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Preserving and reflecting covers by functors. Applications to graded modules

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Abstract

We study $\mathscr{C}$-covers in the context of Grothendieck categories. Namely, we analyse when a functor between two Grothendieck categories preserves or reflects $\mathscr{C}$-covers. We apply our general study to the category of graded modules over a graded ring, by showing that relative injective covers with respect to a torsion theory are preserved and reflected, in some cases, among the categories R-gr, R_1-Mod and R-Mod.

1. Introduction

Let $\mathscr{C}$ be a Grothendieck category and $\mathscr{A}$ a full subcategory of $\mathscr{C}$. The general problem of the existence of $\mathscr{A}$-(pre)covers for every object in $\mathscr{C}$ is an interesting question. For concrete $\mathscr{C}$ and $\mathscr{A}$, the characterization of the existence of $\mathscr{A}$-(pre)covers has allowed to obtain special properties for $\mathscr{C}$. For example, if $\mathscr{A}$ is the full subcategory of injective objects in $\mathscr{C}$, every object in $\mathscr{C}$ has an injective (pre)cover if and only if $\mathscr{C}$ is a locally noetherian category. This result was given by E. Enochs for the case of $\mathscr{C} = R - Mod$ and it has been the first step in a homology theory for modules, using injective resolvents [2, 3, 7, 8]. In [4, 16], the existence of (pre)covers of injectives relative to an hereditary torsion theory was considered.

In Section 2, we find some conditions under which a given functor preserves and (or) reflects (pre)covers. For this, the concept of separable functors (cf. [12]) is fundamental. In Section 3, we apply these results to study the relationship between (relative) injective

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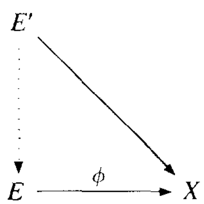
(pre)covers in R-Mod and the category of graded left R-modules for $R = \bigoplus_{g \in G} R_g$ a graded ring. We also obtain precovers in R-Mod from precovers in R_1-Mod.

The study of relative injective (pre)covers could be important for the development of a relative homology theory.

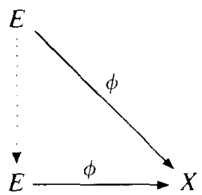
2. General results

In this section $\mathscr{C}$ and $\mathscr{D}$ will denote arbitrary Grothendieck categories. Let $\mathscr{A}$ be a class of objects in $\mathscr{C}$. We recall the definition introduced by Enochs in [2].

Definition 1. Let X be an object of $\mathscr{C}$. We say that E in $\mathscr{A}$ is a $\mathscr{A}$-precover of X if there exists a homomorphism $\phi : E \to X$ such that the triangle



can be completed for each homomorphism $E' \to X$ with E' in $\mathscr{A}$. If the triangle



can be completed only by automorphisms, we say that $\phi : E \to X$ is a $\mathscr{A}$-cover.

Throughout this section let $F : \mathscr{C} \longrightarrow \mathscr{D}$ be a covariant functor and $\mathscr{A} \subseteq \mathscr{C}$ a full subcategory of $\mathscr{C}$ closed under isomorphisms. Suppose that $F(\mathscr{A}) = \{F(A) | A \in \mathscr{A}\}$ is full in $\mathscr{D}$ and closed under isomorphisms.

Definition 2. We say that a covariant functor $F : \mathscr{C} \longrightarrow \mathscr{D}$ preserves (resp. reflect) $\mathscr{A}$-(pre)covers in the case that if $\phi : E \to X$ is a $\mathscr{A}$-(pre)cover, then $F(\phi) : F(E) \to F(X)$ is a $F(\mathscr{A})$-(pre)cover (resp. if $F(\phi) : F(E) \to F(X)$ is a $F(\mathscr{A})$-(pre)cover then $\phi : E \to X$ is a $\mathscr{A}$-(pre)cover).

We are going to study when F preserves or reflects $\mathscr{A}$-(pre)covers.

Proposition 1. *If F is an equivalence of categories, then it preserves and reflects $\mathscr{A}$-covers.*

Proof. Easy. $\square$

However, there exist a more general class of functors that preserve and reflect $\mathscr{A}$-(pre)covers in a separate way.

Recall the concept of separable functor given in [12, Section 1].

Definition 3. A covariant functor $F: \mathscr{C} \longrightarrow \mathscr{D}$ is said to be a separable functor if for all objects M, N in $\mathscr{C}$ there are maps $\varphi^F_{M,N}$:

$$\varphi^F_{M,N}: Hom_{\mathscr{D}}(F(M), F(N)) \longrightarrow Hom_{\mathscr{C}}(M,N),$$

satisfying the following conditions:

SF1. For $\alpha \in Hom_{\mathscr{C}}(M,N)$ we have $\varphi^F_{M,N}(F(\alpha)) = \alpha$.

SF2. Given $M', N' \in \mathscr{C}$, $\alpha \in Hom_{\mathscr{C}}(M,M')$, $\beta \in Hom_{\mathscr{C}}(N,N')$, $f \in Hom_{\mathscr{D}}(F(M), F(N))$, $g \in Hom_{\mathscr{D}}(F(M'),F(N'))$ such that the following diagram is commutative:

$$\begin{array}{ccc} F(M) & \xrightarrow{\ f\ } & F(N) \\ \downarrow{\scriptstyle F(\alpha)} & & \downarrow{\scriptstyle F(\beta)} \\ F(M') & \xrightarrow[\ g\ ]{} & F(N') \end{array}$$

Then the following diagram is also commutative:

$$\begin{array}{ccc} M & \xrightarrow{\ \varphi^F_{M,N}(f)\ } & N \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} \\ M' & \xrightarrow[\ \varphi^F_{M',N'}(g)\ ]{} & N' \end{array}$$

It is clear that SF1 implies that a separable functor is faithful. Conversely, if F is full and faithful (but not necessarily an equivalence), then F is separable. We collect this fact in the next lemma.

Lemma 1. *The following affirmations are equivalent about a covariant functor* $F: \mathscr{C} \longrightarrow \mathscr{D}$.

(a) *F is full and faithful.*

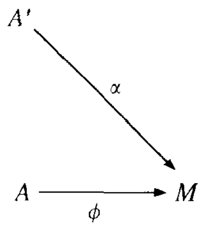
(b) *F is full and separable.*

Proposition 2. *Let $F: \mathscr{C} \longrightarrow \mathscr{D}$ be a covariant functor with $\mathscr{A}$ and $F(\mathscr{A})$ full subcategories in $\mathscr{C}$ and $\mathscr{D}$, respectively. Then,*

(1) *If F is full and faithful, F preserves $\mathscr{A}$-covers.*
(2) *If F is separable, F reflects $\mathscr{A}$-covers.*

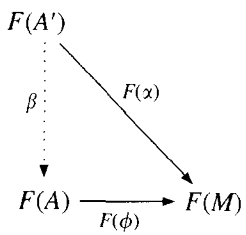
Proof. (1) Is easy, so we will prove (2). For that, we check: $F(\phi) : F(A) \to F(M)$ is a $F(\mathscr{A})$-cover implies $\phi : A \to M$ is a $\mathscr{A}$-cover.

We consider the diagram

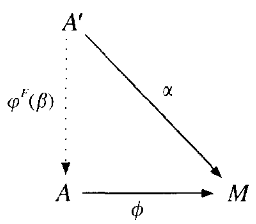


with A' in $\mathscr{A}$.

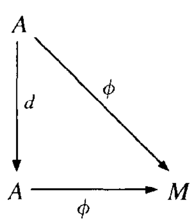
Applying F we have the commutative triangle



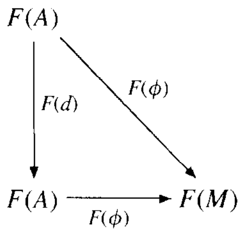
By definition of separability we deduce the commutativity of the triangle



Hence $\phi: A \to M$ is an $\mathscr{A}$-precover. Now, we will see that it is a $\mathscr{A}$-cover. We consider the commutative triangle



Applying F we obtain



$F(d)$ is an isomorphism because $F(\phi)$ is an $F(\mathscr{A})$-cover. Therefore, there exists a $\delta \in Hom_{\mathscr{D}}(F(A), F(A))$ such that $\delta F(d) = F(d)\delta = 1_{F(A)}$. Hence:

$$\varphi^F(\delta)\varphi^F(F(d)) = \varphi^F(F(d))\varphi^F(\delta) = \varphi^F(1_{F(A)}).$$

So $\varphi^F(\delta)d = d\varphi^F(\delta) = 1_A$, and so d is an isomorphism. □

Now, we give the concepts of full, faithful, and separable functor relative to a full subcategory in $\mathscr{C}$.

Definition 4. For $F : \mathscr{C} \longrightarrow \mathscr{D}$ a covariant functor and $\mathscr{A}$ a full subcategory in $\mathscr{C}$ with $F(\mathscr{A})$ full in $\mathscr{D}$, we say that:

(1) F is $\mathscr{A}$-full (resp. $\mathscr{A}$-faithful) in the case that for every $A \in \mathscr{A}$ and $X \in \mathscr{C}$, the abelian group morphisms

$$F(A,X) : Hom_{\mathscr{C}}(A,X) \longrightarrow Hom_{\mathscr{D}}(F(A), F(X))$$

are surjective (resp. injective);

(2) F is $\mathscr{A}$-separable if $F|_{\mathscr{A}} : \mathscr{A} \longrightarrow F(\mathscr{A})$ is a separable functor.

Proposition 3. *Let $F : \mathscr{C} \longrightarrow \mathscr{D}$ be a covariant functor with $\mathscr{A}$ and $F(\mathscr{A})$ full subcategories in $\mathscr{C}$ and $\mathscr{D}$, respectively.*

(1) If F is $\mathscr{A}$-full, then F preserves $\mathscr{A}$-precovers.

(2) If F is $\mathscr{A}$-full and $\mathscr{A}$-faithful, then F preserves $\mathscr{A}$-covers.

(3) If F is $\mathscr{A}$-separable, F reflects $\mathscr{A}$-covers.

Proof. Easy. □

Throughout we will denote by $(F,G) : \mathscr{C} \longrightarrow \mathscr{D}$ an adjoint situation with $\mathscr{A} \subseteq \mathscr{C}$ and $\mathscr{B} \subseteq \mathscr{D}$ full subcategories verifying $F(\mathscr{A}) \subseteq \mathscr{B}$ and $G(\mathscr{B}) \subseteq \mathscr{A}$.

Proposition 4. *If $Y' \in \mathscr{D}$ has a $\mathscr{B}$-precover $\phi : X' \to Y'$, then $G(\phi) : G(X') \to G(Y')$ is a $\mathscr{A}$-precover of $G(Y')$.*

Proof. Let $f : A \to G(Y')$ be a morphism in $\mathscr{C}$. Then, there exists $g : F(A) \to X'$ such that $\phi g = \varepsilon_{Y'} F(f)$. Hence $G(\phi g) = G(\varepsilon_{Y'} F(f)) = f$, and so there exists a morphism in $\mathscr{C}$, namely $G(g)$, such that $G(\phi)G(g) = f$. $\square$

The following proposition is of preparatory nature. In [14, Theorem 1.2] appears the proof of (1) $\Leftrightarrow$ (3) in (a) and (b), respectively, and so we do not give them.

Proposition 5. *Let $(F,G) : \mathscr{C} \longrightarrow \mathscr{D}$ be an adjoint situation with $\varepsilon : FG \longrightarrow 1_{\mathscr{D}}$ and $\eta : 1_{\mathscr{C}} \longrightarrow GF$ the co-unit and the unit of the adjunction, respectively.*

(a) *The following assertions are equivalent.*
  (1) *G is a separable functor.*
  (2) *FG is a separable functor.*
  (3) *ε is, as a natural transformation, a splitting epimorphism; i.e., there exists $\bar{\varepsilon} : 1_{\mathscr{D}} \longrightarrow FG$ such that $\varepsilon\bar{\varepsilon} = 1_{\mathscr{D}}$.*

(b) *The following assertions are equivalent.*
  (1) *F is a separable functor.*
  (2) *GF is a separable functor.*
  (3) *η is, as a natural transformation, a splitting monomorphism; i.e., there exists $\bar{\eta} : GF \longrightarrow 1_{\mathscr{C}}$ such that $\bar{\eta}\eta = 1_{\mathscr{C}}$.*

Proof. (a) First, we will prove (2) $\Rightarrow$ (3). If FG is separable, for each $X, X' \in \mathscr{D}$ there exists $\varphi^{FG}_{X,X'} : Hom_{\mathscr{D}}(FGX, FGX') \longrightarrow Hom_{\mathscr{D}}(X,X')$ verifying the separability conditions. We will see that $\varphi(F * \eta * G)$ ($*$ represents the Yoneda product which throughout we will omit) is the required $\bar{\varepsilon}$. First we will check that $\varphi F\eta G$ is a natural transformation. Let $X,X' \in \mathscr{D}$ and $f \in Hom_{\mathscr{D}}(X,X')$. From the diagram

$$\begin{array}{ccc} GX & \xrightarrow{\eta_{GX}} & GFGX \\ {\scriptstyle G(f)}\downarrow & & \downarrow{\scriptstyle GFG(f)} \\ GX' & \xrightarrow[\eta_{GX'}]{} & GFGX' \end{array}$$

we obtain, applying the functor F

$$\begin{array}{ccc} FGX & \xrightarrow{F\eta_{GX}} & FGFGX \\ {\scriptstyle FG(f)}\downarrow & & \downarrow{\scriptstyle FGFG(f)} \\ FGX' & \xrightarrow[F\eta_{GX'}]{} & FGFGX' \end{array}$$

By separability, we have the commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\varphi\,(F\eta_{GX})} & FGX \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle FG(f)} \\ X' & \xrightarrow[\varphi\,(F\eta_{GX'})]{} & FGX' \end{array}$$

Therefore, $\varphi F\eta G : 1_{\mathscr{S}} \longrightarrow FG$ is a natural transformation.

Now, we will see that $\varepsilon(\varphi F\eta G) = 1_{\mathscr{S}}$. From the commutative diagram

$$\begin{array}{ccc} FGX & \xrightarrow{F\eta_{GX}} & FGFGX \\ {\scriptstyle FG(1)}\downarrow & & \downarrow{\scriptstyle FG(\varepsilon_X)} \\ FGX & \xrightarrow[FG(1)]{} & FGX \end{array}$$

(the commutativity is obtained by $G\varepsilon_X\eta_{GX} = 1_{GX}$ because (F,G) is an adjunction) we obtain by separability

$$\begin{array}{ccc} X & \xrightarrow{\varphi F\eta_{GX}} & FGX \\ {\scriptstyle 1}\downarrow & & \downarrow{\scriptstyle \varepsilon_X} \\ X & \xrightarrow[1]{} & X \end{array}$$

Hence $\varepsilon_X\varphi F\eta_{GX} = 1_X$, $\forall X$ and $\bar{\varepsilon} = \varphi F\eta G$.

(3) $\Rightarrow$ (2) Suppose that there exists a natural transformation $\bar{\varepsilon} : 1_{\mathscr{S}} \longrightarrow FG$ such that $\varepsilon\bar{\varepsilon} = 1_{\mathscr{S}}$. For $X, X' \in \mathscr{S}$ we define

$$\varphi^{FG}_{X,X'} : Hom_{\mathscr{S}}(FGX, FGX') \longrightarrow Hom_{\mathscr{S}}(X, X')$$

by $\varphi^{FG}(g) = \varepsilon_{X'}g\bar{\varepsilon}_X$. We verify SF1 of the definition of separability:

Let $\alpha \in Hom_{\mathscr{S}}(X, X')$, as ε is a natural transformation we have the diagram

$$\begin{array}{ccc} FGX & \xrightarrow{\varepsilon_X} & X \\ {\scriptstyle FG\alpha}\downarrow & & \downarrow{\scriptstyle \alpha} \\ FGX' & \xrightarrow[\varepsilon_{X'}]{} & X' \end{array}$$

We obtain $\varphi^{FG}(FG(\alpha)) = \varepsilon_{X'}FG(\alpha)\bar{\varepsilon}_X = \alpha\varepsilon_X\bar{\varepsilon}_X = \alpha$ and we have SF1.

We will prove SF2. Let $X, X', Y, Y' \in \mathscr{D}$ and we consider the following commutative diagram:

$$\begin{array}{ccc} FGX & \xrightarrow{f} & FGY \\ {\scriptstyle FG\alpha}\downarrow & & \downarrow{\scriptstyle FG\beta} \\ FGX' & \xrightarrow[g]{} & FGY' \end{array}$$

We want to prove that the diagram

$$\begin{array}{ccc} X & \xrightarrow{\varepsilon_Y f \bar{\varepsilon}_X} & Y \\ {\scriptstyle \alpha}\downarrow & & \downarrow{\scriptstyle \beta} \\ X' & \xrightarrow[\varepsilon_{Y'} g \bar{\varepsilon}_{X'}]{} & Y' \end{array}$$

is also commutative. We have the following: $\varepsilon_{Y'} g \bar{\varepsilon}_{X'} \alpha = \varepsilon_{Y'} g FG(\alpha) \bar{\varepsilon}_X = \varepsilon_{Y'} FG(\beta) f \bar{\varepsilon}_X = \beta \varepsilon_Y f \bar{\varepsilon}_X$ as we wanted (these equalities are valid only when $\bar{\varepsilon}$ is a natural transformation and not when $\bar{\varepsilon}_X$ is a splitting epimorphism $\forall X \in \mathscr{D}$).

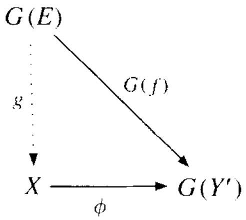
(b) The proof is analogous to (a). For completeness, if GF is separable with a map of separation φ, the natural transformation $\bar{\eta}$ such that $\bar{\eta}\eta = 1$ is $\bar{\eta} = \varphi(G\varepsilon F)$. Conversely, if $\bar{\eta}$ is the natural transformation such that $\bar{\eta}\eta = 1$ then we define φ by $\varphi(f) = \bar{\eta} f \eta$. □

The following corollary is a trivial consequence of the above.

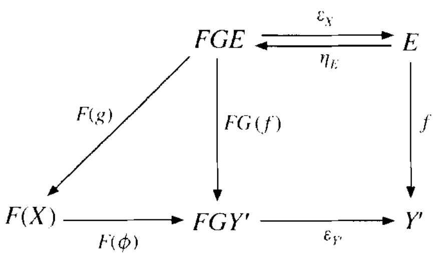
Corollary 1. *Let* $(F, G) : \mathscr{C} \longrightarrow \mathscr{D}$ *be an adjoint situation with* $\varepsilon : FG \longrightarrow 1_{\mathscr{D}}$ *the co-unit of the adjunction. If* FG *is a separable functor, then* $\varepsilon_X : FGX \to X$ *is a splitting epimorphism* $\forall X \in \mathscr{D}$.

Proposition 6. *In the adjoint situation above, if the co-unit of the adjunction* ε *verifies that* $\varepsilon_{Z'} : FGZ' \to Z'$ *is a splitting epimorphism* $\forall Z' \in \mathscr{B}$ *(weaker than* ε *to be a splitting epimorphism as a natural transformation) then: if* $Y' \in \mathscr{D}$ *is such that* $G(Y') \in \mathscr{C}$ *has an* $\mathscr{A}$*-precover* $\phi : X \to G(Y')$*, then* $F(X) \xrightarrow{F(\phi)} FG(Y') \xrightarrow{\varepsilon_{Y'}} Y'$ *is a* $\mathscr{B}$*-precover of* Y'.

Proof. Let $E \in \mathscr{B}$ and $f : E \to Y'$. Applying G we have the commutative diagram



Applying F to the later triangle we can obtain the following diagram with square and triangle commutatives



Hence $\varepsilon_{Y'}F(\phi)F(g)\eta_E = \varepsilon_{Y'}FG(f)\eta_E = f\varepsilon_E\eta_E = f$. Therefore, the morphism $F(g)\eta_E$ makes the diagram commutative and so $\varepsilon_{Y'}F(\phi)$ is a $\mathscr{B}$-precover. □

Now, the following corollary is immediate.

Corollary 2. *In the adjoint situation* (F, G) *suppose that* FG *is a* $\mathscr{B}$*-separable functor, if* $Y' \in \mathscr{D}$ *is such that* $G(Y') \in \mathscr{C}$ *has an* $\mathscr{A}$*-precover* $\phi : X \to G(Y')$, *then* $F(X) \overset{F(\phi)}{\to} FG(Y') \overset{\varepsilon_{Y'}}{\to} Y'$ *is a* $\mathscr{B}$*-precover of* Y'.

Proof. It is routine to check that the co-unit of the adjunction is a splitting epimorphism if we restrict it to $\mathscr{A}$ (routine with the above results). □

In the applications given below, we will concentrate our attention on τ-injective (pre)covers. Remember that if τ is a hereditary torsion theory defined in a Grothendieck category $\mathscr{C}$ (for concepts about torsion theories we will refer to [5, 15]), we say that an object E in $\mathscr{C}$ is τ-injective if $Ext^1_{\mathscr{C}}(X, E) = 0$ for all $X \in \mathscr{T}_\tau$, where $\mathscr{T}_\tau$ denotes the class of all τ-torsion objects. By τ-injective (pre)covers, we mean $\mathscr{A}$-(pre)covers, where $\mathscr{A}$ is the class of τ-injective objects in $\mathscr{C}$.

Theorem 1. *Let* $(F, G) : \mathscr{C} \longrightarrow \mathscr{D}$ *be an adjoint situation and* (η, ε) *the unit and the co-unit of the adjunction, respectively. Let* τ *be a hereditary torsion theory in* $\mathscr{C}$ *and* $\mathscr{A}$ *be a full subcategory in* $\mathscr{D}$. *We denote by* $\mathscr{I}_\tau$ *the class of* τ*-injective objects in* $\mathscr{C}$. *Suppose that it verifies the following conditions.*

(1) $F(\mathscr{I}_\tau) \subseteq \mathscr{A}$, $G(\mathscr{A}) \subseteq \mathscr{I}_\tau$.

(2) *For each* $M \in \mathscr{C}$, *the sequence*

$$0 \to Ker\,\eta(M) \to M \overset{\eta(M)}{\longrightarrow} GF(M) \to Coker\,\eta(M) \to 0$$

has $Ker\,\eta(M) = 0$ *and* $Coker\,\eta(M)$ *τ-torsion free.*

(3) *For each* $D \in \mathscr{D}$, $FG(D) \overset{\varepsilon(D)}{\longrightarrow} D$ *is a splitting epimorphism.*

(4) G *is right exact.*

The following statements are verified.

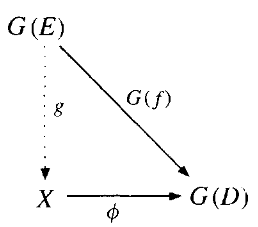
(a) *If every object in* $\mathscr{C}$ *has a τ-injective precover, then every object in* $\mathscr{D}$ *has an* $\mathscr{A}$*-precover.*

(b) *If every object in* $\mathscr{D}$ *in the form* $F(M)$ *(for some* $M \in \mathscr{C}$*) has an epic* $\mathscr{A}$*-precover, then every object in* $\mathscr{C}$ *has an epic τ-injective precover.*

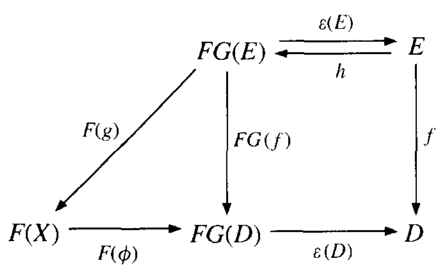
Proof. (a) Let $D \in \mathscr{D}$, then there exists a τ-injective precover in $\mathscr{C}$ in the form $\phi : X \to G(D)$. We will see that

$$F(X) \overset{F(\phi)}{\longrightarrow} FG(D) \overset{\varepsilon(D)}{\longrightarrow} D$$

is an $\mathscr{A}$-precover in $\mathscr{D}$. Let $E \in \mathscr{A}$ and $f : E \to D$ a morphism in $\mathscr{D}$. We have the completed commutative triangle



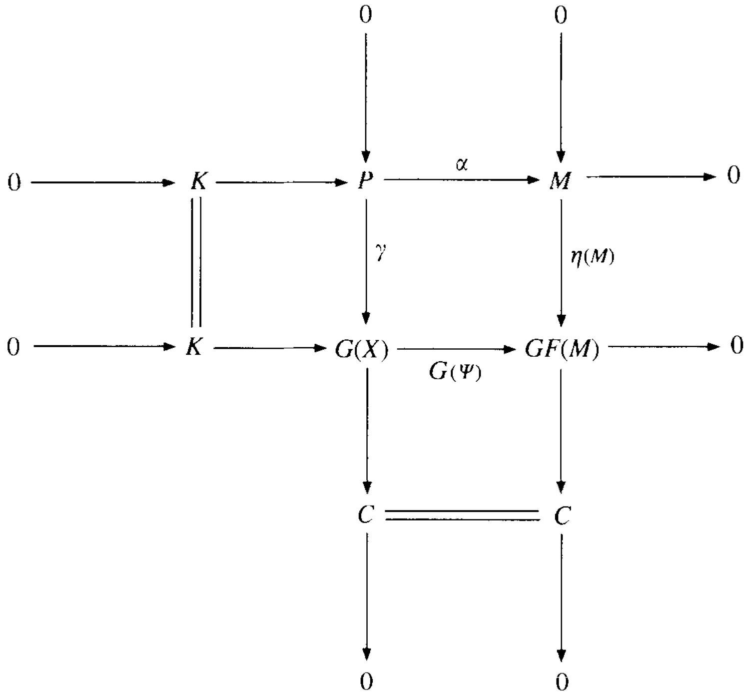
Applying F, we obtain the commutative diagram



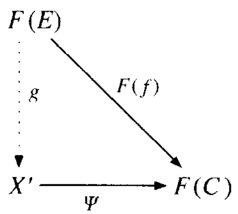
Hence $\varepsilon(D)F(\phi)F(g)h = \varepsilon(D)FG(f)h = f\varepsilon(E)h = f$, therefore there exists $F(g)h : E \to F(X)$ such that $\varepsilon(D)F(\phi)F(g)h = f$ as we wished.

(b) Let $M \in \mathscr{C}$. By hypothesis, there exists an $\mathscr{A}$-precover in $\mathscr{D}$ in the form $X' \overset{\psi}{\to} F(M) \to 0$. Since G is right exact, the sequence $G(X') \overset{G(\psi)}{\longrightarrow} GF(M) \to 0$ is exact.

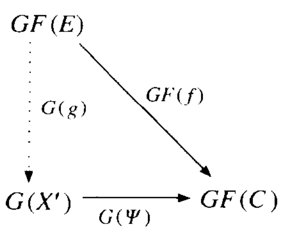
Now, we consider the following pullback diagram:



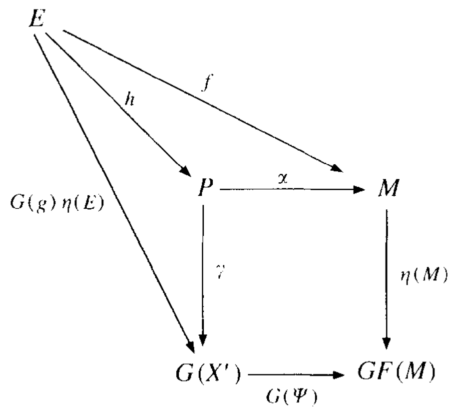
C is τ-torsion free and, by hypothesis, $Ker\,\eta(M) = 0$. Also, since $G(X') \in \mathscr{I}_\tau$ and C is τ-torsion free, it follows that $P \in \mathscr{I}_\tau$. We will see that $P \xrightarrow{\alpha} M \to 0$ is a τ-injective precover. We take $E \in \mathscr{I}_\tau$ and $f : E \to M$ a morphism in $\mathscr{C}$. The following diagram can be completed:



Applying G, we obtain the commutative diagram



By definition of pullback, we obtain the diagram



The commutativity of the diagram is given by: $G(\psi)G(g)\eta(E) = G(\psi g)\eta(E) = GF(f)\eta(E) = \eta(M)f$ (the last equality is obtained because η is a natural transformation) then there exists $h : E \to P$ such that $\alpha h = f$ as we wished. □

Remark. If we take for $\mathscr{I}_\tau$ the class of τ-torsion free τ-injective objects in $\mathscr{C}$, the theorem remains true.

3. Covers of graded modules

Let G be a multiplicative group with identity element 1. A G-graded ring R is a ring with identity 1, together with a direct sum $R = \bigoplus_{g\in G} R_g$ as additive subgroups, such that: $R_g R_h \subseteq R_{gh}$ for all $g, h \in G$. It is well known that R_1 is a subring of R and $1 \in R_1$. If $R_g R_h = R_{gh}$ for all $g, h \in G$, then R is called a strongly graded ring. By a (left) G-graded module we mean a left R-module M with a direct sum decomposition $\bigoplus_{g\in G} M_g$ as additive subgroups, such that $R_g M_h \subseteq M_{gh}$ for all $g, h \in G$. If $N = \bigoplus_{g\in G} N_g$ is another graded R-module, then $Hom_{R-gr}(M,N)$ consists of the R-homomorphism $f : M \to N$ such that $f(M_g) \subseteq N_g$. We denote by $R - gr$ the category of left G-graded modules and G-graded homomorphisms. It is well known that $R - gr$ is a Grothendieck category (cf. [13]). Then it is obvious that we can define torsion theories and injective (pre)covers relative to a torsion theory in $R - gr$.

Following [11, Section 4], we consider the adjunction of functors $(U,F) : R - gr \longrightarrow R - Mod$, where U forgets the gradiation and for $M \in R$-Mod, $F(M) = \bigoplus_{g\in G}({}^gM)$ (where each ${}^gM$ is a copy of M, ${}^gM = \{{}^gm | m \in M\}$) with the structure of R-module given by $r *^g m =^{hg} (rm)$ for $r \in R_h$. The gradiation of $F(M)$ is given by $F(M)_g = ({}^gM)$, $g \in G$. We know that U is the left adjoint of F and, when G is finite, is the right adjoint too.

Between the Grothendieck categories R_1-Mod and R-gr we consider the following functors:

(1) $(-)_1 : R-gr \longrightarrow R_1 - Mod$, $(M)_1 = M_1$.

(2) $Coind: R_1 - Mod \longrightarrow R-gr$, $Coind(N) = \bigoplus_{g\in G} Coind(N)_g$, with $Coind(N)_g = \{f \in Hom_{R_1}({}_{R_1}R_R, N) \mid f(R_h) = 0 \ : \forall h \neq g^{-1}\}$.

(3) $Ind: R_1 - Mod \longrightarrow R - gr$, $Ind(N) = R \otimes_{R_1} N$.

It is known [9] that *Ind* is the left adjoint of $(-)_1$ and *Coind* is the right adjoint of $(-)_1$.

Let τ be a rigid torsion theory on R-gr (for the definition of rigid torsion theory see [11, Section 4]), and $\bar{\tau}$ the induced torsion theory on R-Mod. We know that [11, Proposition 4.2] $X \in R$-Mod is $\bar{\tau}$-torsion if and only if $F(X) \in R$-gr is τ-torsion.

Let σ be a torsion theory on R_1-Mod. σ is said to be G-stable if, for any σ-torsion R_1-module M, $R_g \otimes_{R_1} M$ is σ-torsion, for all $g \in G$. In [13], it is proved that if R is a strongly graded ring, then there exists a bijective correspondence between rigid torsion theories on R-gr and G-stable torsion theories on R_1-Mod. We will denote by σ^{gr} the corresponding torsion theory on R-gr induced by σ.

In fact, when R is strongly graded, R-gr and R_1-Mod are equivalent categories. The equivalence is given by the functors *Coind* and $(-)_1$; in this case $Ind \cong Coind$, (see [1, Theorem 2.8]).

Proposition 7. *Let $R = \bigoplus_{g\in G} R_g$ a strongly graded ring and κ a G-stable hereditary torsion theory on R_1-Mod.*

Every R_1-module has a κ-injective cover if and only if every graded R-module has a κ^{gr}-injective cover.

Proof. First, we are going to check the following two claims:

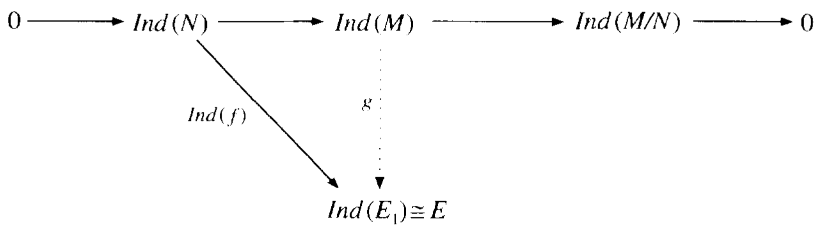
(a) If $E \in R\text{-}gr$ is κ^{gr}-injective, then E_1 is κ-injective.

(b) If $E \in R_1$-Mod is κ-injective, then $Ind(E)$ is κ^{gr}-injective.

(a) Let $E \in R\text{-}gr$ κ^{gr}-injective. We consider an exact sequence in R_1-Mod

$$0 \to N \to M \to M/N \to 0$$

with M/N κ-torsion, and $f : N \to E_1$ an R_1-homomorphism. Applying *Ind*, we can complete the triangle



(Since κ is G-stable, $Ind(M/N)$ is κ^{gr}-torsion. Also Ind is exact in this case). Now, applying $(-)_1$ to the above diagram, we have

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Ind(N)_1 & \longrightarrow & Ind(M)_1 & \longrightarrow & Ind(M/N)_1 & \longrightarrow & 0 \\ & & & \searrow^{Ind(f)_1} & \vdots\, g_1 & & & & \\ & & & & \downarrow & & & & \\ & & & & E_1 & & & & \end{array}$$

Since $Ind(-)_1 \cong 1_{R_1-Mod}$ and $(-)_1 Ind \cong 1_{R-gr}$, the proof of (a) is finished.

The proof of (b) is similar to (a).

Let $M \in R$-gr, then there exists a κ-injective cover for $M_1 : \phi : E \to M_1$ in R_1-Mod. We will see that $Ind(\phi) : Ind(E) \to M$ is a κ^{gr}-injective cover for M in R-gr. By (b), $Ind(E)$ is κ^{gr}-injective. We consider E' a κ^{gr}-injective G-graded R-module and $f : E' \to M$ a G-graded morphism. Then there exists an R_1-morphism $g : E'_1 \to Ind(E)_1 = E$ such that $\phi g = f_1$. Hence $Ind(\phi g) = Ind(f_1)$ and so $Ind(\phi)Ind(g) = f$ as we wished. If $h : Ind(E) \to Ind(E)$ verifies that $Ind(\phi)h = Ind(\phi)$, then $\phi h_1 = \phi$, therefore h_1 is an automorphism. Since Ind is an equivalence of categories, it follows that $Ind(h_1) = h$ is an automorphism.

Conversely, let $M \in R_1$-Mod, then there exists a κ^{gr}-injective cover for $Ind(M)$: $\psi : E \to Ind(M)$. It is easy to check (analogous to the above) that $\Psi_1 : E_1 \to M$ is a κ-injective cover for M in R_1-Mod. $\square$

Now, the following result is immediate.

Corollary 3. *Let $R = \bigoplus_{g \in G} R_g$ a strongly graded ring. The following assertions are equivalent.*

Every R_1-module has an injective cover if and only if every graded R-module has a gr-injective cover.

Proposition 8. *Suppose that $R = \bigoplus_{g \in G} R_g$ is left noetherian and with finite support. If $\phi : E \to M$ is an injective cover in R-Mod with $E, M \in$ R-gr and ϕ a G-graded morphism, then $\phi : E \to M$ is a gr-injective cover in R-gr.*

Proof. By hypothesis, the class of gr-injective objects in R-gr coincides with the class of injective objects (as R-modules) in R-gr, [9, Corollary 2.3]. Hence, let $E' \in R$-gr an injective $=$ gr-injective object and let $g : E' \to M$ be a G-graded morphism. Then there exists a "R-*morphism*" $h : E' \to E$ such that $\phi h = g$. By [13, Lemma I.2.1], there exists a G-graded morphism $h' : E' \to E$ such that $\phi h' = g$ and so $\phi : E \to M$ is a gr-injective precover. If $f : E \to E$ is a G-graded morphism verifying $\phi f = \phi$, then f is an automorphism in R-Mod. Again, by [13, Lemma I.2.1], f is an automorphism in R-gr. Therefore $\phi : E \to M$ is a gr-injective cover. $\square$

Remark. When R is a graded ring with finite support, we have the equivalences [9, p. 124]:

R is left gr-noetherian $\Leftrightarrow$ R_1 is left noetherian and R_g is a finitely generated R_1-module for all $g \in G$ $\Leftrightarrow$ R is left noetherian.

Therefore, when R is a graded ring with finite support, the existence of gr-injective covers for all graded R-module is equivalent to the existence of injective covers for all R-module [2, Theorem 2.1].

Now, we are going to solve the following question: if R is a G-graded ring with finite support, constructing for $M \in R$-gr a gr-injective cover via a known object in R-Mod (using the adjunction (U,F)), and conversely, given $M \in R$-Mod, constructing for $M \in R$-Mod an injective cover with objects in R-gr. We will be able to prove this either in the case that $n = |\{g \in G | R_g \neq 0\}|$ is an invertible element in R or in case F is a separable functor [14, Lemma 3.2].

Proposition 9. *Suppose that $R = \bigoplus_{g\in G} R_g$ is a left noetherian ring with finite support. Let τ be a rigid torsion theory on R-gr and $\bar{\tau}$ the induced torsion theory on R-Mod.*

(a) *If $E \in$ R-gr is τ-injective, then $U(E)$ is $\bar{\tau}$-injective in R-Mod.*

(b) *If $\phi : E \to M$ is a $\bar{\tau}$-injective precover in R-Mod, then $F(\phi) : F(E) \to F(M)$ is a τ-injective precover in R-gr.*

(c) *Suppose that F is a separable functor, if $E \xrightarrow{\phi} F(X)$ is a τ-injective precover in R-gr, then $U(E) \xrightarrow{U(\phi)} UF(X) \xrightarrow{\varepsilon_X} X$ is a $\bar{\tau}$-injective precover, where ε is the co-unit of the adjunction (U,F), $\varepsilon_X({}^g a) = a$, for ${}^g a \in ({}^g X)$ (remember that $F(X) = \bigoplus_{g\in G}({}^g X)$, ${}^g X = X$ for $g \in G$).*

(d) *Suppose that F is a separable functor. Let $X \in$ R-Mod. Then X has a $\bar{\tau}$-injective precover if and only if $F(X)$ has a τ-injective precover.*

Proof. (a) Let $E \in R$-gr be a τ-injective object in R-gr. We consider the short exact sequence:

$$0 \to E \to \mathscr{E}^{gr}(E) \to \mathscr{E}^{gr}(E)/E \to 0,$$

where $\mathscr{E}^{gr}(E)$ is the gr-injective envelope of E in R-gr. Then, applying the forgetful functor U, we obtain the short exact sequence

$$0 \to U(E) \to U(\mathscr{E}^{gr}(E)) \to U(\mathscr{E}^{gr}(E)/E) \to 0.$$

By the hypothesis [9, Corollary 2.3], $U(\mathscr{E}^{gr}(E))$ is an injective R-module. Also, since $\mathscr{E}^{gr}(E)/E$ is τ-torsion-free, then $U(\mathscr{E}^{gr}(E)/E)$ is $\bar{\tau}$-torsion-free (see for example [6, Proposition 2.2]). Now, it is easy to check that $U(\mathscr{E}^{gr}(E))$ is $\bar{\tau}$-injective, with the last two considerations.

(b) and (c) follow by Propositions 4 and 6, respectively. (d) is a consequence of (b) and (c). □

Theorem 2. *Suppose that $R = \bigoplus_{g\in G} R_g$ is a left noetherian ring and F a separable functor. Let τ be a rigid torsion theory on R-gr and $\bar{\tau}$ the induced torsion theory on*

R-Mod. Then, every graded R-module has a τ-injective precover in R-gr if and only if every R-module has a $\bar{\tau}$-injective precover in R-Mod.

Proof. Suppose that every R-module has a $\bar{\tau}$-injective precover in R-Mod. Let M be an object in R-gr. Then $U(M)$ has a $\bar{\tau}$-injective precover $\phi : E \to U(M)$. By part (b) of Proposition 9, $F(\phi) : F(E) \to FU(M)$ is a τ-injective precover in R-gr. Since M is a direct summand of $FU(M) \cong \bigoplus_{g\in G} M(g)$, then it is easy to check that $F(E) \xrightarrow{F(\phi)} FU(M) \xrightarrow{p_1} M$ (where $p_1 : FU(M) \to M$ is the natural projection of $FU(M)$ over M) is a τ-injective precover.

Conversely, suppose that every graded R-module has a τ-injective precover in R-gr and let us consider $N \in R$-Mod. Then $F(N)$ has a τ-injective precover in R-gr $\psi : E \to F(N)$. By part (c) of Proposition 9, $U(E) \xrightarrow{U(\psi)} UF(N) \xrightarrow{\varepsilon_N} N$ is a $\bar{\tau}$-injective precover of N. □

Remark. If we also impose the condition that G is finite in Theorem 2, then U is the right adjoint of F and the proof of the theorem is easier. In particular, in this case, it can be proved that if $\phi : E \to M$ is a τ-injective precover in R-gr, then $U(\phi) : U(E) \to U(M)$ is a $\bar{\tau}$-injective precover in R-Mod.

Finally, we give the most general result about lifting of relative injective covers involving the three categories R-Mod, R-gr and R_1-Mod that we have been able to obtain.

Theorem 3. *Let $R = \bigoplus_{g\in G} R_g$ a strongly graded ring. Let τ be a rigid torsion theory on R-gr. Suppose that R is left noetherian and that F is a separable functor. The following assertions are equivalent.*

(i) *Every graded R-module has a τ-injective precover in R-gr.*
(ii) *Every R-module has a $\bar{\tau}$-injective precover in R-Mod.*
(iii) *Every R_1-module has a τ_1-injective precover in R_1-Mod.*

Proof. The proof follows by Theorem 2 and Proposition 7. □

Example. Let G be a finite group and let S be a left noetherian ring. Suppose that the order of G is an invertible element of S. Then SG (the group ring of S over G) is in the conditions of Theorem 3. (See [14] for the separability of F in this case.)

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