pure projective. But then the Kulikov property forces *every* R-module to be pure projective, i.e. R is pure semisimple. For an Artin algebra this implies finite representation type (cf. [2]). Thus R is a Köthe ring.

Next let R be local with radical $\mathfrak{m}$ and residue-class field k. Then $R/\mathfrak{m}^2$ is an artinian ring with Kulikov property. Thus $R/\mathfrak{m}^2$ is a Köthe ring, as above; in particular $R/\mathfrak{m}^2$ is principal, i.e. $\dim_k \mathfrak{m}/\mathfrak{m}^2 = 1$. By a classical result this implies that R is principal, the powers $\mathfrak{m}^n$ ($n \geq 0$) and zero being the only ideals of R (cf. [15]). So R is either a Köthe ring (if it's Krull dimension is $\dim R = 0$) or a discrete valuation ring (if $\dim R = 1$).

Finally let R be arbitrary but not artinian. Using induction, it is enough to show that R has a direct factor which is a Dedekind domain. Since R is not artinian there is a prime ideal $\mathfrak{p}$ of R which is not maximal. By what we showed above $R/\mathfrak{p}$ is a noetherian integral domain such that al it's localizations are discrete valuation rings, hence $R/\mathfrak{p}$ is a Dedekind domain. To show that $R/\mathfrak{p}$ is a direct factor of R it is enough to show that $R/\mathfrak{p}$ is projective as an R-module. So take any maximal ideal $\mathfrak{m}$ of R, with $\mathfrak{m} \in \operatorname{support}(R/\mathfrak{p})$. Then $\mathfrak{p} \subsetneq \mathfrak{m}$ and $\mathfrak{p}_\mathfrak{m} \subsetneq \mathfrak{m}_\mathfrak{m} \subset R_\mathfrak{m}$, hence $\dim R_\mathfrak{m} \geq 1$. But using again the above local result we get $\dim R_\mathfrak{m} \leq 1$. Therefore $\dim R_\mathfrak{m} = 1$, hence $\mathfrak{p}_\mathfrak{m} = 0$ and $(R/\mathfrak{p})_\mathfrak{m} = R_\mathfrak{m}$. So the R-module $R/\mathfrak{p}$ is projective since $(R/\mathfrak{p})_\mathfrak{m}$ is a projective $R_\mathfrak{m}$-module for every maximal ideal $\mathfrak{m}$.

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